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INEQUALITIES FOR THE GENERALIZED DERIVATIVE OF A COMPLEX POLYNOMIAL

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Abstract. We extend the well known Bernstein inequality to the generalized derivative of a polynomial introduced by Sz-Nagy [**13**] with restricted zeros and obtain some new inequalities with respect to sup-norm. The obtained results include some known inequalities of Turán, Malik and Govil as special cases for the ordinary derivative.

1. Introduction

One of the interesting problems related to polynomials was posed by Mendeleev [**10**], which says how large is the size of modulus of the derivative of $P(x)$ on a given interval? This extremal problem was solved by Markov [**9**] for real polynomial of degree at most n in a generalized form. He proved that if $P(x) = \sum_{\nu=0}^{n} a_\nu x^\nu$ is an algebraic polynomial of degree $\leqslant n$ with real coefficients, then

$$\max_{-1\leqslant x\leqslant 1} |P'(x)| \leqslant n^2 \max_{-1\leqslant x\leqslant 1} |P(x)|. \tag{1.1}$$

Several years later, an analogue of (1.1) for the unit disc in the complex plane was formulated by Bernstein [**11**] which is commonly known as Bernstein's theorem for trigonometric polynomials. It states that if $t(\theta)$ is a trigonometric polynomial of degree at most n, then

$$\max_{-\pi\leqslant \theta\leqslant \pi} |t'(\theta)| \leqslant n^2 \max_{-\pi\leqslant \theta\leqslant \pi} |t(\theta)|. \tag{1.2}$$

The above inequalities show how fast a polynomial of degree at most n or its derivative can change. Various analogues of these inequalities are known in which the underlying intervals, the sup-norms, and the family of functions are replaced by more general sets, norms, and families of functions, respectively. Several monographs and papers have been published on Markov's and Bernstein's inequalities and their generalizations. For instance one can see Borwein and Erdélyi [**2**], Lorentz et al. [**7**] and Milovanović and Rassias [**12**]. Let $\mathcal{P}_n$ denote the space of all complex

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polynomials of degree less than or equal to n, then for $P \in \mathcal{P}_n$, inequality (1.2) is equivalent to

$$\max_{|z|=1} |P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1.3}$$

Inequality (1.3) is best possible with equality holding for polynomial $P(z) = \lambda z^n$, λ being a complex number. In this direction Turán [**16**] considered the class of polynomials vanishing inside the disc $|z| \leqslant 1$, and proved that if $P \in \mathcal{P}_n$ has all zeros in $|z| \leqslant 1$, then we have

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Aziz [**1**] proved that inequality (1.4) holds true without any restriction on the zeros of polynomials which are self-inversive or self-reciprocal. As an extension of (1.4) to polynomials $P \in \mathcal{P}_n$ having all their zeros in the disc $|z| \leqslant k$, $k \leqslant 1$, Malik [**8**] proved that

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.5}$$

The case when $k \geqslant 1$ of inequality (1.5) was considered by Govil [**6**] who proved that, if $P \in \mathcal{P}_n$ has all its zeros in the disc $|z| \leqslant k$ with $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \geqslant \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{1.6}$$

These inequalities are fundamental and play a very significant role for the proofs of many inverse theorems in polynomial approximation theory. The study of Bernstein-type inequalities which relate the norm of a polynomial to that of its derivative and their various versions are classical topics in analysis. These inequalities have been extended widely in the literature from ordinary derivative to polar derivative of complex polynomials. For more information related to this area one can refer to a very recent monograph due to Gardner, Govil and Milovanović [**5**]. In this paper, we approach this study and extend the above inequalities to the generalized derivative of a polynomial introduced by Sz-Nagy [**13**] and obtain some new interesting inequalities which include the above inequalities as particular cases.

