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RECURSION FORMULAS FOR H_1–H_7 HORN HYPERGEOMETRIC FUNCTIONS

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Abstract. We derive the recursion formulas for Horn hypergeometric functions H_1 to H_7. These recursion formulas help us write these functions as a combination of themselves.

Introduction

The Horn functions initially enumerated by Jakob Horn in 1931. They were corrected by Borngasser in 1933. These functions are 34 distinct convergent hypergeometric series of order two out of which 14 are complete series and 20 are confluent series. The importance of Horn functions is supported by their basic role in Complexity theory.

Recently, Opps et al. [**1**] have obtained some recursion formulas for F_2 by the contiguous relation of the Gauss hypergeometric series ${}_2F_1$, and then applied the relations to the radiation field problem. In [**6**], Wang has established recursion formulas for Appell hypergeometric functions and in [**3**], Sahin and Agha have obtained recursion formulas for G_1 and G_2. Also recursion is one of the central ideas of computer science.

Here we construct various recursion formulas for Horn hypergeometric functions H_1–H_7.

The Horn hypergeometric functions [**4, 5**] H_1–H_7 are defined as follows:

$$H_1(\alpha,\beta,\gamma;\delta;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_{m+n}(\gamma)_n}{(\delta)_m}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m-n}(\beta)_m(\gamma)_n(\delta)_n}{(\epsilon)_m}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_3(\alpha,\beta;\gamma;x,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!},$$

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$$H_4(\alpha,\beta;\gamma,\delta;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_m(\delta)_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_5(\alpha,\beta;\gamma;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m+n}(\beta)_{n-m}}{(\gamma)_n}\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_6(\alpha,\beta,\gamma;x,y) = \sum_{m,n=0}^{\infty} (\alpha)_{2m-n}(\beta)_{n-m}(\gamma)_n\frac{x^m}{m!}\frac{y^n}{n!},$$

$$H_7(\alpha,\beta,\gamma;\delta;x,y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2m-n}(\beta)_n(\gamma)_n}{(\delta)_m}\frac{x^m}{m!}\frac{y^n}{n!}.$$

We recall that Gamma function is defined in [**2**] by

$$\Gamma(n) = \int_0^{\infty} t^{n-1}e^{-t}dt, \quad Re(n) > 0,$$

and the Pochhammer symbol $(\lambda)_n$ is given by

$$(\lambda)_0 := 1 \quad \text{and} \quad (\lambda)_n := \lambda(\lambda+1)\dots(\lambda+n-1) = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}, \qquad n = 1,2,\dots$$

Here we give seven sections to present the recursion formulas of Horn hypergeometric functions H_1–H_7 with all parameters.

1. Recursion formulas of H_1

THEOREM 1.1. *Recursion formulas for the function H_1 are as follows:*

$$\begin{aligned}(1.1)\quad H_1(\alpha+l,\beta,\gamma;\delta;x,y) &= H_1(\alpha,\beta,\gamma;\delta;x,y)\\ &+ \frac{x\beta}{\delta}\sum_{k=1}^{l} H_1(\alpha+k,\beta+1,\gamma;\delta+1;x,y)\\ &- y\beta\gamma\sum_{k=1}^{l}(3-\alpha-k)_{-2}\, H_1(\alpha-2+k,\beta+1,\gamma+1;\delta;x,y),\end{aligned}$$

$$\begin{aligned}(1.2)\quad H_1(\alpha-l,\beta,\gamma;\delta;x,y) &= H_1(\alpha,\beta,\gamma;\delta;x,y)\\ &- \frac{x\beta}{\delta}\sum_{k=0}^{l-1} H_1(\alpha-k,\beta+1,\gamma;\delta+1;x,y)\\ &+ y\beta\gamma\sum_{k=0}^{l-1}(3-\alpha+k)_{-2}\, H_1(\alpha-2-k,\beta+1,\gamma+1;\delta;x,y).\end{aligned}$$

PROOF. From the definition of the function H_1 and transformation

$$(\alpha+1)_{m-n} = (\alpha)_{m-n}\Big(1+\frac{m}{\alpha}-\frac{n}{\alpha}\Big)$$

we can get the following relation:

$$(1.3)\qquad H_1(\alpha+1,\beta,\gamma;\delta;x,y) = H_1(\alpha,\beta,\gamma;\delta;x,y)$$

$$+\frac{x\beta}{\delta}H_1(\alpha+1,\beta+1,\gamma;\delta+1;x,y)$$
$$-\,y\beta\gamma(2-\alpha)_{-2}\,H_1(\alpha-1,\beta+1,\gamma+1;\delta;x,y).$$

