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CENTRAL AUTOMORPHISMS OF ZAPPA–SZÉP PRODUCTS OF TWO CYCLIC GROUPS

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Abstract. The central automorphism group of the Zappa–Szép product of two cyclic groups of orders m and p^2 is calculated, where p is a prime.

1. Introduction

Let $\mathrm{Aut}(G)$ be the group of automorphisms of a group G. Then, $\theta \in \mathrm{Aut}(G)$ is called a central automorphism of G if $g^{-1}\theta(g) \in Z(G)$ for all $g \in G$, where $Z(G)$ denotes the center of the group G. In fact, the set $\mathrm{Aut}_c(G)$ of all central automorphisms of the group G is a normal subgroup of $\mathrm{Aut}(G)$. Apparently, $\mathrm{Aut}_c(G) = C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$(the centralizer of $\mathrm{Inn}(G)$ in the group $\mathrm{Aut}(G)$), where $\mathrm{Inn}(G)$ denotes the group of inner automorphisms of the group G. Thus, central automorphisms play an important role in the investigation of the group $\mathrm{Aut}(G)$. The study of central automorphisms of a group has been an interest to the algebraists (see [**2**, **3**, **5**–**8**, **11**]).

Zappa [**15**] was the first to study the Zappa–Szép product of two groups which was also studied by J. Szép in a series of papers. The Zappa–Szép product is a natural generalization of the semidirect product of two groups. Let H and K be two subgroups of a group G. Then, G is called the internal Zappa–Szép product of H and K if $G = HK$ and $H \cap K = \{1\}$. If G is the internal Zappa–Szép product of H and K, then $kh = \sigma(k,h)\tau(k,h)$, where $\sigma(k,h) \in H$ and $\tau(k,h) \in K$. This determines the maps $\sigma\colon K \times H \to H$ and $\tau\colon K \times H \to K$ defined by $\sigma(k,h) = \sigma_k(h)$ and $\tau(k,h) = \tau_h(k)$ for all $h \in H$ and $k \in K$ respectively. These maps are called the matched pair of groups and satisfy the following conditions (see [**4**])

(C1) $\sigma_1(h) = h$ and $\tau_1(k) = k$, (C4) $\tau_h(kk') = \tau_{\sigma_{k'}(h)}(k)\tau_h(k')$,

(C2) $\sigma_k(1) = 1 = \tau_h(1)$, (C5) $\sigma_k(hh') = \sigma_k(h)\sigma_{\tau_h(k)}(h')$,

(C3) $\sigma_{kk'}(h) = \sigma_k(\sigma_{k'}(h))$, (C6) $\tau_{hh'}(k) = \tau_{h'}(\tau_h(k)$,

for all $h, h' \in H$ and $k, k' \in K$.

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Now, let H and K be two groups, $\sigma\colon K\times H\to H$ and $\tau\colon K\times H\to K$ be two maps which satisfy the above conditions. Then, the set $H\times K$ with the binary operation defined by $(h,k)(h',k')=(h\sigma_k(h'),\tau_{h'}(k)k')$ forms a group called the external Zappa–Szép product of H and K. The internal Zappa–Szép product is isomorphic to the external Zappa–Szép product (see [**4**, Proposition 2.4]). We will identify the external Zappa–Szép product with the internal Zappa–Szép product. The Zappa–Szép product of the groups H and K is denoted by $H\bowtie K$.

In this paper, we compute the central automorphism groups of groups which are the Zappa–Szép products of two cyclic groups of orders m and p^2, where p is a prime. The Zappa–Szép products of semigroups is a source of new and interesting examples of C^*-algebras (see [**1**, **14**]). One can construct new examples of finite group C^*-algebras using the results mentioned in this paper. The terminology used in this paper is the same as in [**9**] and [**10**]. Throughout the paper, $\mathbb{Z}_n$ denotes the cyclic group of order n. Let U and V be groups. Then $\mathrm{Hom}(U,V)$ and $\mathrm{Epi}(U,V)$ denote the groups of all group homomorphisms and onto group homomorphisms from U to V, respectively. If $U=V$, then we simply write $\mathrm{Epi}(U)$.

2. Structure of the central automorphism group

Let G be the Zappa–Szép product of two groups H and K. Let U,V and W be any groups. Then $\mathrm{Map}(U,V)$ denotes the set of all maps from the group U to the group V. If $\phi,\psi\in\mathrm{Map}(U,V)$ and $\eta\in\mathrm{Map}(V,W)$, then $\phi+\psi\in\mathrm{Map}(U,V)$ is defined by $(\phi+\psi)(u)=\phi(u)\psi(u)$, $\eta\phi\in\mathrm{Map}(U,W)$ is defined by $\eta\phi(u)=\eta(\phi(u))$, $\sigma_\phi(\psi)\in\mathrm{Map}(U,V)$ is defined by $(\sigma_\phi(\psi))(u)=\sigma_{\phi(u)}(\psi(u))$ and $\tau_\phi(\psi)\in\mathrm{Map}(U,V)$ is defined by $(\tau_\phi(\psi))(u)=\tau_{\phi(u)}(\psi(u))$, for all $u\in U$. Let $\ker(\sigma)=\{k\in K\mid \sigma_k(h)=h \text{ for all } h\in H\}$ and $\mathrm{Fix}(\sigma)=\{h\in H\mid \sigma_k(h)=h, \text{ for all } k\in K\}$. Similarly, we define the sets $\ker(\tau)$ and $\mathrm{Fix}(\tau)$. In this section, we study the structure of the central automorphism group of G.

Proposition 2.1 ([**10**, Corollary 2.1]). *Let G be the Zappa–Szép product of two abelian groups H and K. Then $Z(G)=H^*\times K^*$, where $H^*=\mathrm{Fix}(\sigma)\cap\ker(\tau)\cap Z(H)$ and $K^*=\mathrm{Fix}(\tau)\cap\ker(\sigma)\cap Z(K)$.*

Let $\mathcal{A}_c$ be the set of all matrices of the form $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, where $\alpha\in\mathrm{Epi}(H)$, $\beta\in\mathrm{Hom}(K,H^*)$, $\gamma\in\mathrm{Hom}(H,K^*)$ and $\delta\in\mathrm{Epi}(K)$ satisfy the following conditions,

(A1) $h^{-1}\alpha(h)\in H^*$,
(A2) $k^{-1}\delta(k)\in K^*$,
(A3) $\beta(k)\sigma_{\delta(k)}(\alpha(h))=\alpha(\sigma_k(h))\beta(\tau_h(k))$,
(A4) $\tau_{\alpha(h)}(\delta(k))\gamma(h)=\gamma(\sigma_k(h))\delta(\tau_h(k)$,
(A5) for any $h'k'\in G$, there exists a unique $h\in H$ and $k\in K$ such that $h'=\alpha(h)\beta(k)$ and $k'=\gamma(h)\delta(k)$.