Definition 1.1 (Sz-Nagy generalized derivative). Given a polynomial $P(z) = c(z-z_1)(z-z_2)\cdots(z-z_n)$ of degree n and an n-tuple $\gamma := (\gamma_1, \gamma_2, \dots, \gamma_n)$ of non-negative real numbers not all zero, Sz-Nagy [**13**] introduced a generalized derivative of $P(z)$ defined by

$$P^{\gamma}(z) := P(z) \sum_{j=1}^{n} \frac{\gamma_j}{z-z_j} = \sum_{j=1}^{n} \gamma_j P_j(z), \tag{1.7}$$

where $P_j(z) = c\prod_{i=1, i\neq j}^{n}(z-z_i)$ for $1 \leqslant j \leqslant n$. Note that the ordinary derivative $P'(z)$ of $P(z)$ can be obtained from $P^{\gamma}(z)$ by taking $\gamma_j = 1$ for $j = 1, 2, \dots, n$, that is

$$P^{\gamma}(z) = P'(z) \quad \text{for } \gamma = (1, 1, \dots, 1).$$

Throughout the paper we shall use the following notations

$$S = \{\gamma : \gamma = (\gamma_1, \gamma_2, \dots, \gamma_n), \gamma_i \geqslant 0 \ \forall\ i = 1, 2, \dots, n\} \quad \text{and} \quad \Lambda := \sum_{j=1}^{n} \gamma_j.$$

Many results pertaining to ordinary derivative of a polynomial have been extended to the generalized derivative of a polynomial. For instance, Diaz–Barrero and Egozcue [**3**] have extended a well known Gauss–Lucas theorem to the above generalized derivative of a polynomial. Rather et al. [**15**] have also presented a proof of the Gauss–Lucas theorem for the generalized derivative of a polynomial which is much simpler than the one given by Diaz–Barrero and Egozcue [**3**]. In [**15**] the authors have also extended Jensen's theorem to the Sz-Nagy generalized derivative of a polynomial.

2. Lemmas

In this section, we mention the following lemmas that will be used to prove our main results in the next section. The first lemma is due to Rather, Gulzar and Bhat [**15**].

Lemma 2.1. *If $P \in \mathcal{P}_n$ and $0 \neq \gamma \in S$ and $P^{\gamma}(z)$ is defined as in* (1.7)*, then all the zeros of $P^{\gamma}(z)$ lie in the convex hull of the zeros of $P(z)$.*

The following lemma is a simple consequence of maximum modulus principle (see [**14**, Vol. I, p. 137]).

Lemma 2.2. *If $P \in \mathcal{P}_n$, then for $R \geqslant 1$*

$$\max_{|z|=1} |P(Rz)| \leqslant R^n \max_{|z|=1} |P(z)|.$$

The next lemma is due to Aziz [**1**].

Lemma 2.3. *If $P \in \mathcal{P}_n$ has all its zeros in the disc $|z| \leqslant k$, $k \geqslant 1$, then*

$$\max_{|z|=k} |P(z)| \geqslant \frac{2k^n}{1+k^n} \max_{|z|=1} |P(z)|.$$

The following lemma is due to Frappier, Rahman and Ruscheweyh [**4**].

Lemma 2.4. *If $P \in \mathcal{P}_n$, then for $k \geqslant 1$ we have*

$$\max_{|z|=k} |P(z)| \leqslant k^n \max_{|z|=1} |P(z)| - \phi(k)|P(0)|,$$

where

$$\phi(k) = \begin{cases} k^n - k^{n-1} & \text{if} \quad n > 1 \\ k - 1 & \text{if} \quad n = 1. \end{cases} \tag{2.1}$$

Lemma 2.5. *If $P(z) = a\prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree n and $Q(z) = z^n\overline{P(1/\bar{z})} = \bar{a}\prod_{j=1}^{n}(1 - z\bar{z}_j)$ and $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$ is n-tuple of non-negative real numbers with $\gamma \neq 0$, then*

$$z^{n-1}\overline{P^{\gamma}(1/\bar{z})} \equiv \Lambda Q(z) - zQ^{\gamma}(z) \quad \textit{and} \quad z^{n-1}\overline{Q^{\gamma}(1/\bar{z})} \equiv \Lambda P(z) - zP^{\gamma}(z),$$

where $P^\gamma(z)$ *is defined as in* (1.7) *and* $Q^\gamma(z) = Q(z)\sum_{j=1}^n \gamma_j \bar{z}_j/(z\bar{z}_j - 1)$.

