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RATE OF CONVERGENCE BY KANTOROVICH TYPE OPERATORS INVOLVING ADJOINT BERNOULLI POLYNOMIALS

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Abstract. We introduce a sequence of positive linear operators involving adjoint Bernoulli polynomials of the first kind, and we focus on the approximation properties of these operators. One of the main objectives is to get estimates for the order of approximation by means of first-order modulus of continuity, the Lipschitz condition, first modulus of derivative and a combination of first-order modulus of continuity and extended second-order modulus. Further, we give Voronovskaya type and Grüss–Voronovskaya type asymptotic results. Finally, we give two examples for error estimation by using Maple software.

1. Introduction

There are many various proofs of the Weierstrass theorem on the density of algebraic and trigonometric polynomials on compact intervals in $\mathbb{R}$. But, the remarkable proofs of the theorem related to the approximation of a continuous function by polynomials use some sequences of linear positive operators. The classical operators of Bernstein, Gauss–Weierstrass, Vallee–Poussin, Jackson, and Landau are good examples of this situation. However, the simplest proof of the Weierstrass theorem was given by Bernstein in [**6**]. Let $C([0,1])$ be the set of continuous functions on $[0,1]$. For $f \in C([0,1])$, Bernstein defined the following linear positive operators:

$$B_n(f;x) := \sum_{k=0}^{\infty} f\Big(\frac{k}{n}\Big) b_{n,k}(x),$$

where for $n,k \in \mathbb{N} \cup \{0\}$, $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is called the Bernstein basis polynomial of degree n. For some pioneering studies on generalizations of Bernstein operators and their approximation properties in approximation theory, one may refer to [**8, 15, 16, 24, 28**]. Bernstein polynomials and Bernstein-type operators are

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studied not only in approximation theory, but also in analytic number theory and combinatorics (e.g. [**1**, **20**–**23**]).

Classes of polynomials are important in approximation theory, one of which is the class of Appell polynomials. The Appell polynomial families $\{p_k(x)\}_{k=0}^{\infty}$ are introduced by means of the exponential generating function of the type

$$A(t)e^{xt} = \sum_{k=0}^{\infty} p_k(x)\frac{t^k}{k!}, \tag{1.1}$$

where $A(t) = \sum_{k=0}^{\infty} \alpha_k \frac{t^k}{k!}$, $A(0) \neq 0$, is an analytic function at $t = 0$, and $\alpha_k := p_k(0)$ [**3**]. Jakimovski and Leviatan constructed the linear positive operator using the Appell polynomials in [**12**]. For similar studies, see [**2**, **4**, **11**, **17**, **25**–**27**]. The case of $A(t) = \frac{t}{e^t-1}$ in (1.1), polynomials $\{p_k(x)\}_{k=1}^{\infty}$ turn out the Bernoulli polynomials. In [**18**], Natalini and Ricci introduced the adjunction property for the Appell polynomials set and gave the particular case of adjoint Appell Bernoulli polynomials. According to [**18**], simply, adjoint Appell polynomials are defined by changing $A(t)$ with $1/A(t)$ in the generating function. The adjoint Bernoulli polynomials $\{\tilde{\beta}_k(x)\}_{k=1}^{\infty}$ are defined by means of the exponential generating function of the type Equation (8) in [**18**]

$$\frac{e^t-1}{t}e^{xt} = \sum_{k=0}^{\infty} \tilde{\beta}_k(x)\frac{t^k}{k!}. \tag{1.2}$$

The Taylor expansion of the expression $\frac{e^t-1}{t}e^{xt}$ is

$$\begin{aligned} 1 &+ \Big(x+\frac{1}{2}\Big)t + \Big(x^2+x+\frac{1}{3}\Big)\frac{t^2}{2!} + \Big(x^3+\frac{3}{2}x^2+x+\frac{1}{4}\Big)\frac{t^3}{3!} \\ &+ \Big(x^4+2x^3+2x^2+x+\frac{1}{5}\Big)\frac{t^4}{4!} + O(t^5), \end{aligned}$$

where fancy O denotes big-O notation.

