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SOME NEW RESULTS ON ABSOLUTE MATRIX SUMMABILITY OF INFINITE SERIES AND FOURIER SERIES

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Abstract. Two known results dealing with absolute summability of infinite series and trigonometric Fourier series are generalized to the $|A, p_n, \beta; \delta|_k$ summability method.

1. Introduction

Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where $A_n(s) = \sum_{v=0}^{n} a_{nv} s_v$, $n = 0, 1, \ldots$ The series $\sum a_n$ is said to be summable $|A, p_n, \beta; \delta|_k$, $k \geqslant 1$, $\delta \geqslant 0$ and β is a real number, if (see [**16**])

$$\sum_{n=1}^{\infty} \Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)} |A_n(s) - A_{n-1}(s)|^k < \infty,$$

where (p_n) is a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-k} = p_{-k} = 0, \quad k \geqslant 1).$$

If we take $\beta = 1$, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$, then $|A, p_n, \beta; \delta|_k$ summability is the same as $|\bar{N}, p_n|_k$ summability method (see [**1**]). For any sequence (λ_n), it should be noted that $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, $\Delta^0\lambda_n = \lambda_n$, $\Delta^k\lambda_n = \Delta\Delta^{k-1}\lambda_n$ for $k = 1, 2, \ldots$ (see [**9**]) and (t_n) is the n-th $(C, 1)$ mean of the sequence (na_n), i.e., $t_n = \frac{1}{n+1}\sum_{v=1}^{n} v a_v$.

Also, if we write $X_n = \sum_{v=0}^{n} \frac{p_v}{P_v}$, then (X_n) is a positive increasing sequence tending to infinity as $n \to \infty$. A sequence (λ_n) is said to be of bounded variation, denoted by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| < \infty$.

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2. Known Result

There are many papers on absolute summability of infinite and Fourier series, some of them are [**3**–**7**, **10**]. Among them, in [**3**], the following theorem on absolute Riesz summability of the series $\sum a_n\lambda_n$ has been proved.

THEOREM 2.1. *Let (p_n) be a sequence of positive numbers such that*

$$P_n = O(np_n) \quad as \quad n \to \infty.$$

If the conditions

$$\lambda_m = o(1) \quad as \quad m \to \infty, \tag{2.1}$$

$$\sum_{n=1}^{m} nX_n|\Delta^2\lambda_n| = O(1) \quad as \quad m \to \infty, \tag{2.2}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n}\frac{|t_n|^k}{X_n^{k-1}} = O(X_m) \quad as \quad m \to \infty$$

hold, then the series $\sum a_n\lambda_n$ is summable $|\bar{N}, p_n|_k$, $k \geqslant 1$.

3. Main Result

We generalize Theorem 2.1 for a general matrix summability method. For some other papers on matrix summability of infinite and Fourier series, we can refer to [**11**–**15**, **17**].

Before giving the main result, let us introduce some further notations. Given a normal matrix $A = (a_{nv})$, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, \dots \tag{3.1}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, \dots \tag{3.2}$$

$$\bar{\Delta}A_n(s) = A_n(s) - A_{n-1}(s) = \sum_{v=0}^{n} \hat{a}_{nv}a_v. \tag{3.3}$$

THEOREM 3.1. *Let $A = (a_{nv})$ be a positive normal matrix such that*

$$\overline{a}_{n0} = 1, \ n = 0, 1, \dots, \tag{3.4}$$

$$a_{n-1,v} \geqslant a_{nv}, \ for \ n \geqslant v + 1, \tag{3.5}$$

$$a_{nn} = O\Big(\frac{p_n}{P_n}\Big), \tag{3.6}$$

$$|\hat{a}_{n,v+1}| = O(v|\Delta_v(\hat{a}_{nv})|). \tag{3.7}$$

If conditions (2.1), (2.2) *and*

$$\sum_{n=1}^{m} \Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)-k} \frac{|t_n|^k}{X_n^{k-1}} = O(X_m), m \to \infty \tag{3.8}$$

$$\sum_{n=v+1}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)-k+1}|\Delta_v\hat{a}_{nv}| = O\Big(\Big(\frac{P_v}{p_v}\Big)^{\beta(\delta k+k-1)-k}\Big), m\to\infty \tag{3.9}$$

are satisfied, then the series $\sum a_n\lambda_n$ *is summable* $|A,p_n,\beta;\delta|_k$, $k\geqslant 1$, $\delta\geqslant 0$ *and* $-\beta(\delta k+k-1)+k>0$.