Proof. We have

$$\sum_{j=1}^n \gamma_j Q(z) - zQ^\gamma(z) = Q(z)\sum_{j=1}^n \left(\gamma_j - \frac{\gamma_j z\bar{z}_j}{z\bar{z}_j - 1}\right) = -Q(z)\sum_{j=1}^n \frac{\gamma_j}{z\bar{z}_j - 1}. \tag{2.2}$$

Since $P^\gamma(z) = P(z)\sum_{j=1}^n \frac{\gamma_j}{z-z_j}$, therefore,

$$z^{n-1}\overline{P^\gamma(1/\bar{z})} = -z^n\overline{P(1/\bar{z})}\sum_{j=1}^n \frac{\gamma_j}{z\bar{z}_j - 1} = -Q(z)\sum_{j=1}^n \frac{\gamma_j}{z\bar{z}_j - 1}. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain $z^{n-1}\overline{P^\gamma(1/\bar{z})} \equiv \sum_{j=1}^n \gamma_j Q(z) - zQ^\gamma(z)$

The second part follows on the same lines and the lemma is proved. □

3. Main results and proofs

We extend inequalities (1.4), (1.5) and (1.6) to the Sz-Nagy's generalized derivative of a polynomial defined in (1.7) and obtain some interesting inequalities concerning the derivatives of a polynomial. We first present the following extension of inequality (1.3).

Theorem 3.1. *If* $P \in \mathcal{P}_n$ *has all its zeros* $z_1, z_2, \ldots, z_n$ *in the disc* $|z| \leqslant k$, $k \leqslant 1$ *and* $0 \neq \gamma \in S$, *then*

$$\left(\sum_{j=1}^n \frac{\gamma_j}{1+|z_j|}\right)\max_{|z|=1}|P(z)| \leqslant \max_{|z|=1}|P^\gamma(z)|,$$

where $P^\gamma(z)$ *is defined as in* (1.7). *The result is best possible and the extremal polynomial is* $P(z) = (z+k)^n$.

Proof. Let $P(z) = c\sum_{j=1}^n (z - z_j)$ where $|z_j| \leqslant k$, $j = 1, 2, \ldots, n$, $k \leqslant 1$; then for $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, we have

$$P^\gamma(z) = P(z)\sum_{j=1}^n \frac{\gamma_j}{z - z_j}.$$

This gives for the points $e^{i\theta}, 0 \leqslant \theta \leqslant 2\pi$, which are not the zeros of $P(z)$

$$\frac{e^{i\theta}P^\gamma(e^{i\theta})}{P(e^{i\theta})} = \sum_{j=1}^n \frac{e^{i\theta}\gamma_j}{e^{i\theta} - k_j e^{i\theta_j}}, \quad z_j = k_j e^{\theta_j}, \ \ k_j \leqslant k, \ \ 1 \leqslant j \leqslant n.$$