The importance of the adjoint Bernoulli polynomials for this work is that they are positive on $[0,\infty)$. The classical Bernstein operators are suitable only for continuous functions, to approximate integrable function $f \in L_1([0,1])$. In 1932, Kantorovich [**13**] presented the linear positive operators $K_n\colon L_1([0,1]) \to C([0,1])$ defined for any $f \in L_1([0,1])$ and any non-negative integer n by

$$K_n(f;x) = (n+1)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt. \tag{1.3}$$

Based on (1.2) and (1.3) for $n \in \mathbb{N}$, we define the operators $\tilde{A}_n(f;x)$, as follows.

Definition 1.1. Let $\tilde{A}_n\colon L_1([0,1]) \to C([0,1])$. For $n \in \mathbb{N}$ and $f \in C([0,1])$, the operators $\tilde{A}_n$ defined by

$$\tilde{A}_n(f;x) := n\frac{e^{-nx}}{e-1}\sum_{k=0}^{n}\frac{\tilde{\beta}_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)dt, \tag{1.4}$$

where $\{\tilde{\beta}_k(x)\}_{k=0}^{\infty}$ are adjoint Bernoulli polynomials that are positive on the interval $[0,1]$.

For all $n \in \mathbb{N}$, the operators $\tilde{A}_n$ are positive and linear.

The rest of the paper is organized as follows: In the next section, we give some auxiliary lemmas for the operators $\tilde{A}_n$. In the third section, we investigate the uniform approximation of the sequence $\{\tilde{A}_n(f;x)\}_{n=0}^{\infty}$, and then we estimate the rate of convergence of the operators $\tilde{A}_n$ with the aid of the first-order modulus of continuity, the Lipschitz condition, first modulus of derivative and a combination of first-order modulus of continuity and extended second-order modulus. In the fourth section, we give the Voronovskaya type and Grüss–Voronovskaya type theorems for the operators $\tilde{A}_n$. Finally, we give some numerical estimations for the rate of approximation of $\tilde{A}_n$ to a given function using the first-order modulus of continuity and extended second-order modulus, and then graphically illustrate them.

2. Auxiliary Results

In this part, we give the following lemmas which are used in the sequel. The following lemma gives the moments of the operators $\tilde{A}_n(f;x)$.

Lemma 2.1. *For all $x \in [0,1]$ and $n \in \mathbb{N}$, the operators $\tilde{A}_n$ satisfy*

$$\begin{aligned}
\tilde{A}_n(e_0;x) &= 1, \\
\tilde{A}_n(e_1;x) &= x + \frac{e+1}{2n(e-1)}, \\
\tilde{A}_n(e_2;x) &= x^2 + \frac{2e}{n(e-1)}x + \frac{4e-1}{3n^2(e-1)}, \\
\tilde{A}_n(e_3;x) &= x^3 + \frac{1}{n(e-1)}\Big\{\frac{9e-3}{2}x^2 + \frac{13e-1}{2n}x + \frac{11e+1}{4n^2}\Big\}, \\
\tilde{A}_n(e_4;x) &= x^4 + \frac{1}{n(e-1)}\Big\{(8e-4)x^3 + \frac{21e-3}{n}x^2\Big\} \\
&\quad + \frac{1}{n(e-1)}\Big\{\frac{22e-48}{n^2}x + \frac{41e-1}{5n^3}\Big\},
\end{aligned} \tag{2.1}$$

where $e_j(t) = t^j \in C([0,1])$, $j = \overline{0,4}$.

