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ON EMBEDDING OF $\mathcal{F}$-HEDGEHOGS IN FUNCTION SPACES

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Abstract. For a filter $\mathcal{F}$, $S_{\mathcal{F}} = \{\infty\} \cup \{(n,m) : n, m \in \mathbb{N}\}$ be the $\mathcal{F}$-hedgehog ($\mathcal{F}$-fan) of spininess ω where each (n,m) is isolated in $S_{\mathcal{F}}$ and a basic open neighborhood of ∞ is of the form $N(\varphi) = \{\infty\} \cup \{(n,m) : n \in \mathbb{N}, m \in \varphi(n)\}$ for function $\varphi\colon \mathbb{N} \to \mathcal{F}$. We study some connections among the $\mathcal{F}^*$-Menger property and an embedding of $\mathcal{F}$-hedgehog $S_{\mathcal{F}}$ into function spaces for any P-filter $\mathcal{F}$.

1. Introduction

A space X is said to be Menger [**9**] if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X, there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is a cover X. A space X is said to have *countable fan-tightness* [**1**] if whenever $A_n \subset X$ and $x \in \bar{A}_n$ $(n \in \mathbb{N})$, there are finite sets $F_n \subset A_n$ such that $x \in \overline{\bigcup\{F_n : n \in \mathbb{N}\}}$.

Let $S_\omega = \{\infty\} \cup \{(n,m) : n, m \in \mathbb{N}\}$ be the sequential hedgehog (sequential fan) of spininess ω, where each (n,m) is isolated in S_ω and a basic open neighborhood of ∞ is of the form $N(\varphi) = \{\infty\} \cup \{(n,m) : n \in \mathbb{N}, m \geqslant \varphi(n)\}$ for a function $\varphi\colon \mathbb{N} \to \mathbb{N}$. Obviously S_ω does not have countable fan-tightness.

Archangel'skiĭ [**1**] proved that every finite power of X is Menger if, and only if, $C_p(X)$ has countable fan-tightness. Hence, if every finite power of X is Menger, S_ω cannot be embedded into $C_p(X)$. A. V. Archangel'skii raised following natural question [**2**, Problem II.2.7]: Can S_ω be embedded into $C_p(X)$ for some Menger space X?

Sakai proved (under CH) that there is a Lusin set X (hence X is Menger) such that S_ω be embedded into $C_p(X)$ [**10**].

In this paper we study some connections among the $\mathcal{F}^*$-Menger property and an embedding of $\mathcal{F}$-hedgehog $S_{\mathcal{F}}$ into function spaces for any P-filter $\mathcal{F}$.

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2. Main definitions and notation

Throughout this paper, all spaces are assumed to be Tychonoff. The set of positive integers is denoted by $\mathbb{N}$ or ω. The space $P(\mathbb{N})$ splits into two important subspaces: the family of infinite subsets of $\mathbb{N}$, denoted $[\mathbb{N}]^{\infty}$, and the family of finite subsets of $\mathbb{N}$, denoted $[\mathbb{N}]^{<\infty}$.

Let $\mathbb{R}$ be the real line, we put $\mathbb{I} = [0,1] \subset \mathbb{R}$, and let $\mathbb{Q}$ be the rational numbers.

Let $C_p(X)$ denote the space of continuous real-valued functions $C(X)$ on a space X with the topology of pointwise convergence. Let $B_0(X) = C(X)$ and inductively define $B_\alpha(X)$ for each ordinal $\alpha \leqslant \omega_1$ to be the space of pointwise limits of sequences of functions in $\bigcup_{\beta<\alpha} B_\beta(X)$. So $B(X) = \bigcup_{\beta<\omega_1} B_\beta(X)$ a set of all functions of Baire, defined on a Tychonoff space X, provided with the pointwise convergence topology.

We recall that a subset of X that is the complete preimage of zero for a certain function from $C(X)$ is called a zero-set. A subset $O \subseteq X$ is called a cozero-set (or functionally open) of X if $X \setminus O$ is a zero-set.

The family of Baire sets of a space X is the smallest family of sets containing the zero sets of continuous real-valued functions, and closed under countable unions and countable intersections. The Baire sets of X of multiplicative class 0, denoted $Z(X)$, are the zero-sets of continuous real-valued functions. The sets of additive class 0, denoted $CZ(X)$, are the complements of the sets in $Z(X)$.