for all $h,h'\in H$ and $k,k'\in K$. Then, the set $\mathcal{A}_c$ forms a group with the binary operation defined as follows (see [**9**, p. 98]),

$$\begin{pmatrix}\alpha' & \beta'\\ \gamma' & \delta'\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\alpha'\alpha+\beta'\gamma & \alpha'\beta+\beta'\delta\\ \gamma'\alpha+\delta'\gamma & \gamma'\beta+\delta'\delta\end{pmatrix}.$$

Theorem 2.1. *Let G be the Zappa–Szép product of two abelian groups H and K. Let $\mathcal{A}_c$ be as above. Then there is an isomorphism of groups between* $\mathrm{Aut}_c(G)$ *and $\mathcal{A}_c$ given by $\theta \leftrightarrow \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, where $\theta(h) = \alpha(h)\gamma(h)$ and $\theta(k) = \beta(k)\delta(k)$, for all $h \in H$ and $k \in K$.*

Proof. The proof follows along the lines of the proof of [**10**, Theorem 2.2]. □

We identify the central automorphisms of G with the corresponding matrices in $\mathcal{A}_c$. Note that, if $h^{-1}\alpha(h) \in H^*$, then $\tau_{\alpha(h)}(k) = \tau_h(k)$, for all $h \in H$ and $k \in K$. Also, if $k^{-1}\delta(k) \in K^*$, then $\sigma_{\delta(k)}(h) = \sigma_k(h)$, for all $h \in H$ and $k \in K$. Let

$$P = \{\alpha \in \mathrm{Aut}_c(H) \mid \sigma_k(\alpha(h)) = \alpha(\sigma_k(h)), h^{-1}\alpha(h) \in H^* \ \forall\ h \in H, k \in K\},$$
$$Q = \{\beta \in \mathrm{Hom}(K, H^*) \mid \beta(k) = \beta(\tau_h(k)) \ \forall\ h \in H, k \in K\},$$
$$R = \{\gamma \in \mathrm{Hom}(H, K^*) \mid \gamma(\sigma_k(h)) = \gamma(h) \ \forall\ h \in H, k \in K\},$$
$$S = \{\delta \in \mathrm{Aut}_c(K) \mid \tau_h(\delta(k)) = \delta(\tau_h(k)), k^{-1}\delta(k) \in K^* \ \forall\ h \in H, k \in K\}$$

be subsets of the group $\mathrm{Aut}_c(G)$. Then one can easily check that P, Q, R and S are all subgroups of the group $\mathrm{Aut}_c(G)$. Let

$$A = \left\{\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} \mid \alpha \in P\right\}, \quad B = \left\{\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix} \mid \beta \in Q\right\},$$
$$C = \left\{\begin{pmatrix}1 & 0\\ \gamma & 1\end{pmatrix} \mid \gamma \in R\right\}, \quad D = \left\{\begin{pmatrix}1 & 0\\ 0 & \delta\end{pmatrix} \mid \delta \in S\right\}.$$

be the corresponding subsets of $\mathcal{A}_c$, where 0 is the trivial group homomorphism and 1 is the identity group automorphism. Then one can easily check that A, B, C and D are subgroups of $\mathcal{A}_c$. Note that A and D normalize B and C.

Theorem 2.2. [**10**, Theorem 2.3] *Let G be the Zappa–Szép product of two groups H and K. Let A, B, C and D be defined as above. Then, if $1 - \beta\gamma \in P$, for all maps β and γ, then $ABCD = \mathcal{A}_c$ and* $\mathrm{Aut}_c(G) \simeq ABCD$.

3. $\mathrm{Aut}_c(\mathbb{Z}_4 \bowtie \mathbb{Z}_m)$

In [**12**], Yacoub classified the groups which are Zappa–Szép products of cyclic groups of order 4 and order m. He listed them as (see [**12**, Conclusion])

$$L_1 = \langle a, b \mid a^m = 1 = b^4, ab = ba^r, r^4 \equiv 1 \pmod{m}\rangle,$$
$$L_2 = \langle a, b \mid a^m = 1 = b^4, ab = b^3a^{2t+1}, a^2b = ba^{2s}\rangle,$$

where in L_2, m is even. Note that, the group L_1 may be isomorphic to the group L_2 depending on the values of m, r and t (see [**12**, Theorem 5]). Clearly, L_1 is a semidirect product. Throughout this section G will denote the group L_2 and we will be only concerned about groups L_2 which are Zappa–Szép products but not a semidirect product. Let $H = \langle b\rangle$, $K = \langle a\rangle$ and the mutual actions of H and K be defined by $\sigma_a(b) = b^3, \tau_b(a) = a^{2t+1}$ along with $\sigma_{a^2}(b) = b$ and $\tau_b(a^2) = a^{2s}$, where t and s are the integers satisfying the conditions

(G1) $2s^2 \equiv 2 \pmod{m}$, (G3) $2(t+1)(s-1) \equiv 0 \pmod{m}$,

(G2) $4t(s+1) \equiv 0 \pmod{m}$, (G4) $\gcd\big(s, m/2\big) = 1$.

Proposition 3.1. *If G is the group defined above, then $Z(G) = \ker(\tau)\operatorname{Fix}(\tau)$, where*

$$\ker(\tau) = \begin{cases} \{1, b^2\}, & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \{1\}, & \text{if } 2t(s+1) \not\equiv 0 \pmod{m} \end{cases},$$

$$\operatorname{Fix}(\tau) = \begin{cases} \langle a^2 \rangle, & \text{if } s \equiv 1 \pmod{\frac{m}{2}} \\ \left\{ a^l \mid l \equiv 0 \pmod{\frac{m}{\gcd(m, s-1)}} \right\}, & \text{if } s \not\equiv 1 \pmod{\frac{m}{2}}. \end{cases}$$