By applying this relation to function H_1 with the parameter α+2, we have

$$H_1(\alpha+2,\beta,\gamma;\delta;x,y)=H_1(\alpha+1,\beta,\gamma;\delta;x,y)+\frac{x\beta}{\delta}H_1(\alpha+2,\beta+1,\gamma;\delta+1;x,y)$$
$$-y\beta\gamma(1-\alpha)_{-2}\,H_1(\alpha,\beta+1,\gamma+1;\delta;x,y)$$
$$=H_1(\alpha,\beta,\gamma;\delta;x,y)+\frac{x\beta}{\delta}[H_1(\alpha+1,\beta+1,\gamma;\delta+1;x,y)+H_1(\alpha+2,\beta+1,\gamma;\delta+1;x,y)]$$
$$-y\beta\gamma[(2-\alpha)_{-2}\,H_1(\alpha-1,\beta+1,\gamma+1;\delta;x,y)+(1-\alpha)_{-2}\,H_1(\alpha,\beta+1,\gamma+1;\delta;x,y)].$$

If we compute the function H_1 with the parameter $\alpha+l$ by relation (1.3) for l times, we find the formula given by (1.1).

Replacing α by α-1 in the relation (1.3), we get

$$H_1(\alpha-1,\beta,\gamma;\delta;x,y)=H_1(\alpha,\beta,\gamma;\delta;x,y)$$
$$-\frac{x\beta}{\delta}H_1(\alpha,\beta+1,\gamma;\delta+1;x,y)$$
$$+\,y\beta\gamma(3-\alpha)_{-2}\,H_1(\alpha-2,\beta+1,\gamma+1;\delta;x,y).$$

If we apply this relation to the function H_1 with the parameter $\alpha-l$ for l times, we obtain the recursion formula (1.2) similar to the proof of formula (1.1). □

Theorem 1.2. *For the function H_1, we have*

$$H_1(\alpha,\beta+l,\gamma;\delta;x,y)=H_1(\alpha,\beta,\gamma;\delta;x,y)$$
$$+\frac{x\alpha}{\delta}\sum_{k=1}^{l}H_1(\alpha+1,\beta+k,\gamma;\delta+1;x,y)$$
$$+\,y\gamma(\alpha)_{-1}\sum_{k=1}^{l}H_1(\alpha-1,\beta+k,\gamma+1;\delta;x,y),$$

$$H_1(\alpha,\beta-l,\gamma;\delta;x,y)=H_1(\alpha,\beta,\gamma;\delta;x,y)$$
$$-\frac{x\alpha}{\delta}\sum_{k=0}^{l-1}H_1(\alpha+1,\beta-k,\gamma;\delta+1;x,y)$$
$$-\,y\gamma(\alpha)_{-1}\sum_{k=0}^{l-1}H_1(\alpha-1,\beta-k,\gamma+1;\delta;x,y),$$

$$H_1(\alpha,\beta,\gamma+l;\delta;x,y)=H_1(\alpha,\beta,\gamma;\delta;x,y)$$
$$+\,y\beta(\alpha)_{-1}\sum_{k=1}^{l}H_1(\alpha-1,\beta+1,\gamma+k;\delta;x,y),$$

$$H_1(\alpha,\beta,\gamma-l;\delta;x,y)=H_1(\alpha,\beta,\gamma;\delta;x,y)$$

$$- y\beta(\alpha)_{-1}\sum_{k=0}^{l-1} H_1(\alpha-1,\beta+1,\gamma-k;\delta;x,y),$$

$$H_1(\alpha,\beta,\gamma;\delta-l;x,y) = H_1(\alpha,\beta,\gamma;\delta;x,y) + x\alpha\beta\sum_{k=0}^{l-1}(2-\delta+k)_{-2}H_1(\alpha+1,\beta+1,\gamma;\delta+1-k;x,y).$$

Now, we present the recursion formulas for other Horn hypergeometric functions. We omit the proof of the given below theorems.