The symbol $\mathbf{0}$ stands for the constant function to 0. A basic open neighborhood of $\mathbf{0}$ is of the form $[F,(-\epsilon,\epsilon)] = \{f \in C(X) : f(F) \subset (-\epsilon,\epsilon)\}$, where $F \in [X]^{<\omega}$ and $\epsilon > 0$.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set.

• Then $S_1(\mathcal{A},\mathcal{B})$ denote the following hypothesis:

For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, $B_n \in A_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

• The symbol $S_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ denote the following hypothesis:

For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, $B_n \subset A_n$ is finite, and $\bigcup\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

• $U_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$: For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, $B_n \subset A_n$ is finite, and $\{\bigcup B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

An open cover $\mathcal{U}$ of a space X is:

• an *ω-cover* if X does not belong to $\mathcal{U}$ and every finite subset of X is contained in a member of $\mathcal{U}$. Note that if $\mathcal{U}$ is an ω-cover of a set X and $X \notin \mathcal{U}$, then each finite subset of X is contained in infinitely many members of $\mathcal{U}$.

• a *γ-cover* if it is infinite and each $x \in X$ belongs to all but finitely many elements of $\mathcal{U}$. Note that every γ-cover contains a countably γ-cover.

• a *γ_F-shrinkable* if it is an γ-cover $\mathcal{U}$ of co-zero sets of X and there exists a γ-cover $\{F(U) : U \in \mathcal{U}\}$ of zero-sets of X with $F(U) \subset U$ for every $U \in \mathcal{U}$.

For a topological space X we denote:

- $\mathcal{O}$—the family of all open covers of X.
- Ω—the family of all open ω-covers of X.

- Γ—the family of all countable open γ-covers of X.
- Γ_F—the family of all countable γ_F-shrinkable covers of X.
- $\mathcal{B}$—the family of all countable Baire covers of X.
- $\mathcal{B}_\Gamma$—the family of all countable Baire γ-covers of X.
- $\mathcal{B}_\Omega$—the family of all countable Baire ω-covers of X.
- $S_1(\mathcal{O},\mathcal{O})$ denote the Rothberger property.
- $S_{\text{fin}}(\mathcal{O},\mathcal{O})$ denotes the Menger property.
- $U_{\text{fin}}(\mathcal{O},\Gamma)$ denotes the Hurewicz property.

Let X be a topological space, and $x \in X$. A subset A of X *converges* to x, $x = \lim A$, if A is infinite, $x \notin A$, and for each neighborhood U of x, $A \setminus U$ is finite.

- $\Omega_x = \{A \subseteq X : x \in \bar{A} \setminus A\}$.
- $\Omega_x^\omega = \{A \subseteq X : |A| = \aleph_0 \text{ and } x \in \bar{A} \setminus A\}$.
- $\Gamma_x = \{A \subseteq X : x = \lim A\}$.
- $\Gamma_x^\omega = \{A \subseteq X : |A| = \aleph_0 \text{ and } x = \lim A\}$.

3. An embedding of sequential hedgehogs in function spaces

THEOREM 3.1. [**10**, Theorem 3.2] *The following conditions are equivalent for a space X:*

(1) *S_ω cannot be embedded into $C_p(X)$.*
(2) *X has property $S_{\text{fin}}(\Gamma_F,\Omega)$.*

Let $\mathcal{P}$ be a topological property. Arhangel'skiĭ calls X *projectively* $\mathcal{P}$ if every second countable continuous image of X is $\mathcal{P}$.

By Theorem 3.1 and [**8**, Theorem 11.1], we have the following result.

THEOREM 3.2. *The following conditions are equivalent for a space X:*

(1) *S_ω cannot be embedded into $C_p(X)$.*
(2) *X has property projectively $S_{\text{fin}}(\Gamma,\Omega)$.*
(3) *$C_p(X)$ has property $S_{\text{fin}}(\Gamma_x^\omega,\Omega_x^\omega)$.*
(4) *$C_p(X)$ has property $S_{\text{fin}}(\Gamma_x,\Omega_x)$.*

Note that, if every finite power of X is projectively Menger, then X is projectively $S_{\text{fin}}(\Omega,\Omega)$ in [**10**, Proposition 4.4].