Proof. If we take the derivative of both sides of (1.2) with respect to t, then we get

$$\sum_{k=0}^{\infty} k t^{k-1} \frac{\tilde{\beta}_k(x)}{k!} = \frac{1}{t^2}\Big[(-1+t(x+1))e^{t(x+1)} + (-tx+1)e^{tx}\Big]. \tag{2.2}$$

Substituting $t=1$ and $x = nx$ in (2.2), we have

$$\sum_{k=0}^{\infty} k \frac{\tilde{\beta}_k(nx)}{k!} = e^{nx}nx(e-1) + e^{nx}.$$

After continuing similar operations, we get

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{\tilde{\beta}_k(nx)}{k!}&=e^{nx}(e-1),\\ \sum_{k=0}^{\infty}k\frac{\tilde{\beta}_k(nx)}{k!}&=nxe^{nx}(e-1)+e^{nx},\\ \sum_{k=0}^{\infty}k^2\frac{\tilde{\beta}_k(nx)}{k!}&=n^2x^2e^{nx}(e-1)+nxe^{nx}(e+1)+e^{nx}(e-1),\\ \sum_{k=0}^{\infty}k^3\frac{\tilde{\beta}_k(nx)}{k!}&=n^3x^3e^{nx}(e-1)+n^2x^2e^{nx}3e+nxe^{nx}(4e-1)+e^{nx}(e+1),\\ \sum_{k=0}^{\infty}k^4\frac{\tilde{\beta}_k(nx)}{k!}&=n^4x^4(e-1)+n^3x^3e^{nx}(6e-2)+n^2x^2e^{nx}(13e-1)\\ &\quad+nxe^{nx}(11e+1)+e^{nx}(4e-1).\end{aligned}$$

Finally, using above equalities and (1.4), we obtain the desired results. □

LEMMA 2.2. *The central moments for the operators $\tilde{A}_n(f;x)$ for $n\in\mathbb{N}$ are*

$$\tilde{A}_n(e_1-x;x)=\frac{e+1}{2(e-1)n},\tag{2.3}$$

$$\tilde{A}_n((e_1-x)^2;x)=\frac{x}{n}+\frac{4e-1}{3(e-1)n^2},\tag{2.4}$$

$$\tilde{A}_n((e_1-x)^3;x)=\frac{3+5e}{2(e-1)n^2}x+\frac{11e+1}{4(e-1)n^3},\tag{2.5}$$

$$\tilde{A}_n((e_1-x)^4;x)=\frac{3x^2}{n^2}+\frac{11e-49}{n^3}x+\frac{41e-1}{5(e-1)n^4}.\tag{2.6}$$

PROOF. Using the linearity property of $\tilde{A}_n$, we get

$$\begin{aligned}\tilde{A}_n((e_1-x)^4;x)&=\tilde{A}_n(e_4;x)-4x\tilde{A}_n(e_3;x)+6x^2\tilde{A}_n(e_2;x)\\ &\quad-4x^3\tilde{A}_n(e_1;x)+x^4\tilde{A}_n(e_0;x).\end{aligned}$$

Using Lemma 2.1, we arrive at (2.6). For equations (2.3)–(2.5), the proof method is similar. □

LEMMA 2.3. *For all $x\in[0,1]$ and $n\in\mathbb{N}$, the operators $\tilde{A}_n$ satisfy*

$$\tilde{A}_n(e_1-x;x)\leqslant\gamma_n,\quad \tilde{A}_n((e_1-x)^2;x)\leqslant\xi_n,$$

where

$$\gamma_n:=\frac{11}{10n},\tag{2.7}$$

$$\xi_n:=\frac{1}{n}+\frac{192}{100n^2}.\tag{2.8}$$

PROOF. With the help of any calculator, one determines the expressions on the right-hand sides of (2.3) and (2.4) cannot exceed (2.7) and (2.8), respectively, for all $x \in [0,1]$ and $n \in \mathbb{N}$. □

3. Approximation properties of the operators $\tilde{A}_n$

In this section, we investigate the approximation properties of the operators $\tilde{A}_n$ and give some required notions.

THEOREM 3.1. *For any $f \in C([0,1])$, $\lim_{n\to\infty} \tilde{A}_n(f;x) = f(x)$ uniformly on $[0,1]$.*

PROOF. From Lemma 2.1, we have $\lim_{n\to\infty} \tilde{A}_n((e_1 - x)^2; x) = 0$, uniformly on $[0,1]$. According to the Korovkin theorem [**14**], the desired result is obtained. □

DEFINITION 3.1 (cf. [**9**]). The first-order modulus of continuity is defined by

$$\omega_1(f,\delta) := \sup\{|f(t) - f(x)|,\ t,x \in [0,1],\ |t-x| \leqslant \delta\},$$

where $\delta \geqslant 0$, $f \in C([0,1])$.

