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DOMAIN OF THE CESÀRO MEAN OF ORDER α IN MADDOX'S SPACE $\ell(p)$

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Abstract. The sequence space $\ell(p)$ was defined by I. J. Maddox, *Spaces of strongly summable sequences*, Quart. J. Math. Oxford (2), **18** (1967), 345–355. Here, we introduce the paranormed Cesàro sequence space $\ell(C_\alpha, p)$ of order α, of non-absolute type as the domain of Cesàro mean C_α of order α and prove that the spaces $\ell(C_\alpha, p)$ and $\ell(p)$ are linearly paranorm isomorphic. Besides this, we compute the α-, β- and γ-duals of the space $\ell(C_\alpha, p)$ and construct the basis of the space $\ell(C_\alpha, p)$ together with the characterization of the classes of matrix transformations from the space $\ell(C_\alpha, p)$ into the spaces ℓ_∞ of bounded sequences and f of almost convergent sequences, and any given sequence space Y, and from a given sequence space Y into the sequence space $\ell(C_\alpha, p)$. Finally, we emphasize on some geometric properties of the space $\ell(C_\alpha, p)$.

1. Introduction

We denote the space of all sequences of complex entries by ω. Any vector subspace of ω is called a *sequence space*. We shall write ℓ_∞, c and c_0 for the spaces of all bounded, convergent and null sequences, respectively. Also by bs, cs, ℓ_1 and ℓ_p, we denote the spaces of all bounded, convergent, absolutely and p-absolutely convergent series, respectively.

A linear topological space X over the real field $\mathbb{R}$ is said to be a paranormed space if there is a subadditive function $g\colon X \to \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$ and scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X. Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear space $\ell(p)$ was defined by Maddox in [**17**] (see also Nakano [**21**] and Simons

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[**24**]), as follows:

$$\ell(p)=\left\{x=(x_k)\in\omega:\sum_k |x_k|^{p_k}<\infty\right\}\quad\text{with}\quad 0<p_k\leqslant H<\infty$$

which is a complete space paranormed by

$$g_1(x)=\left(\sum_k |x_k|^{p_k}\right)^{1/M}.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. We shall assume throughout that $p_k^{-1}+(p_k')^{-1}=1$ provided $1<\inf p_k\leqslant H<\infty$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$, where $\mathbb{N}=\{0,1,2,\dots\}$.

The *multiplier space* $S(X,Y)$ of the sequence spaces X and Y is defined by

$$S(X,Y)=\{z=(z_k)\in\omega: xz=(x_kz_k)\in Y \text{ for all } x\in X\}. \tag{1.1}$$

With the notation of (1.1), the α-, β- and γ-duals X^α, X^β and X^γ of a sequence space X are defined by $X^\alpha=S(X,\ell_1)$, $X^\beta=S(X,cs)$, and $X^\gamma=S(X,bs)$.

If a sequence space X paranormed by g contains a sequence (b_k) with the property that for every $x\in X$ there is a unique sequence of scalars (α_k) such that $\lim_{n\to\infty} g\big(x-\sum_{k=0}^n \alpha_k b_k\big)=0$, then (b_k) is called a *Schauder basis* (or briefly *basis*) for X. The series $\sum_k \alpha_k b_k$ which has the sum x is then called the expansion of x with respect to (b_k) and written as $x=\sum_k \alpha_k b_k$.

Let X, Y be any two sequence spaces and $A=(a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k,n\in\mathbb{N}$. Then, we say that A defines a *matrix transformation* from X into Y and denote it by writing $A\colon X\to Y$, if for every sequence $x=(x_k)\in X$ the sequence $Ax=\{(Ax)_n\}$, the A-transform of x, is in Y; where

$$(Ax)_n=\sum_k a_{nk}x_k\quad\text{for each}\quad n\in\mathbb{N}. \tag{1.2}$$

By $(X:Y)$, we denote the class of all matrices A such that $A\colon X\to Y$. Thus, $A\in(X:Y)$ if and only if the series on the right side of (1.2) converges for each $n\in\mathbb{N}$ and every $x\in X$, and we have $Ax\in Y$ for all $x\in X$. Also, we write $A_n=(a_{nk})_{k\in\mathbb{N}}$ for the sequence in the n-th row of A.