COROLLARY 3.1. *If every finite power of X is projectively Menger, then the following conditions are equivalent:*

(1) *S_ω cannot be embedded into $C_p(X)$.*
(2) *X has property projectively $S_{\text{fin}}(\Omega,\Omega)$.*

COROLLARY 3.2. [**10**, Proposition 4.12] *Every finite power of X is projectively Menger if, and only if, for any $n \in \mathbb{N}$, S_ω cannot be embedded into $C_p(X^n)$.*

We summarize implications in the following diagram.

$$X \text{ is projectively } S_{\text{fin}}(\Omega,\Omega)$$
$$\Downarrow$$
$$X \text{ is } S_{\text{fin}}(\Gamma_F,\Omega)$$
$$\Updownarrow$$

$$S_\omega \not\subset C_p(X) \Leftrightarrow X \text{ is projectively } S_{\text{fin}}(\Gamma, \Omega)$$
$$\Downarrow$$
$$X \text{ is projectively Menger}$$

Diagram 1.

Lemma 3.1. [**3**, Lemma 80] *Let $X = \{x\} \cup \{x_{n,m} : n, m \in \mathbb{N}\}$ be a Hausdorff space such that $x_{n,m} \to x$ $(m \to \infty)$ for each $n \in \mathbb{N}$, and for any $\varphi \in \mathbb{N}^{\mathbb{N}}$, $x \notin \overline{\{x_{n,m} : n \in \mathbb{N}, m \leqslant \varphi(n)\}}$. Then S_ω can be embedded into X.*

Theorem 3.3. *The following conditions are equivalent for a space X:*

1. *S_ω cannot be embedded into $B(X)$.*
2. *X has property $S_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$.*
3. *X has property $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$.*
4. *$B(X)$ has property $S_{\text{fin}}(\Gamma_x, \Omega_x)$.*

Proof. (3) $\Rightarrow$ (1). Let $S_\omega = \{\mathbf{0}\} \cup \{f_{n,k} : n, k \in \mathbb{N}\} \subseteq B(X)$, where $f_{n,k} \to \mathbf{0}$ $(k \to \infty)$. For each $n, k \in \mathbb{N}$, we put $U_{n,k} = \{x \in X : |f_{n,k}(x)| < \frac{1}{n}\}$. Each $U_{n,k}$ is a Baire set in X. Let $\mathcal{U}_n = \{U_{n,k} : k \in \mathbb{N}\}$. If $I = \{n \in \mathbb{N} : X \in \mathcal{U}_n\}$ is infinite, some sequence $\{f_{n,k_n} : n \in I\}$ converges to $\mathbf{0}$ uniformly. This is a contradiction, so without loss of generality, we may assume $U_{n,k} \neq X$ for each $n, k \in \mathbb{N}$. We can easily check that the condition $f_{n,k} \to \mathbf{0}$ $(k \to \infty)$ implies that $\mathcal{U}_n$ is a Baire γ-cover of X. Then, by (3), there is $\{U_{n,k_n} : n \in \mathbb{N}\}$ a ω-cover of X such that $U_{n,k_n} \in \mathcal{U}_n$ for each $n \in \mathbb{N}$. Then $\mathbf{0} \in \overline{\{f_{n,k_n} : n \in \mathbb{N}\}}$, this is a contradiction.

(1) $\Rightarrow$ (3). Let $\mathcal{U}_n = \{U_{n,k} : k \in \mathbb{N}\}$ be a Baire γ-cover of X for each $n \in \mathbb{N}$ and $\mathcal{U}_\varphi = \{U_{n,k} : n \in \mathbb{N}, k \leqslant \varphi(n)\}$ is not an ω-cover of X for any $\varphi \in \mathbb{N}^{\mathbb{N}}$. For each $n, k \in \mathbb{N}$, we take a Baire function $f_{n,k} : X \to [0, 1]$ such that $f_{n,k}(x) = 0$ for all $x \in U_{n,k}$ and $f_{n,k} = 1$ for all $x \in X \setminus U_{n,k}$. Then $f_{n,k} \to \mathbf{0}$ $(k \to \infty)$. Let $\varphi \in \mathbb{N}$. Since $\mathcal{U}_\varphi$ is not an ω-cover of X, there is a finite subset $F \subset X$ such that F is not contained in any member of $\mathcal{U}_\varphi$. Then we can easily check $\{f \in B(X) : f(F) \subset (-\frac{1}{2}, \frac{1}{2})\} \cap \{f_{n,k} : n \in \mathbb{N}, k \leqslant \varphi(n)\} = \emptyset$. By Lemma 3.1, S_ω can be embedded into $\{\mathbf{0}\} \cup \{f_{n,m} : n, m \in \mathbb{N}\} \subset B(X)$.