2. Recursion formulas of H_2

In this section we will present the recursion formulas of Horn hypergeometric function H_2 as follows:

Theorem 2.1.

$$H_2(\alpha+l,\beta,\gamma,\delta;\epsilon;x,y) = H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y) + \frac{x\beta}{\epsilon}\sum_{k=1}^{l} H_2(\alpha+k,\beta+1,\gamma,\delta;\epsilon+1;x,y) - y\gamma\delta\sum_{k=1}^{l}(3-\alpha-k)_{-2}\, H_2(\alpha-2+k,\beta,\gamma+1,\delta+1;\epsilon;x,y),$$

$$H_2(\alpha-l,\beta,\gamma,\delta;\epsilon;x,y) = H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y) - \frac{x\beta}{\epsilon}\sum_{k=0}^{l-1} H_2(\alpha-k,\beta+1,\gamma,\delta;\epsilon+1;x,y) + y\gamma\delta\sum_{k=0}^{l-1}(3-\alpha+k)_{-2}\, H_2(\alpha-2-k,\beta,\gamma+1,\delta+1;\epsilon;x,y),$$

$$H_2(\alpha,\beta+l,\gamma,\delta;\epsilon;x,y) = H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y) + \frac{x\alpha}{\epsilon}\sum_{k=1}^{l} H_2(\alpha+1,\beta+k,\gamma,\delta;\epsilon+1;x,y),$$

$$H_2(\alpha,\beta-l,\gamma,\delta;\epsilon;x,y) = H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y) - \frac{x\alpha}{\epsilon}\sum_{k=0}^{l-1} H_2(\alpha+1,\beta-k,\gamma,\delta;\epsilon+1;x,y),$$

$$H_2(\alpha,\beta,\gamma+l,\delta;\epsilon;x,y) = H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y) + y\delta(\alpha)_{-1}\sum_{k=1}^{l} H_2(\alpha-1,\beta,\gamma+k,\delta+1;\epsilon;x,y),$$

$$\begin{aligned}H_2(\alpha,\beta,\gamma-l,\delta;\epsilon;x,y) &= H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y)\\ &\quad - y\delta(\alpha)_{-1}\sum_{k=0}^{l-1} H_2(\alpha-1,\beta,\gamma-k,\delta+1;\epsilon;x,y),\end{aligned}$$

$$\begin{aligned}H_2(\alpha,\beta,\gamma,\delta+l;\epsilon;x,y) &= H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y)\\ &\quad + y\gamma(\alpha)_{-1}\sum_{k=1}^{l} H_2(\alpha-1,\beta,\gamma+1,\delta+k;\epsilon;x,y),\end{aligned}$$

$$\begin{aligned}H_2(\alpha,\beta,\gamma,\delta-l;\epsilon;x,y) &= H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y)\\ &\quad - y\gamma(\alpha)_{-1}\sum_{k=0}^{l-1} H_2(\alpha-1,\beta,\gamma+1,\delta-k;\epsilon;x,y),\end{aligned}$$

$$\begin{aligned}H_2(\alpha,\beta,\gamma,\delta;\epsilon-l;x,y) &= H_2(\alpha,\beta,\gamma,\delta;\epsilon;x,y)\\ &\quad + x\alpha\beta\sum_{k=0}^{l-1}(2-\epsilon+k)_{-2}\, H_2(\alpha+1,\beta+1,\gamma,\delta;\epsilon+1-k;x,y).\end{aligned}$$

3. Recursion formulas of H_3

In this section, we give some recursion formulas for the function H_3. We present the recursion formulas for the function H_3 about the parameter α, β and γ.

Theorem 3.1.