The first-order modulus of continuity satisfies [**9**]:

$$|f(t) - f(x)| \leqslant \omega_1(f;|t-x|), \tag{3.1}$$

$$\omega_1(f;m\delta) \leqslant (1+m)\omega_1(f;\delta), \quad m \geqslant 0. \tag{3.2}$$

The following theorem gives the rate of convergence of the operators $\tilde{A}_n$.

THEOREM 3.2. *For any $f \in C([0,1])$ and each $x \in [0,1]$, the operators $\tilde{A}_n$ satisfy $|\tilde{A}_n(f;x) - f(x)| \leqslant 2\omega_1\big(f; \sqrt{\varsigma_n(x)}\big)$, where $\varsigma_n(x) := \tilde{A}_n((t-x)^2;x)$.*

PROOF. Using linearity property of the operators $\tilde{A}_n$ and (2.1), we write

$$|\tilde{A}_n(f;x) - f(x)| \leqslant n\frac{e^{-nx}}{e-1}\sum_{k=0}^{n}\frac{\tilde{\beta}_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)|dt. \tag{3.3}$$

On the other hand, with the aid of (3.1) and (3.2), we get

$$|f(t) - f(x)| \leqslant \omega_1(f;|t-x|) \leqslant (1+\delta^{-2}(t-x)^2)\omega_1(f;\delta). \tag{3.4}$$

For $|t-x| \leqslant \delta$, (3.4) is clear. For $|t-x| \geqslant \delta$, using (3.2), we get

$$(1+m)\omega_1(f;\delta) \leqslant (1+m^2)\omega_1(f;\delta), \tag{3.5}$$

where we substitute $m = \delta^{-1}(t-x)$ in (3.5) (cf. [**5**]).

Using (3.4) in (3.3), we get

$$\begin{aligned}|\tilde{A}_n(f;x) - f(x)| &\leqslant n\frac{e^{-nx}}{e-1}\sum_{k=0}^{n}\frac{\tilde{\beta}_k(nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}} (1+\delta^{-2}(t-x)^2)\omega_1(f;\delta)dt \\ &\leqslant (\tilde{A}_n(e_0;x) + \delta^{-2}\tilde{A}_n((e_1-x)^2;x))\omega_1(f;\delta),\end{aligned} \tag{3.6}$$

for any $\delta > 0$ and each $x \in [0,1]$.

From (2.1) and (2.4), we obtain $|\tilde{A}_n(f;x) - f(x)| \leqslant (1+\delta^{-2}\varsigma_n(x))\omega_1(f;\delta)$, for any $\delta > 0$ and each $x \in [0,1]$. Since $\tilde{A}_n$ are positive operators, it follows that

$\varsigma_n(x) \geqslant 0$ for each $x \in [0,1]$. In this case, if one chooses $\delta := \sqrt{\varsigma_n(x)}$ in (3.6), the desired result is obtained. □

DEFINITION 3.2. Lipschitz class of order α, denotes $\mathrm{Lip}_1(\alpha; K)$ ($0 < \alpha \leqslant 1$, $K > 0$), is defined by

$$\mathrm{Lip}_1(\alpha; K) := \{f \in C([0,1]) : |f(t) - f(x)| \leqslant K|t-x|^\alpha,\ t, x \in [0,1]\}.$$

THEOREM 3.3. *Let $f \in \mathrm{Lip}_1(\alpha; K)$. For $x \in [0,1]$, we have*

$$|\tilde{A}_n(f; x) - f(x)| \leqslant K\sqrt{\varsigma_n^\alpha(x)}.$$

PROOF. Since the monotonicity property of $\tilde{A}_n$ follows from (3.3), the following inequality holds $|\tilde{A}_n(f;x) - f(x)| \leqslant K\tilde{A}_n(|t-x|^\alpha; x)$. Using the Hölder inequality, we get $|\tilde{A}_n(f;x) - f(x)| \leqslant K(\tilde{A}_n((e_1 - x)^2; x))^{\alpha/2}$. Hence, we obtained the desired result. □

Let f' be the derivative of f and let $\omega_1(f'; \delta)$ be the first modulus of the derivative. In the following theorem, we give an estimate for the differentiable function f (cf. [**19**]).