(2) $\Leftrightarrow$ (3). By Theorem 9 in [**11**], $S_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega) = S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$.

(3) $\Leftrightarrow$ (4). By Theorem 6.1 in [**7**], $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega) = S_{\text{fin}}(\Gamma_x, \Omega_x)$. □

By [**11**, Theorem 6], $S_1(\mathcal{B}_\Gamma, \mathcal{B}) = S_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B})$. Note also that, if all finite powers of X have property $S_1(\mathcal{B}_\Gamma, \mathcal{B})$, then X has property $S_{\text{fin}}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ [**11**, Theorem 20].

Corollary 3.3. *If all finite powers of X have property $S_1(\mathcal{B}_\Gamma, \mathcal{B})$ then the following conditions are equivalent:*

1. *S_ω cannot be embedded into $B(X)$.*
2. *X has property $S_{\text{fin}}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$.*

Corollary 3.4. *Every finite power of X have property $S_1(\mathcal{B}_\Gamma, \mathcal{B})$ if, and only if, for any $n \in \mathbb{N}$, S_ω cannot be embedded into $B(X^n)$.*

We summarize implications in the following diagram.

$$\begin{gathered} X \text{ is } S_{\text{fin}}(\mathcal{B}_\Omega, \mathcal{B}_\Omega) \\ \Downarrow \\ S_\omega \not\subset B(X) \Leftrightarrow X \text{is } S_{\text{fin}}(\mathcal{B}_\Gamma, \mathcal{B}_\Omega) \\ \Downarrow \\ X \text{ is } S_1(\mathcal{B}_\Gamma, \mathcal{B}) \end{gathered}$$

Diagram 2.

PROPOSITION 3.1. *There is a space X such that S_ω can be embedded into $B(X)$, but S_ω cannot be embedded into $C_p(X)$.*

PROOF. Let X be the real line $\mathbb{R}$ with the usual topology. By [**4**, Theorem 2.2], every σ-compact topological space is a member of class $S_{\text{fin}}(\Omega, \Omega)$. Hence, X has the property $S_{\text{fin}}(\Gamma, \Omega)$. By Theorem 3.2, S_ω cannot be embedded into $C_p(X)$. Since X has not property $S_1(\Gamma, \Omega)$, it has not property $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Omega)$. Hence, by Theorem 3.3, S_ω can be embedded into $B(X)$. □

4. An embedding of $\mathcal{F}$-hedgehogs in function spaces

For sets $a, b \in [\mathbb{N}]^\infty$, we write $a \subseteq^* b$ if the set $a \setminus b$ is finite. A semifilter [**12**] is a set $S \subseteq [\mathbb{N}]^\infty$ such that, for each set $s \in S$ and each set $b \in [\mathbb{N}]^\infty$ with $s \subseteq^* b$, we have $b \in S$. Important examples of semifilters include the maximal semifilter $[\mathbb{N}]^\infty$, the minimal semifilter cF of all cofinite sets, and every nonprincipal ultrafilter on $\mathbb{N}$.

By filter we mean a semifilter closed under finite intersections.

An infinite set $B \subseteq \mathbb{N}$ is said to be a *pseudointersection* of a family $\mathcal{A} \subseteq \mathcal{F}$ if $B \subseteq^* A$ for any $A \in \mathcal{A}$. By P-filter we mean a semifilter $\mathcal{F}$ closed under countable pseudointersection, i.e. if $\mathcal{A} = \{A_n : A_n \in \mathcal{F}, n \in \mathbb{N}\}$ and B is a pseudointersection of $\mathcal{A}$ then $B \in \mathcal{F}$.