$$\begin{aligned}H_3(\alpha+l,\beta;\gamma;x,y) &= H_3(\alpha,\beta;\gamma;x,y)\\ &\quad + \frac{2x}{\gamma}\sum_{k=1}^{l}(\alpha+k)H_3(\alpha+1+k,\beta;\gamma+1;x,y)\\ &\quad + \frac{y\beta}{\gamma}\sum_{k=1}^{l} H_3(\alpha+k,\beta+1;\gamma+1;x,y),\end{aligned}$$

$$\begin{aligned}H_3(\alpha-l,\beta;\gamma;x,y) &= H_3(\alpha,\beta;\gamma;x,y)\\ &\quad - \frac{2x}{\gamma}\sum_{k=0}^{l-1}(\alpha-k)H_3(\alpha+1-k,\beta;\gamma+1;x,y)\\ &\quad - \frac{y\beta}{\gamma}\sum_{k=0}^{l-1} H_3(\alpha-k,\beta+1;\gamma+1;x,y),\end{aligned}$$

$$\begin{aligned}H_3(\alpha,\beta+l;\gamma;x,y) &= H_3(\alpha,\beta;\gamma;x,y)\\ &\quad + \frac{y\alpha}{\gamma}\sum_{k=1}^{l} H_3(\alpha+1,\beta+k;\gamma+1;x,y),\end{aligned}$$

$$H_3(\alpha,\beta-l;\gamma;x,y)=H_3(\alpha,\beta;\gamma;x,y)-\frac{y\alpha}{\gamma}\sum_{k=0}^{l-1}H_3(\alpha+1,\beta-k;\gamma+1;x,y),$$

$$\begin{aligned}H_3(\alpha,\beta;\gamma-l;x,y)&=H_3(\alpha,\beta;\gamma;x,y)\\&+x(\alpha)_2\sum_{k=0}^{l-1}(2-\gamma+k)_{-2}\,H_3(\alpha+2,\beta;\gamma+1-k;x,y)\\&+y\alpha\beta\sum_{k=0}^{l-1}(2-\gamma+k)_{-2}\,H_3(\alpha+1,\beta+1;\gamma+1-k;x,y).\end{aligned}$$

4. Recursion formulas of H_4

THEOREM 4.1. *Recursion formulas for the function H_4 are as follows:*

$$\begin{aligned}H_4(\alpha+l,\beta;\gamma,\delta;x,y)&=H_4(\alpha,\beta;\gamma,\delta;x,y)\\&+\frac{2x}{\gamma}\sum_{k=1}^{l}(\alpha+k)H_4(\alpha+1+k,\beta;\gamma+1,\delta;x,y)\\&+\frac{y\beta}{\delta}\sum_{k=1}^{l}H_4(\alpha+k,\beta+1;\gamma,\delta+1;x,y),\end{aligned}$$

$$\begin{aligned}H_4(\alpha-l,\beta;\gamma,\delta;x,y)&=H_4(\alpha,\beta;\gamma,\delta;x,y)\\&-\frac{2x}{\gamma}\sum_{k=0}^{l-1}(\alpha-k)H_4(\alpha+1-k,\beta;\gamma+1,\delta;x,y)\\&-\frac{y\beta}{\delta}\sum_{k=0}^{l-1}H_4(\alpha-k,\beta+1;\gamma,\delta+1;x,y),\end{aligned}$$

$$H_4(\alpha,\beta+l;\gamma,\delta;x,y)=H_4(\alpha,\beta;\gamma,\delta;x,y)+\frac{y\alpha}{\delta}\sum_{k=1}^{l}H_4(\alpha+1,\beta+k;\gamma,\delta+1;x,y),$$

$$H_4(\alpha,\beta-l;\gamma,\delta;x,y)=H_4(\alpha,\beta;\gamma,\delta;x,y)-\frac{y\alpha}{\delta}\sum_{k=0}^{l-1}H_4(\alpha+1,\beta-k;\gamma,\delta+1;x,y),$$

$$\begin{aligned}H_4(\alpha,\beta;\gamma-l,\delta;x,y)&=H_4(\alpha,\beta;\gamma,\delta;x,y)\\&+x(\alpha)_2\sum_{k=0}^{l-1}(2-\gamma+k)_{-2}\,H_4(\alpha+2,\beta;\gamma+1-k,\delta;x,y),\end{aligned}$$

$$\begin{aligned}H_4(\alpha,\beta;\gamma,\delta-l;x,y)&=H_4(\alpha,\beta;\gamma,\delta;x,y)\\&\quad+y\alpha\beta\sum_{k=0}^{l-1}(2-\delta+k)_{-2}\,H_4(\alpha+1,\beta+1;\gamma,\delta+1-k;x,y).\end{aligned}$$