Proof. For all $0 \leqslant i \leqslant 3$ and $0 \leqslant l \leqslant m-1$, we have

$$\sigma_{a^l}(b^i) = \begin{cases} b^{-i}, & \text{if } l \text{ is odd} \\ b^i, & \text{if } l \text{ is even}, \end{cases}$$

and using $(C6)$, and [**9**, Lemma 3.1],

$$\tau(b^i)(a^l) = \begin{cases} a^l, & \text{if } i = 0 \\ a^{2t+1+(l-1)s}, & \text{if } i = 1 \text{ and } l \text{ is odd} \\ a^{2t+2ts+l}, & \text{if } i = 2 \text{ and } l \text{ is odd} \\ a^{4t+1+2ts+(l-1)s}, & \text{if } i = 3 \text{ and } l \text{ is odd} \\ a^{ls}, & \text{if } i = 1, 3 \text{ and } l \text{ is even} \\ a^l, & \text{if } i = 2 \text{ and } l \text{ is even}. \end{cases}$$

Now, one can easily observe that $\ker(\sigma) = \{a^l \mid l \text{ is even}\} = \langle a^2 \rangle$, $\operatorname{Fix}(\sigma) = \{1, b^2\}$,

$$\ker(\tau) = \begin{cases} \{1, b^2\}, & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \{1\}, & \text{if } 2t(s+1) \not\equiv 0 \pmod{m}, \end{cases}$$

and

$$\operatorname{Fix}(\tau) = \begin{cases} \langle a^2 \rangle, & \text{if } s \equiv 1 \pmod{\frac{m}{2}} \\ \left\{ a^l \mid l \equiv 0 \pmod{\frac{m}{\gcd(m, s-1)}} \right\}, & \text{if } s \not\equiv 1 \pmod{\frac{m}{2}}. \end{cases}$$

Clearly, $\operatorname{Fix}(\sigma) \cap \ker(\tau) = \ker(\tau)$ and $\operatorname{Fix}(\tau) \cap \ker(\sigma) = \operatorname{Fix}(\tau)$. Since both H and K are cyclic groups, $Z(H) = H$ and $Z(K) = K$. Thus, $H^* = \ker(\tau)$ and $K^* = \operatorname{Fix}(\tau)$. Hence, the result follows from Proposition 2.1. □

Proposition 3.2. *Let G be the group defined as above. Then*

(i) $A \simeq \begin{cases} \mathbb{Z}_2, & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \{1\}, & \text{if } 2t(s+1) \not\equiv 0 \pmod{m}, \end{cases}$

(ii) $B \simeq \begin{cases} \mathbb{Z}_2, & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \{1\}, & \text{if } 2t(s+1) \not\equiv 0 \pmod{m}, \end{cases}$

(iii) $C \simeq \mathbb{Z}_2$.

Proof. (i) Let $\alpha \in P$. Then $\alpha \in \operatorname{Aut}_c(H)$. Hence, by Proposition 3.1, (i) holds.

(ii) Let $\beta \in Q$. Then $\operatorname{Im}(\beta) \leqslant \{1, b^2\}$. Now, one can easily observe that $\beta(k) = \beta(\tau_h(k))$ holds for all $h \in H$ and $k \in K$. Hence, by Proposition 3.1, (ii) holds.

(iii) Let $\gamma \in R$ be defined by $\gamma(b) = a^\lambda$, where $0 \leqslant \lambda \leqslant m-1$. Then using, $\gamma(b) = \gamma(\sigma_a(b))$, we get $a^\lambda = \gamma(b) = \gamma(\sigma_a(b)) = \gamma(b^3) = a^{3\lambda}$. Thus $2\lambda \equiv 0 \pmod{m}$ which implies that $\lambda \equiv 0 \pmod{\frac{m}{2}}$. Therefore, $\lambda \in \{0, \frac{m}{2}\}$. Hence, $C \simeq \langle\gamma\rangle \simeq \mathbb{Z}_2$. □

LEMMA 3.1. *Let $\alpha \in P$, $\beta \in Q$, $\gamma \in R$ and $\delta \in S$. Then*

$$\text{(i) } \alpha\beta = \beta = \beta\delta, \quad \text{(ii) } \gamma\alpha = \gamma = \delta\gamma, \quad \text{(iii) } \beta\gamma = 0 = \gamma\beta.$$

PROOF. Let the maps $\alpha \in P$, $\beta \in Q$, $\gamma \in R$ and $\delta \in S$ be defined as $\alpha(b) = b^i$, $\beta(a) = b^j$, $\gamma(b) = a^\lambda$ and $\delta(a) = a^r$, where $i \in \{1,3\}$, $j \in \{0,2\}$, $\lambda \in \{0, \frac{m}{2}\}$ and $r \in U(m)$. Then for all $h \in H$ and $k \in K$, we have

(i) $\alpha\beta(a) = \alpha(\beta(a)) = \alpha(b^j) = b^{ij} = b^j = \beta(a)$. Thus $\alpha\beta = \beta$. Also, $\beta\delta(a) = \beta(\delta(a)) = \beta(a^r) = b^{rj} = b^j$, as r is odd. Therefore, $\beta\delta = \beta$.

(ii) $\gamma\alpha(b) = \gamma(b^i) = a^{i\lambda} = a^\lambda = \gamma(b)$. Thus $\gamma\alpha = \gamma$. Now, $\delta\gamma(b) = \delta(a^\lambda) = a^{r\lambda} = a^\lambda$, as r is odd. Therefore, $\delta\gamma = \gamma$.

(iii) $\beta\gamma(b) = \beta(a^\lambda) = a^{j\lambda} = 1$. Thus $\beta\gamma = 0$. Now, $\gamma\beta(a) = \gamma(b^j) = b^{j\lambda} = 1$. Hence, $\gamma\beta = 0$. □

THEOREM 3.1. *Let A, B, C and D be defined as above. Then* $\operatorname{Aut}_c(G) \simeq A \times B \times C \times D$.

PROOF. By Lemma 3.1 (iii), we get $1 - \beta\gamma = 1 \in P$. Therefore, by Theorem 2.2, $\operatorname{Aut}_c(G) \simeq ABCD$,

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha'\alpha & \beta + \beta' \\ \gamma' + \gamma & \delta'\delta \end{pmatrix}.$$

Also,

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' & \beta' + \beta \\ \gamma + \gamma' & \delta\delta' \end{pmatrix}.$$

Since A, B, C, and D are abelian groups, we get

$$\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix}.$$

Hence, $\mathcal{A}_c$ is an abelian group and $\operatorname{Aut}_c(G) \simeq \mathcal{A}_c \simeq A \times B \times C \times D$. □

Now, we will find the structure of the group $\operatorname{Aut}_c(G)$. For this, we first take t such that $\gcd(t, m) = 1$ and then we take t such that $\gcd(t, m) > 1$.