Therefore

$$\begin{aligned}\operatorname{Re}\frac{e^{i\theta}P^{\gamma}(e^{i\theta})}{P(e^{i\theta})} &= \sum_{j=1}^{n}\operatorname{Re}\frac{e^{i\theta}\gamma_j}{e^{i\theta}-k_je^{i\theta_j}} = \sum_{j=1}^{n}\operatorname{Re}\frac{\gamma_j}{1-k_je^{i(\theta_j-\theta)}}\\ &= \sum_{j=1}^{n}\gamma_j\operatorname{Re}\frac{1-k_je^{-i(\theta_j-\theta)}}{|1-k_je^{i(\theta_j-\theta)}|^2} = \sum_{j=1}^{n}\gamma_j\frac{1-k_j\cos(\theta_j-\theta)}{1+k_j^2-2k_j\cos(\theta_j-\theta)}\\ &\geqslant \sum_{j=1}^{n}\frac{\gamma_j}{1+k_j}\end{aligned}$$

for the points $e^{i\theta}$, $0\leqslant\theta\leqslant 2\pi$, which are not the zeros of $P(z)$. This implies

$$\Big|\frac{e^{i\theta}P^{\gamma}(e^{i\theta})}{P(e^{i\theta})}\Big| \geqslant \operatorname{Re}\frac{e^{i\theta}P^{\gamma}(e^{i\theta})}{P(e^{i\theta})} \geqslant \sum_{j=1}^{n}\frac{\gamma_j}{1+|z_j|}$$

for the points $e^{i\theta}$, $0\leqslant\theta\leqslant 2\pi$, which are not the zeros of $P(z)$. Equivalently

$$|P^{\gamma}(e^{i\theta})| \geqslant \sum_{j=1}^{n}\frac{\gamma_j}{1+|z_j|}|P(e^{i\theta})| \tag{3.1}$$

for the points $e^{i\theta}$, $0\leqslant\theta\leqslant 2\pi$, which are not the zeros of $P(z)$. Since the inequality (3.1) also holds for the points $e^{i\theta}$, $0\leqslant\theta\leqslant 2\pi$ which are the zeros of $P(z)$, therefore it follows that

$$|P^{\gamma}(e^{i\theta})| \geqslant \sum_{j=1}^{n}\frac{\gamma_j}{1+|z_j|}|P(e^{i\theta})| \quad \text{for all } z \text{ on} \quad |z|=1.$$

Hence

$$\max_{|z|=1}|P^{\gamma}(z)| \geqslant \sum_{j=1}^{n}\frac{\gamma_j}{1+|z_j|}\max_{|z|=1}|P(z)|. \qquad \square$$

Using the given condition on the zeros of $P(z)$ in Theorem 3.1, one can easily obtain the following result.

Corollary 3.1. *If $P\in\mathcal{P}_n$ has all its zeros in the disc $|z|\leqslant k$, $k\leqslant 1$ and $0\neq\gamma\in S$, then*

$$\Lambda\max_{|z|=1}|P(z)| \leqslant (1+k)\max_{|z|=1}|P^{\gamma}(z)|, \tag{3.2}$$

where $P^{\gamma}(z)$ is defined as in (1.7). *The result is best possible and equality in* (3.2) *holds for $P(z)=(z+k)^n$.*

Remark 3.1. If we take $\gamma_j=1$ for all $j=1,2,\dots,n$ in (3.2), then we obtain inequality (1.5).

Similarly if we choose $\gamma=(0,\dots,\gamma_j,\dots,0)$ with $\gamma_j=1$ for $1\leqslant j\leqslant n$ in (3.2), we get the following result.

COROLLARY 3.2. *If $P \in \mathcal{P}_n$ has all its zeros $z_1, z_2, \dots, z_n$ in the disc $|z| \leqslant k$, $k \leqslant 1$, then*

$$\max_{|z|=1}\Big|\frac{P(z)}{z-z_j}\Big| \geqslant \frac{1}{1+k}\max_{|z|=1}|P(z)|$$

for $j = 1, 2, \dots, n$.

The result is best possible as shown by $P(z) = (z+k)^n$.

Next, we present the following analogous result for the class of polynomials having all their zeros in $|z| \leqslant k$ where $k \geqslant 1$.