Let $\alpha\in\mathbb{R}$ with $\alpha>-1$. The Cesàro matrix of order α or, in short, the C_α-matrix is defined by the matrix $C_\alpha=(c_{nk}^{(\alpha)})$ which is given by

$$c_{nk}^{(\alpha)}=\begin{cases}\dfrac{\binom{n-k+\alpha-1}{n-k}}{\binom{n+\alpha}{n}}, & 0\leqslant k\leqslant n,\\ 0, & \text{otherwise}\end{cases}$$

for all $n,k\in\mathbb{N}$. Then, the inverse $C_\alpha^{-1}=(\tilde{c}_{nk}^{(\alpha)})$ of the C_α-matrix is determined by

$$\tilde{c}_{nk}^{(\alpha)}=\begin{cases}\binom{n-k-\alpha-1}{n-k}\binom{k+\alpha}{k}, & \max\{0,n-\alpha\}\leqslant k\leqslant n,\\ 0, & k>n\end{cases}$$

for all $n, k \in \mathbb{N}$, where $\alpha \in \mathbb{N}$. We should note here that the reader can refer to Malkowsky and Rakocevic [**19**, pp. 28–44] for some details related to the Cesàro methods of order greater than -1.

1.1. The spaces f and f_0. Now, we may give a short survey on the concept of *almost convergence* which is a generalization of the ordinary convergence. Banach [**3**] proved the existence of a functional L on the space ℓ_∞ satisfying the following conditions for all $x, y \in \ell_\infty$ and all scalars λ and μ:

(i) $L(\lambda x + \mu y) = \lambda L(x) + \mu L(y)$.
(ii) $x_k \geqslant 0$ for all $k \in \mathbb{N}$ implies $L((x_k)_{k=0}^\infty) \geqslant 0$.
(iii) $L((x_{n+k})_{k=0}^\infty) = L((x_k)_{k=0}^\infty)$ for all $n \in \mathbb{N}$.
(iv) $L(e) = 1$, where $e = (1, 1, 1, \dots, 1, \dots)$.

Lorentz [**16**] defined a *Banach limit* to be any functional on ℓ_∞ satisfying the conditions in (i)–(iv), and a sequence $x = (x_k) \in \ell_\infty$ is said to be almost convergent to the generalized limit α if all Banach limits of x are coincide and are equal to α, [**16**]. This is denoted by $\text{f-lim}\, x_k = \alpha$. The *shift operator* P is defined on ω by $P_n(x) = x_{n+1}$ for all $n \in \mathbb{N}$. Let P^i be the composition of P with itself i times and write for a sequence $x = (x_k)$

$$t_{mn}(x) = \frac{1}{m+1} \sum_{i=0}^{m} P_n^i(x) \quad \text{for all} \quad m, n \in \mathbb{N}.$$

Lorentz [**16**] proved that $\text{f-lim}\, x_k = \alpha$ if and only if $\lim_{m\to\infty} t_{mn}(x) = \alpha$, uniformly in n. It is well-known that a convergent sequence is almost convergent such that its ordinary and generalized limits are equal. For more detail on the Banach limit, the reader may refer to Çolak and Çakar [**11**], and Das [**12**]. Therefore, we define the spaces f_0 and f of almost null and almost convergent sequences by

$$f_0 := \left\{ x = (x_k) \in \ell_\infty : \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = 0 \text{ uniformly in } n \right\},$$

$$f := \left\{ x = (x_k) \in \ell_\infty : \exists \alpha \in \mathbb{C} \text{ such that } \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_{n+k}}{m+1} = \alpha \text{ uniformly in } n \right\}.$$

One can easily see that the inclusions $c_0 \subset f_0$, $c \subset f$, and $f_0 \subset f$ are strictly hold.

2. The Cesàro sequence space $\ell(C^\alpha, p)$ of order α

In this section, we define the Cesàro sequence space $\ell(C_\alpha, p)$ and prove that $\ell(C_\alpha, p)$ is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \leqslant H < \infty$ for all $k \in \mathbb{N}$. Finally, we give the basis for the space $\ell(C_\alpha, p)$.