DEFINITION 4.1. Let $\mathcal{F}$ be a filter. A sequence $(x_n : n \in \mathbb{N})$ of elements of a topological space X $\mathcal{F}^*$-*converges* to $x \in X$, written $x_n \overset{\mathcal{F}^*}{\to} x$, if

(1) for every neighborhood U of x, we have $\{n \in \mathbb{N} : x_n \in U\} \in \mathcal{F}$,
(2) for every $F \in \mathcal{F}$ there is a neighborhood U of x such that $\{n \in \mathbb{N} : x_n \in U\} = F$.

For a filter $\mathcal{F}$, $S_\mathcal{F} = \{\infty\} \cup \{(n, m) : n, m \in \mathbb{N}\}$ be the $\mathcal{F}$-hedgehog ($\mathcal{F}$-fan) of spininess ω, where each (n, m) is isolated in $S_\mathcal{F}$ and a basic open neighborhood of ∞ is of the form $N(\varphi) = \{\infty\} \cup \{(n, m) : n \in \mathbb{N}, m \in \varphi(n)\}$ for function $\varphi : \mathbb{N} \to \mathcal{F}$.

First, we note that the topology of $S_\mathcal{F}$ can be characterized by the following conditions:

(a) the points of $\mathbb{N} \times \mathbb{N}$ are isolated,
(b) for every $n \in \mathbb{N}$, the sequence $((n, m) : m \in \mathbb{N})$ $\mathcal{F}^*$-converges to $\{\infty\}$,
(c) if $A \subset \mathbb{N} \times \mathbb{N}$, and for each $n \in \mathbb{N}$ there is $B_n \in \mathcal{F}$ such that $A \cap \{(n, m) : m \in B_n\} = \emptyset$, then $\{\infty\} \notin \bar{A}$.

We need the following lemma, similar to Lemma 79 in [**3**].

LEMMA 4.1. *Let $\mathcal{F} \subseteq [\mathbb{N}]^\infty$ be a P-filter and let $X = \{x_{n,m} : n, m \in \mathbb{N}\} \cup \{p\}$ be a Hausdorff space such that*

(1) *all points* $x_{n,m}$ *and* p *are distinct,*
(2) *for every* $n, m, k \in \mathbb{N}$, $x_{n,m} \notin \overline{\{x_{n_i,j} : 1 \leqslant i \leqslant k, j \in \mathbb{N}\} \setminus \{x_{n,m}\}}$,
(3) *for every* $n \in \mathbb{N}$, $\sigma_n = (x_{n,m} : m \in \mathbb{N})$ $\mathcal{F}^*$*-converges to* p*, and*
(4) *if* $A \subset X \setminus \{p\}$*, and for each* $n \in \mathbb{N}$ *there is* $B_n \in \mathcal{F}$ *such that* $A \cap \{x_{n,m} : m \in B_n\} = \emptyset$*, then* $p \notin \bar{A}$.

Then X *contains a subspace homeomorphic to* $S_{\mathcal{F}}$.

Proof. For $n \in \mathbb{N}$, denote $S_n = \{x_{n,m} : m \in \mathbb{N}\}$. For every n, m, there are disjoint neighborhoods $O_{n,m} \ni q_{n,m}$ and $N_{n,m} \ni p$. Hence, there exists $P_{n,m} = \{F^i_{n,m} : i \in \mathbb{N}, F^i_{n,m} \in \mathcal{F}\}$ such that $\{p\} \cup \{x_{n,i} : n \in \mathbb{N}, i \in F^i_{n,m}\} \subseteq N_{n,m}$. Since $\mathcal{F}$ is a P-filter, there is a pseudointersection $B \in \mathcal{F}$ of $\{F^i_{n,m} : i, n, m \in \mathbb{N}\}$. Since $B \subseteq^* F^i_{n,m}$ for any $i, n, m \in \mathbb{N}$, there is a function $\varphi : \mathbb{N} \to \mathcal{F} \cap B$ such that for every n, m, there are at most finitely many k such that $N_{n,m} \cap S_k \not\supseteq \{x_{k,l} : l \in \varphi(k) \subseteq B \cap F^l_{n,m}\}$. Denote the set of all these k by $K_{n,m}$.