5. Recursion formulas of H_5

THEOREM 5.1. *The following recursion formulas hold true for the Horn hypergeometric function H_5:*

$$\begin{aligned}H_5(\alpha+l,\beta;\gamma;x,y)&=H_5(\alpha,\beta;\gamma;x,y)\\&\quad+2x(\beta)_{-1}\sum_{k=1}^{l}(\alpha+k)H_5(\alpha+1+k,\beta-1;\gamma;x,y)\\&\quad+\frac{y\beta}{\gamma}\sum_{k=1}^{l}H_5(\alpha+k,\beta+1;\gamma+1;x,y),\end{aligned}$$

$$\begin{aligned}H_5(\alpha-l,\beta;\gamma;x,y)&=H_5(\alpha,\beta;\gamma;x,y)\\&\quad-2x(\beta)_{-1}\sum_{k=0}^{l-1}(\alpha-k)H_5(\alpha+1-k,\beta-1;\gamma;x,y)\\&\quad-\frac{y\beta}{\gamma}\sum_{k=0}^{l-1}H_5(\alpha-k,\beta+1;\gamma+1;x,y),\end{aligned}$$

$$\begin{aligned}H_5(\alpha,\beta+l;\gamma;x,y)&=H_5(\alpha,\beta;\gamma;x,y)\\&\quad+\frac{y\alpha}{\gamma}\sum_{k=1}^{l}H_5(\alpha+1,\beta+k;\gamma+1;x,y)\\&\quad-x(\alpha)_2\sum_{k=1}^{l}(3-\beta-k)_{-2}\,H_5(\alpha+2,\beta-2+k;\gamma;x,y),\end{aligned}$$

$$\begin{aligned}H_5(\alpha,\beta-l;\gamma;x,y)&=H_5(\alpha,\beta;\gamma;x,y)\\&\quad-\frac{y\alpha}{\gamma}\sum_{k=0}^{l-1}H_5(\alpha+1,\beta-k;\gamma+1;x,y)\\&\quad+x(\alpha)_2\sum_{k=0}^{l-1}(3-\beta+k)_{-2}\,H_5(\alpha+2,\beta-2-k;\gamma;x,y),\end{aligned}$$

$$\begin{aligned}H_5(\alpha,\beta;\gamma-l;x,y)&=H_5(\alpha,\beta;\gamma;x,y)\\&\quad+y\alpha\beta\sum_{k=0}^{l-1}(2-\gamma+k)_{-2}\,H_5(\alpha+1,\beta+1;\gamma+1-k;x,y).\end{aligned}$$

6. Recursion formulas of H_6

THEOREM 6.1. *Recursion formulas for the function H_6 are as follows:*

$$\begin{aligned}
H_6(\alpha+l,\beta,\gamma;x,y) &= H_6(\alpha,\beta,\gamma;x,y)\\
&\quad + 2x(\beta)_{-1}\sum_{k=1}^{l}(\alpha+k)H_6(\alpha+1+k,\beta-1,\gamma;x,y)\\
&\quad - y\beta\gamma\sum_{k=1}^{l}(3-\alpha-k)_{-2}\,H_6(\alpha-2+k,\beta+1,\gamma+1;x,y),
\end{aligned}$$

$$\begin{aligned}
H_6(\alpha-l,\beta,\gamma;x,y) &= H_6(\alpha,\beta,\gamma;x,y)\\
&\quad - 2x(\beta)_{-1}\sum_{k=0}^{l-1}(\alpha-k)H_6(\alpha+1-k,\beta-1,\gamma;x,y)\\
&\quad + y\beta\gamma\sum_{k=0}^{l-1}(3-\alpha+k)_{-2}\,H_6(\alpha-2-k,\beta+1,\gamma+1;x,y),
\end{aligned}$$

$$\begin{aligned}
H_6(\alpha,\beta+l,\gamma;x,y) &= H_6(\alpha,\beta,\gamma;x,y)\\
&\quad + y\gamma(\alpha)_{-1}\sum_{k=1}^{l}H_6(\alpha-1,\beta+k,\gamma+1;x,y)\\
&\quad - x(\alpha)_2\sum_{k=1}^{l}(3-\beta-k)_{-2}\,H_6(\alpha+2,\beta-2+k,\gamma;x,y),
\end{aligned}$$