Let X be any sequence space. Then, the domain X_A of an infinite matrix A in X is defined by

$$X_A = \{x = (x_k) \in \omega : Ax \in X\}. \tag{2.1}$$

In [**10**], Choudhary and Mishra have defined the sequence space $\overline{\ell(p)}$ consisting of all sequences whose B-transforms are in the space $\ell(p)$, where $B = (b_{nk})$ is defined by

$$b_{nk} = \begin{cases} 1, & 0 \leqslant k \leqslant n \\ 0, & k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Başar and Altay [**6**] have examined the space $bs(p)$ which is formerly defined by Başar [**5**] as the set of all series whose sequences of partial sums are in the space $\ell_\infty(p)$. With the notation of (2.1), the spaces $\overline{\ell(p)}$ and $bs(p)$ can be redefined by

$$\overline{\ell(p)} = [\ell(p)]_B \quad \text{and} \quad bs(p) = [\ell_\infty(p)]_B.$$

In [**7**], Başar and Altay defined the sequence space $r^q(p)$ consisting of all sequences whose R^q-transforms are in the space $\ell(p)$, where $R^q = (r^q_{nk})$ is the matrix of Riesz mean, that is

$$r^q(p) = \{\ell(p)\}_{R^q} \quad \text{and} \quad r^q_p = (\ell_p)_{R^q}.$$

In [**25**], Wang defined the sequence space $X_{a(p)}$ consisting of all sequences whose N^t-transforms are in ℓ_p and is a Banach space with the norm

$$\|x\|_p = \left(\sum_{k=0}^{\infty} \left| \frac{1}{T_k} \sum_{j=0}^{k} t_{k-j} x_j \right|^p \right)^{1/p} \quad \text{with} \quad 1 \leqslant p < \infty.$$

Yeşilkayagil and Başar [**26, 27**] have defined the sequence space $N^t(p)$ consisting of all sequences whose Nörlund transforms are in the space $\ell(p)$, where $N^t = (a^t_{nk})$ is the matrix of the Nörlund mean, that is

$$N^t(p) = \{\ell(p)\}_{N^t}.$$

Also, Aydın and Başar [**1, 2**], Başar et al. [**8**] and Nergiz and Başar [**22**] gave the domain of some triangle matrices in the sequence space $\ell(p)$. The reader can refer to the monographs [**4**] and [**20**] for the background on the normed and paranormed sequence spaces, and summability theory and related topics.

Now, we introduce the Cesàro sequence space $\ell(C_\alpha, p)$ of order α defined by

$$\ell(C_\alpha, p) := \left\{ x = (x_k) \in \omega : \sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k} < \infty \right\}$$
$$\text{with} \quad 0 < p_k \leqslant H < \infty.$$

It is natural that this space may be also defined with the notation of (2.1) that $\ell(C_\alpha, p) = \{\ell(p)\}_{C_\alpha}$.

Define the sequence $y = (y_k)$, which will be frequently used, by the C_α-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = (C_\alpha x)_k = \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \quad \text{for all} \quad k \in \mathbb{N}. \tag{2.2}$$

THEOREM 2.1. *$\ell(C_\alpha, p)$ is a complete linear metric space paranormed by g_2 defined by*

$$g_2(x) = \left(\sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k} \right)^{1/M}$$

PROOF. Since this can be shown by a routine verification, we omit details. $\square$

REMARK 2.1. One can easily see that the absolute property does not hold on the space $\ell(C_\alpha, p)$, that is $g_2(x) \neq g_2(|x|)$ for at least one sequence in the space $\ell(C_\alpha, p)$ and this says that $\ell(C_\alpha, p)$ is a sequence space of non-absolute type; where $|x| = (|x_k|)$.

THEOREM 2.2. *The Cesàro sequence space of order α, $\ell(C_\alpha, p)$ of non-absolute type is linearly isomorphic to the space $\ell(p)$, where $0 < p_k \leqslant H < \infty$ for all $k \in \mathbb{N}$.*

PROOF. To prove the theorem, we should show the existence of a linear bijection between the spaces $\ell(C_\alpha, p)$ and $\ell(p)$ for $0 < p_k \leqslant H < \infty$. Consider the transformation T defined, with the notation of (2.2),

$$T\colon \ell(C_\alpha, p) \to \ell(p), \quad x \mapsto Tx = y.$$

The linearity of T is clear. Further, it is trivial that $x = \theta$ whenever $Tx = \theta$ and hence T is injective.