THEOREM 3.2. *Let 4 divide m and t be odd such that* $\gcd(t, m) = 1$. *Then*

$$\operatorname{Aut}_c(G) \simeq \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, & \textit{if } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_2 \times \mathbb{Z}_2, & \textit{if } 2t(s+1) \not\equiv 0 \pmod{m}, \gcd(m, s-1) = 2 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, & \textit{if } 2t(s+1) \not\equiv 0 \pmod{m}, \gcd(m, s-1) = 4 \textit{ or } 8. \end{cases}$$

PROOF. Let $\gcd(t, m) = 1$. By using $(G2)$ and $(G3)$, we get, $s \equiv -1 \pmod{\frac{m}{4}}$ and $t \equiv -1 \pmod{\frac{m}{4}}$, respectively. Thus $s, t \in \{\frac{m}{4} - 1, \frac{m}{2} - 1, \frac{3m}{4} - 1, m-1\}$. Note that, if $s \in \{\frac{m}{2} - 1, m-1\}$, then $2t(s+1) \equiv 0 \pmod{m}$. Thus by Proposition 3.2, we get $A \simeq B \simeq C \simeq \mathbb{Z}_2$. Let $\gcd(m, s-1) = u$. Since m and $s-1$ are even, u is even. Also, $u \mid m$ and $u \mid s-1$. Therefore, $u \mid m - 2(s-1) = 2$ or 4. If $u = 2$, then by Proposition 3.1, we get $\mathrm{Fix}(\tau) = \{a^l \mid l \equiv 0 \pmod{\frac{m}{2}}\} = \{1, a^{\frac{m}{2}}\}$. Let $\delta \in S$. Then $a^{-1}\delta(a) \in \mathrm{Fix}(\tau) = \{1, a^{\frac{m}{2}}\}$ which implies that $\delta(a) \in \{a, a^{\frac{m}{2}+1}\}$. Therefore, $D = \langle\delta\rangle \simeq \mathbb{Z}_2$. If $u = 4$, then $4 \mid \frac{m}{2} - 2$. Therefore, $\frac{m}{2} \equiv 2 \pmod 4$ and so $m = 4n$, where $n \equiv 1 \pmod 4$. Then $\mathrm{Fix}(\tau) = \{a^l \mid l \equiv 0 \pmod{\frac{m}{4}}\} = \langle a^{\frac{m}{4}}\rangle$ which implies that $\delta(a) \in \{a, a^{\frac{m}{4}+1}, a^{\frac{m}{2}+1}, a^{\frac{3m}{4}+1}\}$. Since $\frac{m}{4} + 1$ is even, $\delta(a) \notin \{a^{\frac{m}{4}+1}, a^{\frac{3m}{4}+1}\}$. Thus $D \simeq \langle\delta\rangle \simeq \mathbb{Z}_2$. Hence, by Theorem 3.1, $\mathrm{Aut}_c(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now, if $s \in \{\frac{m}{4} - 1, \frac{3m}{4} - 1\}$, then $2t(s+1) \not\equiv 0 \pmod m$. Thus by Proposition 3.2, we get A and B are trivial groups and $C \simeq \mathbb{Z}_2$. Let $\gcd(m, s-1) = u$. Then $u \mid m$ and $u \mid s - 1 = \frac{m}{4} - 2$ which implies that $u \mid m - 4(\frac{m}{4} - 2) = 8$. Therefore, $u = 2$ or 4 or 8.

Now, if $u = 2$, then as above, we get $D \simeq \mathbb{Z}_2$ and so, $\mathrm{Aut}_c(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. If $u = 4$, then by Proposition 3.1, we get $\mathrm{Fix}(\tau) = \{a^l \mid l \equiv 0 \pmod{\frac{m}{4}}\} = \{1, a^{\frac{m}{4}}, a^{\frac{m}{2}}, a^{\frac{3m}{4}}\}$. Thus, $\delta(a) \in \{a, a^{\frac{m}{4}+1}, a^{\frac{m}{2}+1}, a^{\frac{3m}{4}+1}\}$. Note that, $\left(\frac{m}{4} + 1\right)^2 = \frac{1}{2}\left(2\left(\frac{m}{4} - 1\right)^2\right) + m$. Therefore, using $(G1)$, $\left(\frac{m}{4} + 1\right)^2 \equiv 1 \pmod m$. Thus, $D \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

Now, let $u = 8$. Then $8 \mid \frac{m}{4} - 2$ which implies that $\frac{m}{4} \equiv 2 \pmod 8$. Thus $m = 8q$, where $q \equiv 1 \pmod 8$. Since $u = 8$, by Proposition 3.1, $\mathrm{Fix}(\tau) = \{a^l \mid l \equiv 0 \pmod{\frac{m}{8}}\} = \langle a^{\frac{m}{8}}\rangle$ and so, $\delta(a) \in \{a, a^{\frac{m}{8}+1}, a^{\frac{m}{4}+1}, a^{\frac{3m}{8}+1}, a^{\frac{m}{2}+1}, a^{\frac{5m}{8}+1}, a^{\frac{3m}{4}+1}, a^{\frac{7m}{8}+1}\}$. Since $\frac{m}{8} + 1$ is even, $\delta(a) \notin \{a^{\frac{m}{8}+1}, a^{\frac{3m}{8}+1}, a^{\frac{5m}{8}+1}, a^{\frac{7m}{8}+1}\}$. Thus $D \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$. Hence, by Theorem 3.1, we get

$$\mathrm{Aut}_c(G) \simeq \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2, & \text{if } \gcd(m, s-1) = 2 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, & \text{if } \gcd(m, s-1) = 4 \text{ or } 8. \end{cases}$$ □

Theorem 3.3. *Let* $m = 2q$, *where* $q > 1$ *is odd and* $\gcd(t, m) = 1$. *Then,* $\mathrm{Aut}_c(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. Using $(G1), (G2)$, and $(G3)$, we get $s, t \in \left\{\frac{m}{2} - 1, m - 1\right\}$. Then, the result follows on the lines of the proof of Theorem 3.2. □

Now, we will discuss the structure of the automorphism group $\mathrm{Aut}(G)$ in the case when $\gcd(t, m) > 1$.