THEOREM 3.2. *If $P \in \mathcal{P}_n$ has all its zeros in the disc $|z| \leqslant k$, $k \geqslant 1$ and $0 \neq \gamma \in S$, then*

$$\Lambda \max_{|z|=1}|P(z)| \leqslant (1+k^n)\max_{|z|=1}|P^{\gamma}(z)|, \tag{3.3}$$

where $P^{\gamma}(z)$ is defined as in (1.7).

REMARK 3.2. If we take $\gamma = (1, 1, \dots, 1)$ in (3.3), we get (1.6).

If we take $\gamma = (0, \dots, \gamma_j, \dots, 0)$ with $\gamma_j = 1$ for $1 \leqslant j \leqslant n$ in (1.7), we obtain

COROLLARY 3.3. *If $P \in \mathcal{P}_n$ has all its zeros $z_1, z_2, \dots, z_n$ in the disc $|z| \leqslant k$, $k \geqslant 1$, then*

$$\max_{|z|=1}\Big|\frac{P(z)}{z-z_j}\Big| \geqslant \frac{1}{1+k^n}\max_{|z|=1}|P(z)|$$

for $j = 1, 2, \dots, n$.

Instead of proving Theorem 3.2, we prove the following more general result which yields Theorem 3.2 as a special case.

THEOREM 3.3. *If $P \in \mathcal{P}_n$ has all its zeros $z_1, z_2, \dots, z_n$ in the disc $|z| \leqslant k$, $k \geqslant 1$ and $0 \neq \gamma \in S$, then*

$$\sum_{j=1}^{n}\frac{k\gamma_j}{k+|z_j|}\max_{|z|=1}|P(z)| \leqslant \frac{1+k^n}{2}\max_{|z|=1}|P^{\gamma}(z)|,$$

where $P^{\gamma}(z)$ is defined as in (1.7).

PROOF. Let $P(z) = c\sum_{j=1}^{n}(z-z_j)$ where $|z_j| \leqslant k$, $j = 1, 2, \dots, n$, $k \geqslant 1$, then, the polynomial $F(z) = P(kz)$ has all its zeros in $|z| \leqslant 1$. Therefore for $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, we have

$$F^{\gamma}(z) = F(z)\sum_{j=1}^{n}\frac{\gamma_j}{z-w_j} \quad \text{where} \quad w_j = z_j/k \quad \text{and} \quad |w_j| \leqslant 1,\ 1 \leqslant j \leqslant n.$$

Proceeding similarly as in the proof of Theorem 3.1, we obtain

$$|F^{\gamma}(z)| \geqslant \sum_{j=1}^{n}\frac{\gamma_j}{1+|z_j|/k}|F(z)| \quad \text{for} \quad |z| = 1,$$

which implies

$$\max_{|z|=1}|F^\gamma(z)| \geqslant \sum_{j=1}^{n}\frac{k\gamma_j}{k+|z_j|}\max_{|z|=1}|F(z)| = \sum_{j=1}^{n}\frac{k\gamma_j}{k+|z_j|}\max_{|z|=k}|P(z)|.$$

Since $P(z)$ is a polynomial of degree n having all zeros in $|z| \leqslant k$ where $k \geqslant 1$, it follows by Lemma 2.3 that

$$\max_{|z|=1}\left|F(z)\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j/k}\right| \geqslant \sum_{j=1}^{n}\frac{k\gamma_j}{k+|z_j|}\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)|,$$

or equivalently,

$$k\max_{|z|=k}\left|P(z)\sum_{j=1}^{n}\frac{\gamma_j}{z-z_j}\right| \geqslant \sum_{j=1}^{n}\frac{k\gamma_j}{k+|z_j|}\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)|,$$

which implies

$$\max_{|z|=k}|P^\gamma(z)| \geqslant \frac{2k^n}{1+k^n}\sum_{j=1}^{n}\frac{\gamma_j}{k+|z_j|}\max_{|z|=1}|P(z)|.$$

Again since $P^\gamma(z)$ is a polynomial of degree $n-1$, invoking Lemma 2.2 it follows that

$$k^{n-1}\max_{|z|=1}|P^\gamma(z)| \geqslant \frac{2k^n}{1+k^n}\sum_{j=1}^{n}\frac{\gamma_j}{k+|z_j|}\max_{|z|=1}|P(z)|,$$

or equivalently,

$$\max_{|z|=1}|P^\gamma(z)| \geqslant \frac{2k}{1+k^n}\sum_{j=1}^{n}\frac{\gamma_j}{k+|z_j|}\max_{|z|=1}|P(z)|. \qquad \square$$

Next, we establish the following refinement of Theorem 3.3.