Let us take any $y \in \ell(p)$ and define the sequence $x = (x_k)$ by

$$x_k = (\tilde{C}_\alpha y)_k = \sum_{j=0}^{k} \binom{k-j-\alpha-1}{k-j} \binom{j+\alpha}{j} y_j, \tag{2.3}$$

for all $k \in \mathbb{N}$, where $\max\{0, k-\alpha\} \leqslant j$. Then, we have

$$\begin{aligned} g_2(x) &= \left(\sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k} \right)^{1/M} \\ &= \left(\sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} \sum_{i=0}^{j} \binom{j-i-\alpha-1}{j-i} \binom{i+\alpha}{i} y_i \right|^{p_k} \right)^{1/M} \\ &= \left(\sum_k |y_k|^{p_k} \right)^{1/M} \\ &= g_1(y) < \infty. \end{aligned}$$

This means that $x \in \ell(C_\alpha, p)$. Consequently, T is surjective and is paranorm preserving. Hence, T is a linear bijection and this says us that the spaces $\ell(C_\alpha, p)$ and $\ell(p)$ are linearly paranorm isomorphic. $\square$

We determine the basis for the paranormed space $\ell(C_\alpha, p)$.

THEOREM 2.3. *Define the sequence* $b^{(k)}(\alpha) = \{b_n^{(k)}(\alpha)\}_{n\in\mathbb{N}}$ *of the elements of the space* $\ell(C_\alpha, p)$ *for every fixed* $k \in \mathbb{N}$ *by*

$$b_n^{(k)}(\alpha) = \begin{cases} \binom{n-k-\alpha-1}{n-k}\binom{k+\alpha}{k}, & \max\{0, n-\alpha\} \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Then, the sequence $\{b^{(k)}(\alpha)\}_{k\in\mathbb{N}}$ *is a basis for the space* $\ell(C_\alpha, p)$ *and any* $x \in \ell(C_\alpha, p)$ *has a unique representation of the form*

$$x = \sum_k \lambda_k(\alpha) b^{(k)}(\alpha), \tag{2.5}$$

where $\lambda_k(\alpha) = (C_\alpha x)_k$ *for all* $k \in \mathbb{N}$ *and* $0 < p_k \leqslant H < \infty$.

PROOF. It is clear that $\{b^{(k)}(\alpha)\} \subset \ell(C_\alpha, p)$, since

$$C_\alpha b^{(k)}(\alpha) = e^{(k)} \in \ell(p) \quad \text{for all} \quad k \in \mathbb{N}, \tag{2.6}$$

where $e^{(k)}$ is the sequence whose only non-zero term is a 1 in the k-th place for each $k \in \mathbb{N}$ and $0 < p_k \leqslant H < \infty$.

Let $x \in \ell(C_\alpha, p)$ be given. For every non-negative integer m, we put

$$x^{[m]} = \sum_{k=0}^{m} \lambda_k(\alpha) b^{(k)}(\alpha). \tag{2.7}$$

Then, we obtain by applying C_α to (2.7) with (2.6) that

$$C_\alpha x^{[m]} = \sum_{k=0}^{m} \lambda_k(\alpha) C_\alpha b^{(k)}(\alpha) = \sum_{k=0}^{m} (C_\alpha x)_k e^{(k)},$$

$$\{C_\alpha(x - x^{[m]})\}_i = \begin{cases} 0, & 0 \leqslant i \leqslant m, \\ (C_\alpha x)_i, & i > m, \end{cases}$$

where $i, m \in \mathbb{N}$. Given $\varepsilon > 0$, then there is an integer m_0 such that

$$\left[\sum_{i=m+1}^{\infty} |(C_\alpha x)_i|^{p_k}\right]^{1/M} < \varepsilon$$

for all $(m+1) \geqslant m_0$. Hence,

$$g_2\big[C_\alpha(x - x^{[m]})\big] = \left[\sum_{i=m+1}^{\infty} |(C_\alpha x)_i|^{p_k}\right]^{1/M} \leqslant \left[\sum_{i=m_0}^{\infty} |(C_\alpha x)_i|^{p_k}\right]^{1/M} < \varepsilon$$

for all $(m+1) \geqslant m_0$ which proves that $x \in \ell(C_\alpha, p)$ is represented as in (2.5).