THEOREM 3.4. *Let f be differentiable on $[0,1]$ and f' be bounded on $[0,1]$. Then the following estimate holds*

$$|\tilde{A}_n(f;x) - f(x)| \leqslant \sqrt{\xi_n}\Big[|f'(x)| + \frac{5}{4}\omega_1\big(f'; \sqrt{\xi_n}\big)\Big], \tag{3.7}$$

for all $x \in [0,1]$.

PROOF. Using [**19**, Theorem 2.3.8] by $r = 2$, we derive the following estimate:

$$\begin{aligned} |\tilde{A}_n(f;x) - f(x)| &\leqslant |\tilde{A}_n(e_1 - x; x)||f'(x)| \\ &\quad + \Big[\frac{\delta}{4}\tilde{A}_n(e_0; x) + \delta^{-1}\tilde{A}_n((e_1 - x)^2; x)\Big]\omega_1(f'; \delta). \end{aligned} \tag{3.8}$$

If we substitute (2.7), (2.1) and (2.8) in (3.8) and consider $\gamma_n < \xi_n$, then we obtain

$$|\tilde{A}_n(f;x) - f(x)| \leqslant \sqrt{\xi_n}|f'(x)| + \Big[\frac{\delta}{4} + \delta^{-1}\xi_n\Big]\omega_1(f'; \delta).$$

Choosing $\delta := \sqrt{\xi_n}$, we immediately derive (3.7). □

If f is not differentiable, one cannot use the first modulus of the derivative. In [**19**], Păltănea gave the following definition, which is an expanded version of the first modulus of the derivative for the class of arbitrary function.

DEFINITION 3.3. For real-valued function $f\colon [0,1] \to \mathbb{R}$ and any $\delta > 0$,

$$\begin{aligned} \omega_2^d(f;\delta) := \delta\Big\{ \sup\Big|\frac{f(x+t) - f(x)}{t}\Big| - \Big|\frac{f(y+s) - f(x)}{s}\Big|,\ s, t > 0, \\ x, x+t, y+s \in [0,1],\ \max\{x+t, y+s\} - \max\{x, y\} \leqslant \delta\Big\}. \end{aligned}$$

THEOREM 3.5. *For $f \in C([0,1])$, the following inequality holds*

$$|\tilde{A}_n(f;x) - f(x)| \leqslant \frac{\gamma_n}{\sqrt{\xi_n}}\omega_1(f;\delta) + \frac{9}{8}\omega_2^d(f;\delta).$$

PROOF. Using [**19**, Theorem 2.3.7] by $r = 2$, we get

$$\begin{aligned} |\tilde{A}_n(f;x) - f(x)| &\leqslant \delta^{-1}|\tilde{A}_n(e_1 - x;x)|\omega_1(f;\delta) \\ &\quad + \Big[\frac{1}{8}\tilde{A}_n(e_0;x) + \delta^{-2}\tilde{A}_n((e_1 - x)^2;x)\Big]\omega_2^d(f;\delta). \end{aligned} \tag{3.9}$$

Substituting (2.7), (2.1) and (2.8) in (3.9), we get

$$|\tilde{A}_n(f;x) - f(x)| \leqslant \delta^{-1}\gamma_n\omega_1(f;\delta) + \Big[\frac{1}{8} + \delta^{-2}\xi_n\Big]\omega_2^d(f;\delta).$$

Taking $\delta = \sqrt{\xi_n}$, we arrive at the desired result. □

4. Voronovskaya-type and Grüss–Voronovskaya type theorems

In this section we give an adapted version of the asymptotic formula given by Voronovskaya for Bernstein operators in 1932, [**29**], and then give Grüss type Voronovskaya theorem (cf. [**10**] and [**7**]).