Put $Z = \{p\} \cup \{x_{k,l} : l \in \varphi(k)\}$. Put $h(p) = \{\infty\}$ and $h(x_{k,l}) = (k, \psi_k(l))$ whenever $l \in \varphi(k)$ where $\psi_k : \varphi(k) \to \mathbb{N}$ is a monotonic bijection for every $k \in \mathbb{N}$. Then h is an homeomorphism of Z onto $S_{\mathcal{F}}$. We have to check only that the point of $Z \setminus \{p\}$ are isolated in Z. Let $x_{n,m} \in Z$ and $C_{n,m} = \{p\} \cup \{x_{k,l} : k \in K_{n,m}, l \in \varphi(k)\}$. Since $O_{n,m} \cap N_{n,m} = \emptyset$, $O_{n,m} \cap Z \subset C_{n,m}$. By condition (2), all points of $C_{n,m}$ other than p are isolated. □

Definition 4.2. Let X be a topological space and $\mathcal{F} \subseteq [\mathbb{N}]^\infty$ be a filter.

• A cover $\mathcal{V} = (V_n : n \in \mathbb{N})$ is a $\mathcal{F}^*$-γ*-cover*, if it is infinite, each $x \in X$ $\{n : x \in V_n\} \in \mathcal{F}$ and each $F \in \mathcal{F}$ there is $K \in [X]^{<\infty}$ such that $\{n : K \subseteq V_n\} = F$.

• A cover $\{V_n : n \in \mathbb{N}\}$ is called a *refinement* of the cover $\{U_n : n \in \mathbb{N}\}$, if $V_n \subseteq U_n$ for each $n \in \mathbb{N}$. An $\mathcal{F}^*$-γ-cover $\{U_n : n \in \mathbb{N}\}$ is $\mathcal{F}^*$-γ_F*-shrinkable* if there exists a zero-set $\mathcal{F}^*$-γ-cover that is a refinement of $\{U_n : n \in \mathbb{N}\}$.

For a topological space X and a filter $\mathcal{F} \subseteq [\mathbb{N}]^\infty$ we denote:

- $\mathcal{F}^*$-Γ the family of all countable open $\mathcal{F}^*$-γ-covers of X.
- $\mathcal{F}^*$-Γ_F the family of all countable co-zero $\mathcal{F}^*$-γ-shrinkable covers of X.
- $\mathcal{F}^*$-$\Gamma^\omega_x = \{A \subseteq X : |A| = \aleph_0 \text{ and } A \xrightarrow{\mathcal{F}^*} x\}$.

Definition 4.3. Let $\mathcal{F} \subseteq [\mathbb{N}]^\infty$ be a semifilter. A space X is $\mathcal{F}^*$-Menger, if for every sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open $\mathcal{F}^*$-Γ covers of X, there are finite subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\bigcup\{\mathcal{V}_n : n \in \mathbb{N}\}$ is an open cover of X, i.e., X has property $U_{\text{fin}}(\mathcal{F}^*\text{-}\Gamma, \mathcal{O})$.

Definition 4.4. Let $\mathcal{P}$ be a topological property. A space X has property *condensationly* $\mathcal{P}$ if every second countable one-to-one continuous image of X is $\mathcal{P}$.

Note that if X has the property projectively $\mathcal{P}$ then it has the property condensationly $\mathcal{P}$.

Proposition 4.1. *Let* $\mathcal{F}$ *be a filter and* X *has a coarser second countable topology. Then* X *has the property* $S_{\text{fin}}(\mathcal{F}^*\text{-}\Gamma_F, \Omega)$ *if and only if it has the property condensationly* $S_{\text{fin}}(\mathcal{F}^*\text{-}\Gamma, \Omega)$.

PROOF. The proof is similar to the proof of [**8**, Theorem 10.2]. $\square$

THEOREM 4.1. *Let $\mathcal{F}$ be a P-filter. Then the following conditions are equivalent for a space X:*

(1) *$S_{\mathcal{F}}$ cannot be embedded into $C_p(X)$.*
(2) *X has property $S_{\mathrm{fin}}(\mathcal{F}^*\text{-}\Gamma_F, \Omega)$.*
(3) *$C_p(X)$ has property $S_{\mathrm{fin}}(\mathcal{F}^*\text{-}\Gamma^{\omega}_x, \Omega^{\omega}_x)$.*