$$\begin{aligned}
H_6(\alpha,\beta-l,\gamma;x,y) &= H_6(\alpha,\beta,\gamma;x,y)\\
&\quad - y\gamma(\alpha)_{-1}\sum_{k=0}^{l-1}H_6(\alpha-1,\beta-k,\gamma+1;x,y)\\
&\quad + x(\alpha)_2\sum_{k=0}^{l-1}(3-\beta+k)_{-2}\,H_6(\alpha+2,\beta-2-k,\gamma;x,y),
\end{aligned}$$

$$\begin{aligned}
H_6(\alpha,\beta,\gamma+l;x,y) &= H_6(\alpha,\beta,\gamma;x,y)\\
&\quad + y\beta(\alpha)_{-1}\sum_{k=1}^{l}H_6(\alpha-1,\beta+1,\gamma+k;x,y),
\end{aligned}$$

$$\begin{aligned}
H_6(\alpha,\beta,\gamma-l;x,y) &= H_6(\alpha,\beta,\gamma;x,y)\\
&\quad - y\beta(\alpha)_{-1}\sum_{k=0}^{l-1}H_6(\alpha-1,\beta+1,\gamma-k;x,y).
\end{aligned}$$

7. Recursion formulas of H_7

In this part, we will present theorem about the recursion formulas of H_7.

THEOREM 7.1. *Recursion formulas for the function H_7 are as follows:*

$$\begin{aligned}H_7(\alpha+l,\beta,\gamma;\delta;x,y) &= H_7(\alpha,\beta,\gamma;\delta;x,y)\\ &\quad+\frac{2x}{\delta}\sum_{k=1}^{l}(\alpha+k)H_7(\alpha+1+k,\beta,\gamma;\delta+1;x,y)\\ &\quad-y\beta\gamma\sum_{k=1}^{l}(3-\alpha-k)_{-2}\,H_7(\alpha-2+k,\beta+1,\gamma+1;\delta;x,y),\end{aligned}$$

$$\begin{aligned}H_7(\alpha-l,\beta,\gamma;\delta;x,y) &= H_7(\alpha,\beta,\gamma;\delta;x,y)\\ &\quad-\frac{2x}{\delta}\sum_{k=0}^{l-1}(\alpha-k)H_7(\alpha+1-k,\beta,\gamma;\delta+1;x,y)\\ &\quad+y\beta\gamma\sum_{k=0}^{l-1}(3-\alpha+k)_{-2}\,H_7(\alpha-2-k,\beta+1,\gamma+1;\delta;x,y),\end{aligned}$$

$$\begin{aligned}H_7(\alpha,\beta+l,\gamma;\delta;x,y) &= H_7(\alpha,\beta,\gamma;\delta;x,y)\\ &\quad+y\gamma(\alpha)_{-1}\sum_{k=1}^{l}H_7(\alpha-1,\beta+k,\gamma+1;\delta;x,y),\end{aligned}$$

$$\begin{aligned}H_7(\alpha,\beta-l,\gamma;\delta;x,y) &= H_7(\alpha,\beta,\gamma;\delta;x,y)\\ &\quad-y\gamma(\alpha)_{-1}\sum_{k=0}^{l-1}H_7(\alpha-1,\beta-k,\gamma+1;\delta;x,y),\end{aligned}$$

$$\begin{aligned}H_7(\alpha,\beta,\gamma+l;\delta;x,y) &= H_7(\alpha,\beta,\gamma;\delta;x,y)\\ &\quad+y\beta(\alpha)_{-1}\sum_{k=1}^{l}H_7(\alpha-1,\beta+1,\gamma+k;\delta;x,y),\end{aligned}$$

$$\begin{aligned}H_7(\alpha,\beta,\gamma-l;\delta;x,y) &= H_7(\alpha,\beta,\gamma;\delta;x,y)\\ &\quad-y\beta(\alpha)_{-1}\sum_{k=0}^{l-1}H_7(\alpha-1,\beta+1,\gamma-k;\delta;x,y).\end{aligned}$$

$$\begin{aligned}H_7(\alpha,\beta,\gamma;\delta-l;x,y) &= H_7(\alpha,\beta,\gamma;\delta;x,y)\\ &\quad+x(\alpha)_2\sum_{k=0}^{l-1}(2-\delta+k)_{-2}\,H_7(\alpha+2,\beta,\gamma;\delta+1-k;x,y).\end{aligned}$$

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