THEOREM 4.1. *Let $f \in C([0,1])$ and f be differentiable twice on $[0,1]$. Then*

$$\lim_{n\to\infty} n[\tilde{A}_n(f;x) - f(x)] = \frac{e+1}{2(e-1)}f'(x) + \frac{x}{2!}f''(x),$$

for every $x \in [0,1]$.

PROOF. For all $t \in [0,1]$ and a fixed point $x_0 \in [0,1]$, by Taylor's formula we get

$$f(t) - f(x_0) = (t - x_0)f'(x_0) + \frac{(t-x_0)^2}{2!}f''(x_0) + p(t,x_0)(t-x_0)^2, \tag{4.1}$$

where $p(t,x_0)$ is the Peano form of the remainder such that $p(t,x_0) \in C([0,1])$ and $\lim_{t\to x_0} p(t,x_0) = 0$. By (2.1) and (4.1), we have

$$\begin{aligned} n[\tilde{A}_n(f;x_0) - f(x_0)] = f'(x)n\tilde{A}_n(t - x_0;x_0) + \frac{f''(x_0)}{2!}n\tilde{A}_n((t-x_0)^2;x_0) \\ + n\tilde{A}_n(p(t,x_0)(t-x_0)^2;x_0), \end{aligned}$$

for every $n \in \mathbb{N}$.

Using (2.3) and (2.4), we get

$$\lim_{n\to\infty} n\tilde{A}_n(e_1 - x_0;x_0) = \frac{e+1}{2(e-1)}, \tag{4.2}$$

$$\lim_{n\to\infty} n\tilde{A}_n((e_1 - x_0)^2;x_0) = x_0. \tag{4.3}$$

On the other hand, by the Cauchy–Schwarz inequality, it follows that

$$n\tilde{A}_n(p(t,x_0)(e_1 - x_0)^2;x_0) \leqslant \sqrt{n^2\tilde{A}_n((e_1 - x_0)^4;x_0)\tilde{A}_n(p^2(t,x_0);x)}.$$

From (2.6), we get

$$\lim_{n\to\infty} n^2\tilde{A}_n((e_1-x_0)^4;x_0)=3x_0^2. \tag{4.4}$$

The function $\phi(t,x_0)=p^2(t,x_0)$, $t\geqslant 0$, we have $\phi(t,x_0)\in C([0,1])$ and $\lim_{t\to x_0}\phi(t,x_0)=0$, therefore

$$\lim_{n\to\infty}\tilde{A}_n(p^2(t,x_0);x_0)=\lim_{n\to\infty}\tilde{A}_n(\phi(t,x_0);x_0)=\phi(x_0,x_0)=0, \tag{4.5}$$

uniformly with respect to $x_0\in[0,1]$. Taking into account (4.4) and (4.5), we obtain $\lim_{n\to\infty} n\tilde{A}_n(p(t,x_0)(t-x_0)^2;x_0)=0$. By (4.2) and (4.3), we have that

$$\lim_{n\to\infty} n[\tilde{A}_n(f;x)-f(x)]=\frac{e+1}{2(e-1)}f'(x)+\frac{f''(x)}{2!}x. \qquad \square$$

The following theorem is a Voronovskaya-type theorem obtained using the Grüss inequality.

THEOREM 4.2. *If f,g are bounded on $[0,1]$, differentiable in some neighborhood of x and has second derivative $f''(x)$ and $g''(x)$ for some $x\in[0,1]$, then*

$$\lim_{n\to\infty} n\tilde{A}_n(f,g;x)=xf'(x)g'(x),$$

where $\tilde{A}_n(f,g;x)=\tilde{A}_n(fg;x)-\tilde{A}_n(f;x)\tilde{A}_n(g;x)$.