Theorem 3.4. *Let* $m = 2^n$, $n \geqslant 4$ *and* t *be even. Then*

$$\mathrm{Aut}_c(G) \simeq \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}, & \text{if } 2t(s+1) \equiv 0 \pmod m \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}, & \text{if } 2t(s+1) \not\equiv 0 \pmod m. \end{cases}$$

Proof. Let t be even. Then $t + 1$ is odd and $\gcd(m, t+1) = 1$. Therefore, using $(G3)$, we get $s \equiv 1 \pmod{2^{n-1}}$ that is, $s = 1, 2^{n-1} + 1$. Now, using $(G2)$, we get $t \equiv 0 \pmod{2^{n-3}}$. Therefore, $t \in \{2^{n-3}, 2^{n-2}, 3 \cdot 2^{n-3}, 2^{n-1}, 5 \cdot 2^{n-3}, 3 \cdot 2^{n-2}, 7 \cdot 2^{n-3}, 2^n\}$. Note that, for $t = 2^{n-1}$ or $t = 2^n$, G is the semidirect product of H and K. Therefore, $t \in \{2^{n-3}, 2^{n-2}, 3 \cdot 2^{n-3}, 5 \cdot 2^{n-3}, 3 \cdot 2^{n-2}, 7 \cdot 2^{n-3}\}$. Since $s \equiv 1 \pmod{2^{n-1}}$, by Proposition 3.1, $\mathrm{Fix}(\tau) = \langle a^2\rangle$. Therefore, for $\delta \in S$, $a^{-1}\delta(a) \in \langle a^2\rangle$. Thus $\delta(a) = a^l$, where l is odd. Hence, $D \simeq U(2^n) \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}}$.

Note that, for $t = 2^{n-2}, 3 \cdot 2^{n-2}$, $2t(s+1) \equiv 0 \pmod m$. Therefore, by Proposition 3.2, $A \simeq B \simeq C \simeq \mathbb{Z}_2$. Also, note that, for $t \in \{2^{n-3}, 3 \cdot 2^{n-3}, 5 \cdot 2^{n-3}, 7 \cdot 2^{n-3}\}$,

$2t(s+1) \not\equiv 0 \pmod{m}$. Therefore, by Proposition 3.2, A and B are trivial and $C \simeq \mathbb{Z}_2$. Hence, the result holds by Theorem 3.1. □

THEOREM 3.5. *Let $m = 4q$ and $\gcd(t, m) = 2^i d$, where $q > 1$ is odd, $i \in \{0, 1, 2\}$, and d divides q. Then*

$$\mathrm{Aut}_c(G) \simeq \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, & \text{if } d = 1 \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(q), & \text{if } d = q \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d), & \text{if } 1 < d < q. \end{cases} \tag{3.1}$$

PROOF. Let $\gcd(t, m) = 2^i d$, where $i \in \{0, 1, 2\}$, and d divides q. Then, using $(G2)$, $s \equiv -1 \pmod{\frac{q}{d}}$ which implies that $s = l\frac{q}{d} - 1$, where $1 \leqslant l \leqslant 4d$. Since $\gcd(s, \frac{m}{2}) = 1$, s is odd and so, l is even. Using $(G1)$ and $(G3)$, we get $\frac{lq}{2d} - 1 \equiv 0 \pmod{d}$ and $t \equiv \frac{lq}{2d} - 1 \pmod{q}$. Now, one can easily observe that $2t(s+1) \equiv 0 \pmod{m}$. Therefore, by Proposition 3.2, $A \simeq B \simeq C \simeq \mathbb{Z}_2$. Let $\delta \in S$. We have three cases namely, $d = 1$ or $d = q$ or $1 < d < q$.

CASE (i): Let $d = 1$. Then $s = 2q - 1$ and $t \in \{q - 1, 2q - 1, 3q - 1\}$. Clearly, $s \not\equiv 1 \pmod{2q}$ and $\gcd(4q, 2q - 2) = 4$. Therefore, by Proposition 3.1, $\mathrm{Fix}(\tau) = \{1, a^q, a^{2q}, a^{3q}\}$. Since $\delta \in S$ and $q + 1, 3q + 1$ are even, $\delta(a) \in \{a, a^{2q+1}\}$. Thus, $D \simeq \mathbb{Z}_2$.

CASE (ii): Let $d = q$. Then $s = 2q + 1$ and $t = q$, otherwise the group G will be the semidirect product of groups. Therefore, by Proposition 3.1, $\mathrm{Fix}(\tau) = \langle a^2 \rangle$. Since $\delta \in S$, $\delta(a) \in \{a^l \mid l \in U(4q)\}$. Thus, $D \simeq U(4q) \simeq \mathbb{Z}_2 \times U(q)$.

CASE (iii): Let $1 < d < q$. Then $s = \frac{lq}{d} - 1$, $\frac{lq}{2d} - 1 \equiv 0 \pmod{d}$ and $t \equiv \frac{lq}{2d} - 1 \pmod{q}$. Now, one can easily observe that $s \not\equiv 1 \pmod{2q}$ and

$$\gcd(m, s - 1) = \gcd\left(4q, l\frac{q}{d} - 2\right) = 2d \text{ or } 4d.$$

If $\gcd(m, s - 1) = 2d$, then by Proposition 3.1, $\mathrm{Fix}(\tau) = \langle a^{\frac{2q}{d}} \rangle$ and so $\delta(a) \in \{a, a^{\frac{2q}{d}+1}, \ldots, a^{4q - \frac{2q}{d}+1}\}$. Clearly, for all $i \in \{1, \frac{2q}{d} + 1, \ldots, 4q - \frac{2q}{d} + 1\}$, $\gcd(\frac{q}{d}, i) = 1$. Therefore, $i \in U(4q)$ if and only if $i \in U(4d)$. Thus $D \simeq U(4d) \simeq \mathbb{Z}_2 \times U(d)$. If $\gcd(m, s - 1) = 4d$, then using the similar argument, we get $D \simeq \mathbb{Z}_2 \times U(d)$.

Hence, combining all the cases (i)–(iii) and by Theorem 3.1, (3.1) holds. □

THEOREM 3.6. *Let $m = 2q$ and $\gcd(t, m) = 2^i d$, where $q > 1$ is odd, $i \in \{0, 1\}$, and d divides q. Then $\mathrm{Aut}_c(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d)$.*

PROOF. The proof follows on the lines of the proof of Theorem 3.5. □

THEOREM 3.7. *Let $m = 2^n q$, t be even and $\gcd(m, t) = 2^i d$, where $1 \leqslant i \leqslant n$, $n \geqslant 3$, $q > 1$ is odd and d divides q. Then*

$$\mathrm{Aut}_c(G) \simeq \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} \times U(d), & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} \times U(d), & \text{if } 2t(s+1) \not\equiv 0 \pmod{m}. \end{cases}$$

Proof. Case (i): Let $d = q$. Then q divides t and $t+1$ is odd which implies that $\gcd(t+1, m) = 1$. Therefore, using $(G2)$ and $(G3)$, $s \equiv 1 \pmod{\frac{m}{2}}$ and $t \equiv 0 \pmod{2^{n-3}q}$. Hence, using the similar argument as in Theorem 3.4,

$$\operatorname{Aut}_c(G) \simeq \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} \times U(q), & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} \times U(q), & \text{if } 2t(s+1) \not\equiv 0 \pmod{m}. \end{cases}$$