PROOF. (1) $\Rightarrow$ (2). Assume that there is a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ such that, for each n, $\mathcal{U}_n \in \mathcal{F}^*\text{-}\Gamma_F$, and if $\mathcal{W}_n \in [\mathcal{U}_n]^{<\omega}$ for each $n \in \mathbb{N}$ then $\bigcup\{\mathcal{W}_n : n \in \mathbb{N}\} \notin \Omega$. Let $\mathcal{V}_n = \mathcal{U}_1 \cap ... \cap \mathcal{U}_n$ for each $n \in \mathbb{N}$. Since $\mathcal{F}$ is a P-filter, $\mathcal{V}_n \in \mathcal{F}^n\text{-}\Gamma^*_F$ for each $n \in \mathbb{N}$. By Theorem 6 and [**5**, Corollary 7], $\mathcal{F}^n$ is homeomorphic to $\mathcal{F}$ for any $n \in \mathbb{N}$. Hence, $\mathcal{V}_n \in \mathcal{F}\text{-}\Gamma^*_F$ for each $n \in \mathbb{N}$.

Let $\mathcal{V}_n = \{V_{n,m} : m \in \mathbb{N}\}$ and $\mathcal{Z}_n = \{Z_{n,m} : m \in \mathbb{N}\}$ is a zero-set family such that $\mathcal{Z}_n \in \mathcal{F}\text{-}\Gamma$ and $Z_{n,m} \subseteq V_{n,m}$ for each $m \in \mathbb{N}$.

For each $n, m \in \mathbb{N}$, we put

$$f_{n,m}(x) = \begin{cases} 0, & x \in Z_{n,m} \\ n + \frac{1}{m}, & x \in X \setminus V_{n,m}. \end{cases}$$

Consider the set $Y = \{\mathbf{0}\} \cup \{f_{n,m} : n, m \in \mathbb{N}\}$. By construction, the set Y have all conditions in Lemma 4.1.

We check the condition (4). Let $A \subset Y \setminus \{\mathbf{0}\}$, and for each $n \in \mathbb{N}$ there is $B_n \in \mathcal{F}$ such that $A \cap \{f_{n,m} : m \in B_n\} = \emptyset$. Consider a pseudointersection S of $\{B_n : n \in \mathbb{N}\}$. Since $\mathcal{F}$ is a P-filter, $S \in \mathcal{F}$. There exists a neighborhood W_1 of $\mathbf{0}$ such that $|W_1 \cap A \cap \{f_{n,m} : m \in \mathbb{N}\}| < \aleph_0$ for each $n \in \mathbb{N}$. Note that if $\mathcal{O}_n \in [\mathcal{V}_n]^{<\omega}$ for each $n \in \mathbb{N}$ then $\bigcup\{\mathcal{O}_n : n \in \mathbb{N}\} \notin \Omega$. Hence there is a neighborhood W_2 of $\mathbf{0}$ such that $W_2 \cap W_1 \cap A \cap \{f_{n,m} : m \in \mathbb{N}\} = \emptyset$ for each $n \in \mathbb{N}$. Let $W = W_1 \cap W_2$ then $W \cap A = \emptyset$ and $\mathbf{0} \notin \bar{A}$.

(2) $\Rightarrow$ (1). Assume that $S_{\mathcal{F}} = \{\mathbf{0}\} \cup \{f_{n,m} : n, m \in \mathbb{N}\} \subset C_p(X)$, where $f_{n,m}$ $\mathcal{F}^*$-converges to $\mathbf{0}$ $(m \to \infty)$. For each $n, m \in \mathbb{N}$, we put

$U_{n,m} = \{x \in X : |f_{n,m}(x)| < \frac{1}{n}\}$, $Z_{n,m} = \{x \in X : |f_{n,m}(x)| \leqslant \frac{1}{n+1}\}$.