Let us show the uniqueness of the representation for $x \in \ell(C_\alpha, p)$ given by (2.5). Suppose, on the contrary, that there exists a representation $x = \sum_k \mu_k(\alpha) b^{(k)}(\alpha)$. Since the linear transformation T, from $\ell(C_\alpha, p)$ to $\ell(p)$, used in the proof of Theorem 2.2 is continuous we have at this stage that

$$(C_\alpha x)_l = \sum_k \mu_k(\alpha)\{C_\alpha b^{(k)}(\alpha)\}_l = \sum_k \mu_k(\alpha) e_l^{(k)} = \mu_l(\alpha)$$

for all $l \in \mathbb{N}$ which contradicts the fact that $(C_\alpha x)_l = \lambda_l(\alpha)$ for all $l \in \mathbb{N}$. Hence, the representation (2.5) of $x \in \ell(C_\alpha, p)$ is unique. $\square$

3. The α-, β- and γ-duals of the space $\ell(C_\alpha, p)$

In this section, we determine the α-, β-and γ-duals of the space $\ell(C_\alpha, p)$. Firstly, we quote some lemmas which are needed in proving our theorems.

LEMMA 3.1. [**14**, Theorem 5.1.0] *The following statements hold:*

(i) *Let $1 < p_k \leqslant H < \infty$ for every $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} a_{nk} B^{-1} \right|^{p'_k} < \infty.$$

(ii) *Let $0 < p_k \leqslant 1$ for every $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if*

$$\sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N} a_{nk} \right|^{p_k} < \infty.$$

LEMMA 3.2. [**15**, Theorem 1] *The following statements hold:*

(i) *Let $1 < p_k \leqslant H < \infty$ for every $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{n \in \mathbb{N}} \sum_k |a_{nk} B^{-1}|^{p'_k} < \infty. \tag{3.1}$$

(ii) *Let $0 < p_k \leqslant 1$ for every $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if*

$$\sup_{n,k \in \mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{3.2}$$

LEMMA 3.3. [**15**, Theorem 1] *Let $0 < p_k \leqslant H < \infty$ for every $k \in \mathbb{N}$. Then, $A \in (\ell(p) : c)$ if and only if* (3.1), (3.2) *hold and there is $\beta_k \in \mathbb{C}$ such that $a_{nk} \to \beta_k$ for each $k \in \mathbb{N}$, as $n \to \infty$.*

THEOREM 3.1. *Let $1 < p_k \leqslant H < \infty$ for every $k \in \mathbb{N}$, $\max\{0, n-\alpha\} \leqslant k$ and $\max\{0, j-\alpha\} \leqslant k$. Then, define the sets $D_1(p)$, $D_2(p)$ and $D_3(p)$ as follows:*

$$D_1(p) := \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_{N \in \mathcal{F}} \sum_k \left| \sum_{n \in N} \binom{n-k-\alpha-1}{n-k} \binom{k+\alpha}{k} a_n B^{-1} \right|^{p'_k} < \infty \right\},$$

$$D_2(p) := \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \sup_{n \in \mathbb{N}} \sum_k \left| \sum_{j=k}^{n} \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_j B^{-1} \right|^{p'_k} < \infty \right\},$$

$$D_3(p) := \bigcup_{B>1} \left\{ a = (a_k) \in \omega : \lim_{n \to \infty} \sum_{j=k}^{n} \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_j \text{ exists} \right\}.$$

Then, the following statements hold:

(i) $\{\ell(C_\alpha, p)\}^\alpha = D_1(p)$. (ii) $\{\ell(C_\alpha, p)\}^\gamma = D_2(p)$.

(iii) $\{\ell(C_\alpha, p)\}^\beta = D_2(p) \cap D_3(p)$.

Proof. (i) Let us take $a = (a_n) \in \omega$. We easily derive with (2.3) that

$$a_n x_n = \sum_{k=0}^{n} \binom{n-k-\alpha-1}{n-k}\binom{k+\alpha}{k} a_n y_k = (B_\alpha y)_n \quad \text{for all} \quad n \in \mathbb{N}, \tag{3.3}$$

where $B_\alpha = (b_{nk}^{(\alpha)})$ is defined by

$$b_{nk}^{(\alpha)} = \begin{cases} \binom{n-k-\alpha-1}{n-k}\binom{k+\alpha}{k} a_n, & \max\{0, n-\alpha\} \leqslant k \leqslant n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Thus, we observe by combining (3.3) with part (i) of Lemma 3.1 that $ax = (a_n x_n) \in \ell_1$ whenever $x = (x_n) \in \ell(C_\alpha, p)$ if and only if $B_\alpha y \in \ell_1$ whenever $y = (y_n) \in \ell(p)$. This gives the desired result that $\{\ell(C_\alpha, p)\}^\alpha = D_1(p)$.