Case (ii): Let $d \neq q$ and $n-2 \leqslant i \leqslant n$. Then using $(G2)$, $s \equiv -1 \pmod{\frac{q}{d}}$. Thus $s = l\frac{q}{d} - 1$, where $1 \leqslant l \leqslant 2^n d$. Since $\gcd(s, \frac{m}{2}) = 1$, s is odd and so, l is even. Now, using $(G1)$, $\frac{l}{2}\big(\frac{lq}{2d} - 1\big) \equiv 0 \pmod{2^{n-3}d}$ and by $(G3)$, $t \equiv \frac{lq}{2d} - 1 \pmod{2^{n-2}q}$. Since t is even, $\frac{l}{2}$ is odd. Also, one can easily observe that $\gcd(\frac{l}{2}, d) = 1$. Thus, $\frac{lq}{2d} \equiv 1 \pmod{2^{n-3}d}$ and $t \equiv 2^i d \pmod{2^{n-2}q}$. Clearly, $2t(s+1) \equiv 0 \pmod{m}$. Therefore, by Proposition 3.2, $A \simeq B \simeq C \simeq \mathbb{Z}_2$.

Since $d \neq q$, $s \not\equiv 1 \pmod{\frac{m}{2}}$. Also, $\gcd(m, s-1) = \gcd\big(2^n q, 2\big(\frac{lq}{2d} - 1\big)\big) = 2^{n-1}d$ or $2^n d$. Therefore, by Proposition 3.1, $\operatorname{Fix}(\tau) = \langle a^{\frac{2q}{d}} \rangle$ or $\operatorname{Fix}(\tau) = \langle a^{\frac{q}{d}} \rangle$. Let $\delta \in S$. Then, using the similar argument as in the proof of Theorem 3.5 *Case(iii)*, we get $D \simeq U(2^n d)$. Hence, by Theorem 3.1, $\operatorname{Aut}_c(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} \times U(d)$.

Case (iii): Let $d \neq q$ and $i = n-3$. Then using $(G2)$, $s \equiv -1 \pmod{\frac{2q}{d}}$, that is, $s = l\frac{2q}{d} - 1$, where $1 \leqslant l \leqslant 2^{n-1}d$. Now, using $(G1)$ and $(G3)$, $l(l\frac{q}{d} - 1) \equiv 0 \pmod{2^{n-3}d}$ and $(t+1)(l\frac{q}{d} - 1) \equiv 0 \pmod{2^{n-2}q}$. If l is even, then $t \equiv l\frac{q}{d} - 1 \pmod{2^{n-2}q}$ gives that t is odd, which is a contradiction. Therefore, l is odd. Also, one can easily observe that $\gcd(l, d) = 1$. Then, $l\frac{q}{d} - 1 = 2^{n-3}dl'$ and $s = 2^{n-2}dl' + 1$, where $1 \leqslant l' \leqslant \frac{8q}{d}$. Clearly, $\gcd(l', \frac{q}{d}) = 1$. Thus, $(t+1)l' \equiv 0 \pmod{\frac{2q}{d}}$. If l' is odd, then $(t+1) \equiv 0 \pmod{\frac{2q}{d}}$ which implies that t is odd. So, l' is even. Note that, $2t(s+1) \not\equiv 0 \pmod{m}$. Therefore, by Proposition 3.2, A and B are trivial and $C \simeq \mathbb{Z}_2$.

Since $d \neq q$, $s \not\equiv 1 \pmod{\frac{m}{2}}$. Also, $\gcd(m, s-1) = \gcd(2^n q, 2(\frac{lq}{d} - 1)) = 2^{n-1}d$ or $2^n d$. Then using the similar argument as in the Case (ii), we get $D \simeq U(2^n d)$. Hence, by Theorem 3.1, $\operatorname{Aut}_c(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} \times U(d)$.

Note that, for $1 \leqslant i \leqslant n-4$, there is no group G which is the Zappa–Szép product of H and K (see [**9**, Theorem 3.11]). ☐

Theorem 3.8. *Let* $m = 2^n q$*,* t *be odd and* $\gcd(t, m) = d$*, where* $n \geqslant 4$ *and* q *is odd. Then*

$$\operatorname{Aut}_c(G) \simeq \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d), & \textit{if } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d), & \textit{if } 2t(s+1) \not\equiv 0 \pmod{m}. \end{cases}$$

Proof. Using $(G2)$, we have $s \equiv -1 \pmod{2^{n-2}\frac{q}{d}}$ which implies that $s = l2^{n-2}\frac{q}{d} - 1$, where $1 \leqslant l \leqslant 4d$. Since $s - 1 = 2(2^{n-3}\frac{lq}{d} - 1)$ and $2^{n-3}\frac{lq}{d} - 1 \not\equiv 0 \pmod{2^{n-2}q}$, $s \not\equiv 1 \pmod{\frac{m}{2}}$. Now, using $(G1)$, $l(2^{n-3}\frac{lq}{d} - 1) \equiv 0 \pmod{d}$. One can easily observe that $\gcd(l, d) = 1$. Therefore, $2^{n-3}\frac{lq}{d} - 1 = dl'$, where l' is odd and $\gcd(l', \frac{q}{d}) = 1$. Therefore, $\gcd(m, s-1) = 2d$ and so, $\operatorname{Fix}(\tau) = \langle 2^{n-1}\frac{q}{d} \rangle$. Now, let $\delta \in S$. Then $a^{-1}\delta(a) \in \langle 2^{n-1}\frac{q}{d} \rangle$ which implies that $\delta(a) \in \{a^{2^{n-1}i\frac{q}{d}+1} \mid 1 \leqslant i \leqslant 2d\}$.

Clearly, $\gcd(2^{n-1}i\frac{q}{d}+1, \frac{q}{d}) = 1$, for all i. Therefore, $\delta(a) = a^{2^{n-1}i\frac{q}{d}+1}$ if and only if $\gcd(2^{n-1}i\frac{q}{d}+1, d) = 1$. Thus $D \simeq \langle\delta\rangle \simeq U(2d) \simeq \mathbb{Z}_2 \times U(d)$.