PROOF. The first and second derivatives of fg are as follows:

$$(fg)'(x)=f'(x)g(x)+f(x)g'(x),$$
$$(fg)''(x)=f''(x)g(x)+2f'(x)g'(x)+g''(x)f(x).$$

By using the linearity property of the operator $\tilde{A}_n$, and simple computations, we obtain

$$\begin{aligned}\tilde{A}_n(f,g;x)&=\tilde{A}_n(fg;x)-f(x)g(x)-(fg)'(x)\tilde{A}_n(e_1-x;x)\\&\qquad-\frac{(fg)''(x)}{2!}\tilde{A}_n((e_1-x)^2;x)\\&\quad-g(x)\Big[\tilde{A}_n(f;x)-f(x)-f'(x)\tilde{A}_n(e_1-x;x)-\frac{f''(x)}{2!}\tilde{A}_n((e_1-x)^2;x)\Big]\\&\quad-\tilde{A}_n(f;x)\Big[\tilde{A}_n(g;x)-g(x)-g'(x)\tilde{A}_n(e_1-x;x)-\frac{g''(x)}{2!}\tilde{A}_n((e_1-x)^2;x)\Big]\\&\quad+\frac{1}{2}!\tilde{A}_n((e_1-x)^2;x)[f(x)g''(x)+2f'(x)g'(x)-g''(x)\tilde{A}_n(f;x)]\\&\qquad+\tilde{A}_n(e_1-x;x)[f(x)g'(x)-g'(x)\tilde{A}_n(f;x)].\end{aligned}$$

Using (2.3) and (2.4), we get

$$\begin{aligned}\lim_{n\to\infty} n\tilde{A}_n(f,g;x)&=\lim_{n\to\infty} n[\tilde{A}_n(fg;x)-\tilde{A}_n(f;x)\tilde{A}_n(g;x)]\\&=\lim_{n\to\infty} n[\tilde{A}_n(fg;x)-f(x)g(x)]-\frac{e+1}{2(e-1)}(fg)'(x)-\frac{(fg)''(x)}{2!}x\end{aligned}$$

$$- g(x)\Big[\lim_{n\to\infty} n[\tilde{A}_n(f;x) - f(x)] - f'(x)\frac{e+1}{2(e-1)} - \frac{f''(x)}{2!}x\Big]$$
$$- \lim_{n\to\infty} \tilde{A}_n(f;x)\Big[\lim_{n\to\infty} n[\tilde{A}_n(g;x) - g(x)] - g'(x)\frac{e+1}{2(e-1)} - \frac{g''(x)}{2!}x\Big]$$
$$+ \frac{1}{2!}x\big[g''(x)\lim_{n\to\infty}[f(x) - \tilde{A}_n(f;x)] + 2f'(x)g'(x)\big]$$
$$+ g'(x)\frac{e+1}{2(e-1)}\lim_{n\to\infty}[f(x) - \tilde{A}_n(f;x)].$$

Finally, taking into account Theorem 3.1 and 4.1, we complete the proof. □

5. Applications and Examples

In this section, we obtain some upper bounds for the error $\tilde{A}_n(f;x) - f(x)$ in the terms of the first-order modulus of continuity $\omega_1(f;.)$ and extended second-order modulus $\omega_2^d(f;.)$ by using Maple 2021.

Let $E_n := |\tilde{A}_n(f;x) - f(x)|$, $f(x) = xe^{x+1}$ and $n \in \{10, 30, 100, 250\}$.

EXAMPLE 5.1. For some values in $[0,1]$, the absolute error E_n of the operators $\tilde{A}_n$ is computed with the help of the first-modulus of continuity in Table 1 and illustrated graphically in Figure 1.

TABLE 1. Error of approximation operators $\tilde{A}_n$ to $f(x) = xe^{x+1}$ using $\omega_1(f;.)$

x	E_{10}	E_{30}	E_{100}	E_{250}
0.0	2.650265324	1.235906659	0.6071382192	0.3677764872
0.2	3.784531330	1.784724711	0.8826317616	0.5362513930
0.4	5.291142748	2.515979292	1.2503539480	0.7612998148
0.6	7.279375384	3.483551688	1.7376396630	1.0597143150
0.8	9.888640910	4.756239236	2.3794068440	1.4529501090
1.0	13.29647458	6.421719002	3.2201743960	1.9683665460