Each $U_{n,m}$ (resp., $Z_{n,m}$) is a cozero-set (resp., zero-set) in X with $Z_{n,m} \subset U_{n,m}$. Let $\mathcal{U}_n = \{U_{n,m} : m \in \mathbb{N}\}$ and $\mathcal{Z}_n = \{Z_{n,m} : m \in \mathbb{N}\}$. If $I = \{n \in \mathbb{N} : X \in \mathcal{U}_n\}$ is infinite, some sequence $\{f_{n,m_n} : n \in I\}$ converges to $\mathbf{0}$ uniformly. This is a contradiction, so without loss of generality, we may assume $U_{n,m} \neq X$ for each $n, m \in \mathbb{N}$. We can easily check that the condition $f_{n,m}$ $\mathcal{F}$-converges to $\mathbf{0}$ $(m \to \infty)$ implies that $\mathcal{Z}_n \in \mathcal{F}\text{-}\Gamma^*_F$ of X. By condition (2), there is a sequence $(\mathcal{W}_n : n \in \mathbb{N})$ such that, for each n, $\mathcal{W}_n \subset \mathcal{Z}_n$ is finite, and $\bigcup\{\mathcal{W}_n : n \in \mathbb{N}\}$ is an element of Ω. Let $\mathcal{W}_n = \{Z_{n,m_1}, ..., Z_{n,m_{k(n)}}\}$ for each $n \in \mathbb{N}$. Then $\mathbf{0} \in \overline{\{f_{n,m_i} : n \in \mathbb{N}, 1 \leqslant i \leqslant k(n)\}}$. This is a contradiction.

The proof of implication (2$\Leftrightarrow$3) is similar to the proof of Theorem 7.2 in [**6**]. $\square$

COROLLARY 4.1. *Let $\mathcal{F}$ be a P-filter and X has a coarser second countable topology. Then $S_{\mathcal{F}}$ cannot be embedded into $C_p(X)$ if and only if X has property condensationly $S_{\mathrm{fin}}(\mathcal{F}^*\text{-}\Gamma, \Omega)$.*

Theorem 4.2. *The following conditions are equivalent for a space X:*

(1) $S_{\mathcal{F}}$ *cannot be embedded into* $B(X)$.
(2) X *has property* $S_1(\mathcal{B}_{\mathcal{F}^*-\Gamma}, \mathcal{B}_{\Omega})$.
(3) $B(X)$ *has property* $S_{\text{fin}}(\mathcal{F}^*\text{-}\Gamma_x, \Omega_x)$.

Proof. The proof of implication $(1 \Leftrightarrow 2)$ is similar to the proof of implication $(1 \Leftrightarrow 2)$ of Theorem 4.1. The proof of implication $(2 \Leftrightarrow 3)$ is similar to the proof of implication $(1 \Leftrightarrow 2)$ of [**7**, Theorem 6.1]. □

Corollary 4.2. *Assume that X has property $S_{\text{fin}}(\mathcal{B}_{\Omega}, \mathcal{B}_{\Omega})$ and $\mathcal{F}$ is a P-filter. Then $S_{\mathcal{F}}$ cannot be embedded into $B(X)$.*

We summarize implications observed in this paper (**con.** is an abbreviation for **condensationly**).

$$X \text{ is } S_{\text{fin}}(\mathcal{B}_{\Omega}, \mathcal{B}_{\Omega})$$
$$\Downarrow$$
$$S_{\mathcal{F}} \not\subset B(X) \Leftrightarrow X \text{ is } S_{\text{fin}}(\mathcal{B}_{\mathcal{F}^*-\Gamma}, \mathcal{B}_{\Omega}) \Rightarrow X \text{ is } S_{\text{fin}}(\mathcal{B}_{\mathcal{F}^*-\Gamma}, \mathcal{B})$$
$$\Downarrow$$
$$X \text{ is con. } S_{\text{fin}}(\Omega, \Omega) \Rightarrow S_{\mathcal{F}} \not\subset C_p(X) \Leftrightarrow X \text{ is con. } S_{\text{fin}}(\mathcal{F}^*\text{-}\Gamma, \Omega)$$
$$\Downarrow$$
$$X \text{ is con. } \mathcal{F}^*\text{-Menger}$$

Diagram 3.

Question 1. Assume that all finite powers of X have property condensationly $\mathcal{F}^*$-Menger.

a). Does it follow that X satisfies condensationly $S_{\text{fin}}(\Omega, \Omega)$?
b). Does it follow that $S_{\mathcal{F}}$ cannot be embedded into $C_p(X^n)$ for every $n \in \mathbb{N}$?

Question 2. Assume that all finite powers of X have property $S_{\text{fin}}(\mathcal{B}_{\mathcal{F}^*-\Gamma}, \mathcal{B})$.

a). Does it follow that X satisfies $S_{\text{fin}}(\mathcal{B}_{\Omega}, \mathcal{B}_{\Omega})$?
b). Does it follow that $S_{\mathcal{F}}$ cannot be embedded into $B(X^n)$ for every $n \in \mathbb{N}$?

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