Now, Using $(G3)$, we get

$$(t+1)\Big(\frac{lq}{d}2^{n-3}-1\Big) \equiv 0 \pmod{2^{n-2}q}. \tag{3.2}$$

If l is even, then by (3.2), $t \equiv \frac{lq}{d}2^{n-3}-1 \pmod{2^{n-2}q}$. Note that, $2t(s+1) \equiv 2t(l2^{n-2}\frac{q}{d}) \equiv 0 \pmod{m}$. Therefore, by Proposition 3.2, $A \simeq B \simeq C \simeq \mathbb{Z}_2$. Hence, by Theorem 3.1, $\operatorname{Aut}_c(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d)$.

If l is odd, then using (3.2), $(t+1)dl' \equiv 0 \pmod{2^{n-2}q}$ which implies that $t \equiv -1 \pmod{2^{n-2}\frac{q}{d}}$. Clearly, $2t(s+1) = 2t(l2^{n-2}\frac{q}{d}) \not\equiv 0 \pmod{m}$. Therefore, by Proposition 3.2, A, B are trivial and $C \simeq \mathbb{Z}_2$. Hence, by Theorem 3.1, $\operatorname{Aut}_c(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d)$. □

Theorem 3.9. *Let $m = 8q$, t be odd, and $\gcd(t, m) = d$, where $q > 1$ is odd. Then*

$$\operatorname{Aut}_c(G) \simeq \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d), & \text{if } 2t(s+1) \equiv 0 \pmod{m} \\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d), & \text{if } 2t(s+1) \not\equiv 0 \pmod{m}. \end{cases}$$

Proof. Using $(G2)$, we have $s \equiv -1 \pmod{2\frac{q}{d}}$ which implies that $s = 2l\frac{q}{d}-1$, where $1 \leqslant l \leqslant 4d$. Now, using $(G1)$, $l\big(\frac{lq}{d}-1\big) \equiv 0 \pmod{d}$. Clearly, $\gcd(l, d) = 1$. Therefore, $\frac{lq}{d}-1 \equiv 0 \pmod{d}$. Using $(G3)$, we get

$$(t+1)\Big(\frac{lq}{d}-1\Big) \equiv 0 \pmod{2q}. \tag{3.3}$$

Case (i): If l is even, then by (3.3), $t \equiv \frac{lq}{d}-1 \pmod{2q}$. Note that, $2t(s+1) \equiv 2t\big(2\frac{lq}{d}\big) \equiv 0 \pmod{m}$. Therefore, by Proposition 3.2, $A \simeq B \simeq C \simeq \mathbb{Z}_2$. Now, $s-1 = 2\big(\frac{lq}{d}-1\big) \not\equiv 0 \pmod{4q}$. Also, one can easily observe that $\gcd(m, s-1) = \gcd\big(8q, 2\big(\frac{lq}{d}-1\big)\big) = 2d$. Therefore, $\operatorname{Fix}(\tau) = \langle a^{\frac{4q}{d}}\rangle$. Let $\delta \in S$. Then $a^{-1}\delta(a) \in \langle a^{\frac{4q}{d}}\rangle$ which implies that $\delta(a) \in \{a^{\frac{4iq}{d}+1} \mid 1 \leqslant i \leqslant 2d\}$. Clearly, $\gcd\big(i, \frac{4q}{d}\big) = 1$, for all i. Therefore, $\delta(a) = a^{2^{n-1}i\frac{q}{d}+1}$ if and only if $\gcd(2^{n-1}i\frac{q}{d}+1, d) = 1$. Thus $D \simeq \langle\delta\rangle \simeq U(2d) \simeq \mathbb{Z}_2 \times U(d)$. Hence, by Theorem 3.1, $\operatorname{Aut}_c(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d)$.

Case (ii): If l is odd, then $\frac{lq}{d}-1 \equiv 0 \pmod{d}$ which implies that $\frac{lq}{d}-1 = dl'$, where l' is even and $\gcd(l', \frac{q}{d}) = 1$. Therefore, by the congruence relation (3.3), $t \equiv -1 \pmod{\frac{q}{d}}$. Clearly, $2t(s+1) \not\equiv 0 \pmod{m}$. Therefore, by Proposition 3.2, A, B are trivial and $C \simeq \mathbb{Z}_2$. Let $\delta \in S$. One can easily observe that $s \equiv 1 \pmod{\frac{m}{2}}$ if and only if $d = q$. In this case, $D \simeq U(8q) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(q)$.

Let $d \neq q$. Then $s \not\equiv 1 \pmod{\frac{m}{2}}$. Now, $\gcd(m, s-1) = \gcd(8q, 2(\frac{lq}{d}-1)) = \gcd(8q, 2dl') = 4d$ or $8d$. Then using the similar argument as in the proof of Theorem 3.7, we get $D \simeq U(8d) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d)$. Hence, by Theorem 3.1, $\operatorname{Aut}_c(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times U(d)$. □

4. $\mathrm{Aut}_c(\mathbb{Z}_{p^2} \bowtie \mathbb{Z}_m)$, p is an odd prime

In [**13**], Yacoub classified the groups which are Zappa–Szép products of cyclic groups of order m and order p^2, where p is an odd prime (see [**13**, Conclusion]) as follows.

$$M_1 = \langle a, b \mid a^m = 1 = b^{p^2}, ab = ba^u, u^{p^2} \equiv 1 \pmod m \rangle,$$
$$M_2 = \langle a, b \mid a^m = 1 = b^{p^2}, ab = b^t a, t^m \equiv 1 \pmod{p^2} \rangle,$$
$$M_3 = \langle a, b \mid a^m = 1 = b^{p^2}, ab = b^t a^{pr+1}, a^p b = ba^{p(pr+1)} \rangle,$$

and in M_3, p divides m. The groups M_1 and M_2 may be isomorphic to the group M_3 depending on the values of m, r and t. Clearly, M_1 and M_2 are semidirect products. Throughout this section G will denote the group M_3 and we will be only concerned about groups M_3 which are Zappa–Szép products but not a semidirect product. Let $H = \langle b \rangle$, $K = \langle a \rangle$ and the mutual actions of H and K are defined by $\sigma_a(b) = b^t, \tau_b(a) = a^{pr+1}$ along with $\sigma_{a^p}(b) = b$ and $\tau_b(a^p) = a^{p(pr+1)}$, where t and r are integers satisfying the conditions

(G1) $\gcd(t-1, p^2) = p$, that is, $t = 1 + \lambda p$, where $\gcd(\lambda, p) = 1$,
(G2) $\gcd(r, p) = 1$,
(G3) $p(pr+1)^p \equiv p \pmod m$.