We need the following lemma to prove Theorem 3.1.

Lemma 3.1. [**2**] *Under the conditions of Theorem* 3.1, *we have*

$$nX_n|\Delta\lambda_n| = O(1) \quad as \quad n\to\infty, \tag{3.10}$$

$$\sum_{n=1}^{\infty} X_n|\Delta\lambda_n| < \infty, \tag{3.11}$$

$$X_n|\lambda_n| = O(1) \quad as \quad n\to\infty. \tag{3.12}$$

Proof of Theorem 3.1. Let (I_n) denotes A-transform of the series $\sum a_n\lambda_n$. By (3.3), we obtain $\bar{\Delta}I_n = \sum_{v=0}^{n}\hat{a}_{nv}a_v\lambda_v = \sum_{v=1}^{n}\frac{\hat{a}_{nv}\lambda_v}{v}va_v$. Then, applying Abel's transformation, we get

$$\begin{aligned}\bar{\Delta}I_n &= \sum_{v=1}^{n-1}\Delta_v\Big(\frac{\hat{a}_{nv}\lambda_v}{v}\Big)\sum_{r=1}^{v} ra_r + \frac{\hat{a}_{nn}\lambda_n}{n}\sum_{r=1}^{n} ra_r\\
&= \sum_{v=1}^{n-1}\frac{v+1}{v}\Delta_v(\hat{a}_{nv})\lambda_v t_v + \sum_{v=1}^{n-1}\frac{v+1}{v}\hat{a}_{n,v+1}\Delta\lambda_v t_v\\
&+ \sum_{v=1}^{n-1}\hat{a}_{n,v+1}\lambda_{v+1}\frac{t_v}{v} + \frac{n+1}{n}a_{nn}\lambda_n t_n\\
&= I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}.\end{aligned}$$

For the proof of Theorem 3.1, we show that

$$\sum_{n=1}^{\infty}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)}|I_{n,r}|^k < \infty \quad for \quad r=1,2,3,4.$$

First, by using Hölder's inequality with indices k and k', where $k>1$ and $\frac{1}{k}+\frac{1}{k'}=1$, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)}|I_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)}\Big(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\Big)\\
&\times\Big(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\Big)^{k-1}.\end{aligned}$$

By using (3.1), (3.2), (3.4) and (3.5), we get $\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\leqslant a_{nn}$. Thus, by using (3.6), (3.9), (3.12), we have

$$\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)}|I_{n,1}|^k= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)-k+1}\Big(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_v|^k|t_v|^k\Big)$$
$$= O(1)\sum_{v=1}^{m}|\lambda_v||\lambda_v|^{k-1}|t_v|^k\sum_{n=v+1}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|$$
$$= O(1)\sum_{v=1}^{m}\Big(\frac{P_v}{p_v}\Big)^{\beta(\delta k+k-1)-k}|\lambda_v|\frac{|t_v|^k}{X_v^{k-1}}.$$

By applying Abel's transformation and using conditions (3.8), (3.11) and (3.12), we have

$$\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)}|I_{n,1}|^k = O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\Big(\frac{P_r}{p_r}\Big)^{\beta(\delta k+k-1)-k}\frac{|t_r|^k}{X_r^{k-1}}$$
$$+ O(1)|\lambda_m|\sum_{v=1}^{m}\Big(\frac{P_v}{p_v}\Big)^{\beta(\delta k+k-1)-k}\frac{|t_v|^k}{X_v^{k-1}}$$
$$= O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v + O(1)|\lambda_m|X_m$$
$$= O(1) \quad as \quad m\to\infty.$$

Now, using Hölder's inequality and conditions (3.7), (3.6), (3.9), (3.10), we have

$$\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)}|I_{n,2}|^k = O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)}\Big(\sum_{v=1}^{n-1}v|\Delta_v(\hat{a}_{nv})||\Delta\lambda_v||t_v|\Big)^k$$
$$= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)}\Big(\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k|\Delta_v(\hat{a}_{nv})||t_v|^k\Big)$$
$$\times\Big(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\Big)^{k-1}$$
$$= O(1)\sum_{n=2}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)-k+1}$$
$$\times\Big(\sum_{v=1}^{n-1}(v|\Delta\lambda_v|)^k|\Delta_v(\hat{a}_{nv})||t_v|^k\Big)$$
$$= O(1)\sum_{v=1}^{m}(v|\Delta\lambda_v|)^{k-1}(v|\Delta\lambda_v|)|t_v|^k$$
$$\times\sum_{n=v+1}^{m+1}\Big(\frac{P_n}{p_n}\Big)^{\beta(\delta k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|$$
$$= O(1)\sum_{v=1}^{m}\Big(\frac{P_v}{p_v}\Big)^{\beta(\delta k+k-1)-k}v|\Delta\lambda_v|\frac{|t_v|^k}{X_v^{k-1}}.$$