Theorem 3.4. *If $P \in \mathcal{P}_n$ has all its zeros $z_1, z_2, \dots, z_n$ in the disc $|z| \leqslant k$, $k \geqslant 1$, and $0 \neq \gamma \in S$, then*

$$\max_{|z|=1}|P^\gamma(z)| \geqslant \frac{2k}{1+k^n}\left(\sum_{j=1}^{n}\frac{\gamma_j}{k+|z_j|}\right)\max_{|z|=1}|P(z)| + \frac{\phi(k)}{k^{n-1}}|P^\gamma(0)|,$$

where $P^\gamma(z)$ is defined as in (1.7) and $\phi(k)$ is given by (2.1).

Proof. Proceeding as in the proof of Theorem 3.3, we get

$$\max_{|z|=k}|P^\gamma(z)| \geqslant \frac{2k^n}{1+k^n}\sum_{j=1}^{n}\frac{\gamma_j}{k+|z_j|}\max_{|z|=1}|P(z)|.$$

Since $P^\gamma(z)$ is a polynomial of degree $n-1$, it follows by Lemma 2.4

$$k^{n-1}\max_{|z|=1}|P^\gamma(z)| - \phi(k)|P^\gamma(0)| \geqslant \frac{2k^n}{1+k^n}\sum_{j=1}^{n}\frac{\gamma_j}{k+|z_j|}\max_{|z|=1}|P(z)|$$

Or equivalently

$$\max_{|z|=1}|P^{\gamma}(z)| \geqslant \frac{2k}{1+k^n}\sum_{j=1}^{n}\frac{\gamma_j}{k+|z_j|}\max_{|z|=1}|P(z)| + \frac{\phi(k)}{k^{n-1}}|P^{\gamma}(0)|. \qquad \square$$

The following corollary is an easy consequence of Theorem 3.4

Corollary 3.4. *If $P \in \mathcal{P}_n$ has all its zeros in the disc $|z| \leqslant k$, $k \geqslant 1$ and $0 \neq \gamma \in S$, then*

$$\max_{|z|=1}|P^{\gamma}(z)| \geqslant \frac{\Lambda}{1+k^n}\max_{|z|=1}|P(z)| + \frac{\phi(k)}{k^{n-1}}|P^{\gamma}(0)|,$$

where $P^{\gamma}(z)$ is defined as in (1.7) *and $\phi(k)$ is given by* (2.1).

If we take $\gamma = (1, 1, \dots, 1)$ in Corollary 3.4, we get the following refinement of Govil's inequality (1.6).

Corollary 3.5. *If all the zeros of $P \in \mathcal{P}_n$ lie in the disc $|z| \leqslant k$, $k \geqslant 1$, then*

$$\max_{|z|=1}|P'(z)| \geqslant \frac{n}{1+k^n}\max_{|z|=1}|P(z)| + \frac{\phi(k)}{k^{n-1}}|P'(0)|,$$

where $P^{\gamma}(z)$ is defined as in (1.5) *and $\phi(k)$ is given by* (2.1).

Finally, we prove the following result for self inversive and self reciprocal polynomials.