Proposition 4.1. *Let G be as above. Then $Z(G) = \ker(\tau)\,\mathrm{Fix}(\tau)$, where*

$$\ker(\tau) = \begin{cases} \langle b^p \rangle, & \text{if } (pr+1)^p \equiv 1 \pmod m \\ \{1\}, & \text{if } (pr+1)^p \not\equiv 1 \pmod m \end{cases} \quad \text{and} \quad \mathrm{Fix}(\tau) = \begin{cases} \langle a^{\frac{m}{p}} \rangle, & \text{if } p^2 \mid m \\ \{1\}, & \text{if } p^2 \nmid m. \end{cases}$$

Proof. Using [**9**, Lemma 4.2], if $a^l \in \ker(\sigma)$, then for all j, we have $b^{jt^l} = b^j$ which implies that $j(1+p\lambda)^l \equiv j \pmod{p^2}$. Thus $jpl\lambda \equiv 0 \pmod{p^2}$ and so, $l \equiv 0 \pmod p$. Therefore, $\ker(\sigma) = \langle a^p \rangle$. Now, let $b^j \in \mathrm{Fix}(\sigma)$. Then using the similar argument we have $j \equiv 0 \pmod p$. Thus $\mathrm{Fix}(\sigma) = \langle b^p \rangle$.

Now, let $b^j \in \ker(\tau)$. Then by Lemma [**9**, Lemma 4.2], for all l, we have

$$a^{\frac{jl(l-1)}{2}((pr+1)^{\lambda p}-1) + l(pr+1)^j} = a^l. \tag{4.1}$$

Note that, if $b^j \in H^* \leqslant \mathrm{Fix}(\sigma)$, then $j \equiv 0 \pmod p$. Therefore, for $j \equiv 0 \pmod p$, using (G3), (4.1) holds if and only if $(pr+1)^p \equiv 1 \pmod m$. Thus

$$H^* = \ker(\tau) = \begin{cases} \langle b^p \rangle, & \text{if } (pr+1)^p \equiv 1 \pmod m \\ \{1\}, & \text{if } (pr+1)^p \not\equiv 1 \pmod m. \end{cases}$$

Now, let $a^l \in \mathrm{Fix}(\tau)$. Then for all j, (4.1) holds if and only if $l \equiv 0 \pmod{\frac{m}{p}}$ and p^2 divides m. Then

$$K^* = \mathrm{Fix}(\tau) = \begin{cases} \langle a^{\frac{m}{p}} \rangle, & \text{if } p^2 \mid m \\ \{1\}, & \text{if } p^2 \nmid m. \end{cases}$$

□

PROPOSITION 4.2. *Let G be the group as above. Then*

(i) $A \simeq \begin{cases} \mathbb{Z}_p, & \text{if } (pr+1)^p \equiv 1 \pmod{m} \\ \{1\}, & \text{if } (pr+1)^p \not\equiv 1 \pmod{m}, \end{cases}$ (iii) $C \simeq \begin{cases} \mathbb{Z}_p, & \text{if } p^2 \mid m \\ \{1\}, & \text{if } p^2 \nmid m, \end{cases}$

(ii) $B \simeq \begin{cases} \mathbb{Z}_p, & \text{if } (pr+1)^p \equiv 1 \pmod{m} \\ \{1\}, & \text{if } (pr+1)^p \not\equiv 1 \pmod{m}, \end{cases}$ (iv) $D \simeq \begin{cases} \mathbb{Z}_p, & \text{if } p^2 \mid m \\ \{1\}, & \text{if } p^2 \nmid m. \end{cases}$

PROOF. (i) Let $\alpha \in P$ be defined by $\alpha(b) = b^i$, where $0 \leqslant i \leqslant p^2 - 1$ and $\gcd(p, i) = 1$. Clearly, $\sigma_a(\alpha(b)) = \alpha(\sigma_a(b))$. Now, by Proposition 4.1, we get $b^{-1}\alpha(b) \in \ker(\tau)$. Then, $\alpha(b) = b$, if $(pr+1)^p \not\equiv 1 \pmod{m}$ and $\alpha(b) \in \{b, b^{p+1}, b^{2p+1}, \dots, b^{(p-1)p+1}\}$ if $(pr+1)^p \equiv 1 \pmod{m}$. Hence, (i) holds.

(ii) Let $\beta \in Q$. Then by Proposition 4.1, $\mathrm{Im}(\beta) \leqslant H^* = \ker(\tau)$. Also, one can easily observe that $\beta(a) = \beta(\tau_b(a))$. Hence, by Proposition 4.1, (ii) holds.

(iii) Let $\gamma \in R$. Then by Proposition 4.1, $\mathrm{Im}(\gamma) \leqslant K^* = \mathrm{Fix}(\tau)$. Clearly, $\gamma(\sigma_a(b)) = \gamma(b)$. Hence, by Proposition 4.1, (iii) holds.

(iv) Let $\delta \in S$ be defined by $\delta(a) = a^j$, where $\gcd(j, m) = 1$. Then $a^{-1}\delta(a) \in K^* = \mathrm{Fix}(\tau)$. Thus, by Proposition 4.1, $\delta(a) = a$, if $p^2 \nmid m$ and $\delta(a) \in \{a^{\frac{m}{p}u+1} \mid 0 \leqslant u \leqslant p-1\}$, if $p^2 \mid m$. Also, one can easily check that $\tau_b(\delta(a)) = \delta(\tau_b(a))$. Hence, (iv) holds. □

LEMMA 4.1. *Let $\alpha \in P$, $\beta \in Q$, $\gamma \in R$ and $\delta \in S$. Then*

$$\text{(i) } \alpha\beta = \beta = \beta\delta, \quad \text{(ii) } \gamma\alpha = \gamma = \delta\gamma, \quad \text{(iii) } \beta\gamma = 0 = \gamma\beta.$$

PROOF. The proof is similar to the proof of Lemma 3.1. □

THEOREM 4.1. *Let A, B, C and D be defined as above. Then* $\mathrm{Aut}_c(G) \simeq A \times B \times C \times D$.

PROOF. The proof follows using a similar argument as in the proof of Theorem 3.1. □

THEOREM 4.2. *Let G be the group defined as above. Then*

$$\mathrm{Aut}_c(G) \simeq \begin{cases} \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p, & \text{if } (pr+1)^p \equiv 1 \pmod{m} \text{ and } p^2 \mid m \\ \mathbb{Z}_p \times \mathbb{Z}_p, & \text{if } (pr+1)^p \equiv 1 \pmod{m} \text{ or } p^2 \mid m \\ \{1\}, & \text{otherwise.} \end{cases}$$

PROOF. The proof follows from Proposition 4.2 and Theorem 4.1. □

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