Then, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|I_{n,2}|^k &= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)\sum_{r=1}^{v}\left(\frac{P_r}{p_r}\right)^{\beta(\delta k+k-1)-k}\frac{|t_r|^k}{X_r^{k-1}}\\ &\quad+O(1)m|\Delta\lambda_m|\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k}\frac{|t_v|^k}{X_v^{k-1}}\\ &= O(1)\sum_{v=1}^{m-1}\Delta(v|\Delta\lambda_v|)X_v+O(1)m|\Delta\lambda_m|X_m\\ &= O(1)\sum_{v=1}^{m-1}v|\Delta^2\lambda_v|X_v+O(1)\sum_{v=1}^{m-1}|\Delta\lambda_v|X_v\\ &\quad+O(1)m|\Delta\lambda_m|X_m=O(1)\quad as\quad m\to\infty\end{aligned}$$

by applying Abel's transformation and using conditions (3.8), (2.2), (3.11), (3.10). For $r = 3$, again using Hölder's inequality and conditions (3.7), (3.6), (3.9), (3.12), we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|I_{n,3}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_{v+1}|^k|t_v|^k\right)\\ &\quad\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1}\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})||\lambda_{v+1}|^k|t_v|^k\right)\\ &= O(1)\sum_{v=1}^{m}|\lambda_{v+1}|^{k-1}|\lambda_{v+1}||t_v|^k\\ &\quad\times\sum_{n=v+1}^{m+1}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k+1}|\Delta_v(\hat{a}_{nv})|\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{p_v}\right)^{\beta(\delta k+k-1)-k}|\lambda_{v+1}|\frac{|t_v|^k}{X_v^{k-1}}=O(1),\quad m\to\infty\end{aligned}$$

as in $I_{n,1}$. Finally, we get

$$\begin{aligned}\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)}|I_{n,4}|^k &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k}|\lambda_n|^{k-1}|\lambda_n||t_n|^k\\ &= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{\beta(\delta k+k-1)-k}|\lambda_n|\frac{|t_n|^k}{X_n^{k-1}}\\ &= O(1)\quad as\quad m\to\infty,\end{aligned}$$

as in $I_{n,1}$ and thus the proof is completed. □

If we take $\beta = 1$, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 3.1, then we get Theorem 2.1.

4. A result for Fourier series

Let f be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$. The trigonometric Fourier series of f is defined as

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) = \sum_{n=0}^{\infty} A_n(x).$$

Write

$$\phi(t) = \frac{1}{2}\{f(x+t) + f(x-t)\} \quad \text{and} \quad \phi_1(t) = \frac{1}{t}\int_0^t \phi(u)\, du.$$

If $\phi_1(t) \in \mathcal{BV}(0, \pi)$, then $t_n(x) = O(1)$, where $t_n(x)$ is the n-th $(C, 1)$ mean of the sequence $(nA_n(x))$ (see [**8**]). By using this fact, in [**3**], Bor has proved the following theorem.

THEOREM 4.1. *If $\phi_1(t) \in \mathcal{BV}(0, \pi)$, and the sequences (p_n), (λ_n) and (X_n) satisfy the conditions of Theorem 2.1, then the series $\sum A_n(x)\lambda_n$ is summable $\left|\bar{N}, p_n\right|_k$, $k \geqslant 1$.*

Theorem 4.1 is generalized to the $|A, p_n, \beta; \delta|_k$ summability of the trigonometric Fourier series as in the following form.

THEOREM 4.2. *Let $\phi_1(t) \in \mathcal{BV}(0, \pi)$. If all conditions of Theorem 3.1 are satisfied, then the series $\sum A_n(x)\lambda_n$ is summable $|A, p_n, \beta; \delta|_k$, $k \geqslant 1$, $\delta \geqslant 0$ and $-\beta(\delta k + k - 1) + k > 0$.*

If we take $\beta = 1$, $\delta = 0$ and $a_{nv} = \frac{p_v}{P_n}$ in Theorem 4.2, then we get Theorem 4.1.

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