Theorem 3.5. *If $P \in \mathcal{P}_n$ is self-inversive or self-reciprocal and $0 \neq \gamma \in S$, then*

$$\max_{|z|=1}|P^{\gamma}(z)| \geqslant \frac{1}{2}\sum_{j=1}^{n}\gamma_j \max_{|z|=1}|P(z)|.$$

where $P^{\gamma}(z)$ is defined as in (1.7). *The result is sharp and equality holds if $P(z) = (1+z)^n$.*

Proof. Let $Q(z) = z^n\bar{P}(1/\bar{z})$, then $P(z) = z^n\bar{Q}(1/\bar{z})$, so that by Lemma 2.5

$$|Q^{\gamma}(z)| = \left|\sum_{j=1}^{n}\gamma_j P(z) - zP^{\gamma}(z)\right| \quad \text{for} \quad |z| = 1$$

which gives,

$$|Q^{\gamma}(z)| \geqslant \sum_{j=1}^{n}\gamma_j|P(z)| - |P^{\gamma}(z)| \quad \text{for} \quad |z| = 1$$

or equivalently,

$$|P^{\gamma}(z)| + |Q^{\gamma}(z)| \geqslant \sum_{j=1}^{n}\gamma_j|P(z)| \quad \text{for} \quad |z| = 1.$$

Assume first that $P(z)$ is a self-inversive polynomial of degree n. Then for all complex numbers z one has $Q(z) = z^n\bar{P}(1/\bar{z}) = P(z)$ Using this fact in (3.2), we get

$$2|P^\gamma(z)| \geqslant \sum_{j=1}^{n} \gamma_j |P(z)| \quad \text{for} \quad |z| = 1$$

and hence

$$2 \max_{|z|=1} |P^\gamma(z)| \geqslant \sum_{j=1}^{n} \gamma_j \max_{|z|=1} P(z).$$

This establishes the result for self-inversive polynomials. Next suppose that $P(z)$ is a self-reciprocal polynomial of degree n, then

$$P(z) = z^n P(1/z) = t(z) \quad \text{(say)}$$

so that

$$P^\gamma(z) = t^\gamma(z) = t(z) \sum_{j=1}^{n} \frac{\gamma_j}{z - 1/z_j}.$$

This gives

$$\begin{aligned} z^{n-1} P^\gamma\Big(\frac{1}{z}\Big) &= \frac{P(z)}{z} \sum_{j=1}^{n} \frac{\gamma_j}{1/z - 1/z_j} = P(z) \sum_{j=1}^{n} \frac{-\gamma_j z_j}{z - z_j} \\ &= P(z) \sum_{j=1}^{n} \gamma_j \Big(1 - \frac{z}{z - z_j}\Big) = \sum_{j=1}^{n} \gamma_j P(z) - z P^\gamma(z), \end{aligned}$$

which implies

$$\Big|z^{n-1} P^\gamma\Big(\frac{1}{z}\Big)\Big| = \Big|\sum_{j=1}^{n} \gamma_j P(z) - z P^\gamma(z)\Big| \geqslant \sum_{j=1}^{n} \gamma_j |P(z)| - |P^\gamma(z)| \quad \text{for} \quad |z| = 1$$

or equivalently,

$$|P^\gamma(z)| + \Big|z^{n-1} P^\gamma\Big(\frac{1}{z}\Big)\Big| \geqslant \sum_{j=1}^{n} \gamma_j |P(z)| \quad \text{for} \quad |z| = 1.$$

Hence

$$\begin{aligned} \max_{|z|=1} |P^\gamma(z)| + \max_{|z|=1} |P^\gamma(z)| &\geqslant |P^\gamma(z)| + |z^{n-1} P^\gamma(1/z)| \quad \text{for} \quad |z| = 1 \\ &\geqslant \sum_{j=1}^{n} \gamma_j |P(z)| \quad \text{for} \quad |z| = 1, \end{aligned}$$

which implies

$$\max_{|z|=1} P^\gamma(z) \geqslant \frac{1}{2} \sum_{j=1}^{n} \gamma_j, \max_{|z|=1} P(z). \qquad \square$$

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