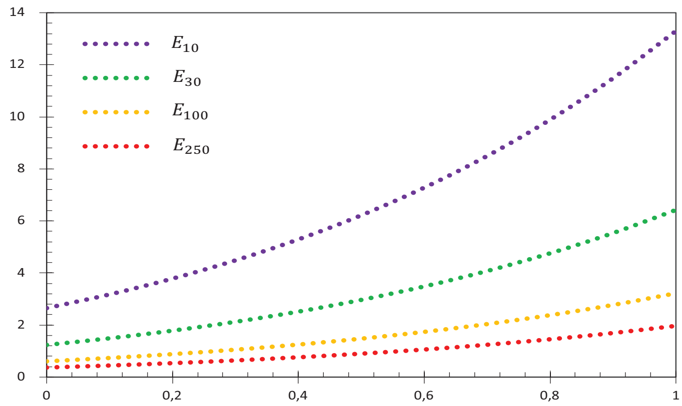

FIGURE 1. Error estimates by means of $\omega_1(f;.)$

EXAMPLE 5.2. The absolute error E_n is obtained by using the extended second-order modulus of continuity in Table 2 and represented graphically in Figure 2, for some values in $[0,1]$.

TABLE 2. Error of approximation operators $\tilde{A}_n$ to $f(x) = xe^{x+1}$ using $\omega_2^d(f;.)$

x	E_{10}	E_{30}	E_{100}	E_{250}
0.0	0.8442376264	0.2224360117	0.0597963438	0.0229862841
0.2	1.1256469900	0.2975942848	0.0801562211	0.0308387404
0.4	1.4902821370	0.3951294256	0.1066003968	0.0410415388
0.6	1.9612014410	0.5212658360	0.1408250063	0.0542505027
0.8	2.5675940030	0.6838872243	0.1849789985	0.0712966459
1.0	3.3463640480	0.8929662361	0.2417814945	0.0932316003

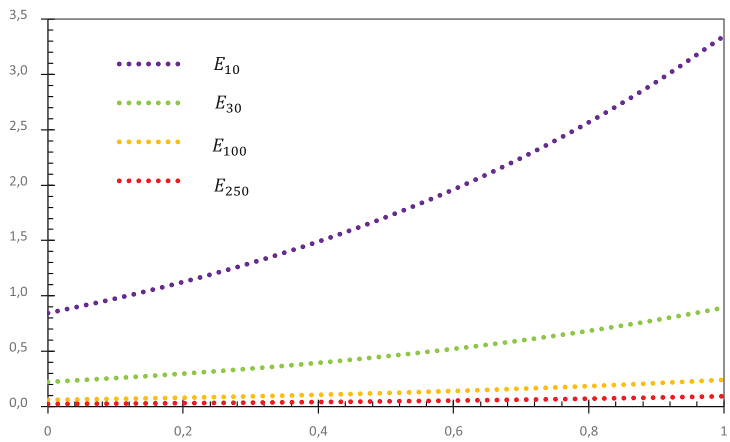

FIGURE 2. Error estimates by means of $\omega_2^d(f;.)$

When we examine the tables and figures in both examples, we notice that the approximation errors of the operators $\tilde{A}_n$ decrease as n increases. Moreover, one observes that estimates with the extended second-order modulus are more refined than estimates using the first-order modulus of continuity.

6. Conclusion

The adjoint Bernoulli polynomials defined by Ricci [**18**] are positive on the positive semi-axis. We have introduced the Kantorovich-type linear positive operators containing these polynomials and proved the uniform convergence of these operators with a Korovkin-type approximation. We have used the first-order modulus of continuity, the Lipschitz class, the first modulus of derivative, and a combination of the first-modulus of continuity and extended second-order modulus to estimate the rate of convergence by operators $\tilde{A}_n$. We have derived asymptotic formulae for determining the uniform order of convergence of operators $\tilde{A}_n$ to a given function f using the idea of the Voronovskaya theorem and Grüss inequality. Finally, we have obtained error estimations for operators $\tilde{A}_n$ with the help of the moduli ω_1 and ω_2^d, and then we have presented the numerical results via tables and figures.

For further studies, one may define various types of operators based on adjoint Appell polynomials and determine the order of approximation of these operators in terms of various types of moduli.

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