(ii) Consider the equality

$$\begin{aligned} \sum_{k=0}^{n} a_k x_k &= \sum_{k=0}^{n}\sum_{j=0}^{k} \binom{k-j-\alpha-1}{k-j}\binom{j+\alpha}{j} a_k y_j \\ &= \sum_{k=0}^{n}\sum_{j=k}^{n} \binom{j-k-\alpha-1}{j-k}\binom{k+\alpha}{k} a_j y_k \\ &= (E_\alpha y)_n \end{aligned} \tag{3.4}$$

for all $n \in \mathbb{N}$, where $E_\alpha = (e_{nk}^{(\alpha)})$ is defined by

$$e_{nk}^{(\alpha)} = \begin{cases} \sum_{j=k}^{n} \binom{j-k-\alpha-1}{j-k}\binom{k+\alpha}{k} a_j, & \max\{0, j-\alpha\} \leqslant k \leqslant n, \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$. Thus, we deduce from part (i) of Lemma 3.2 with (3.4) that $ax = (a_k x_k) \in bs$ whenever $x = (x_k) \in \ell(C_\alpha, p)$ if and only if $E_\alpha y \in \ell_\infty$ whenever $y = (y_k) \in \ell(p)$. Therefore, we obtain from part (i) of Lemma 3.2 that $\{\ell(C_\alpha, p)\}^\gamma = D_2(p)$.

(iii) We see from Lemma 3.3 that $ax = (a_k x_k) \in cs$ whenever $x = (x_k) \in \ell(C_\alpha, p)$ if and only if $E_\alpha y \in c$ whenever $y = (y_k) \in \ell(p)$. Therefore, we derive that $\{\ell(C_\alpha, p)\}^\beta = D_2(p) \cap D_3(p)$. □

Theorem 3.2. *Let $0 < p_k \leqslant 1$ for every $k \in \mathbb{N}$. Define the sets $D_4(p)$ and $D_5(p)$ by*

$$D_4(p) := \left\{ a = (a_k) \in \omega : \sup_{N \in \mathcal{F}} \sup_{k \in \mathbb{N}} \left| \sum_{n \in N} \binom{n-k-\alpha-1}{n-k}\binom{k+\alpha}{k} a_n \right|^{p_k} < \infty \right\},$$

$$D_5(p) := \left\{ a = (a_k) \in \omega : \sup_{n,k \in \mathbb{N}} \left| \sum_{j=k}^{n} \binom{j-k-\alpha-1}{j-k}\binom{k+\alpha}{k} a_j \right|^{p_k} < \infty \right\}.$$

Then, the following statements hold:

(i) $\{\ell(C_\alpha, p)\}^\alpha = D_4(p)$. (ii) $\{\ell(C_\alpha, p)\}^\gamma = D_5(p)$.

(iii) $\{\ell(C_\alpha, p)\}^\beta = D_3(p) \cap D_4(p)$.

PROOF. This is easily obtained by proceeding as in the proof of Theorem 3.1, by using Lemma 3.3 and the second parts of Lemmas 3.1, 3.2 instead of the first parts. So, we omit details. □

4. Some matrix transformations related to the sequence space $\ell(C_\alpha, p)$

In the present section, we characterize the classes $(\ell(C_\alpha,p):\ell_\infty)$, $(\ell(C_\alpha,p):f)$, $(\ell(C_\alpha,p):Y)$ and $(Y:\ell(C_\alpha,p))$ of matrix transformations, where Y denotes any given sequence space. Since $Y_A \cong Y$ for any triangle A and any sequence space Y, it is trivial that the equivalence "$x \in Y_A$ if and only if $y = Ax \in Y$" holds.

For simplicity in notation, in this section we use the notation

$$a(n,k,m) = \frac{1}{m+1}\sum_{i=0}^{m} a_{n+i,k}$$

for all $n,k,m \in \mathbb{N}$. Throughout this section, we assume that the entries of the infinite matrices $A = (a_{nk})$ and $F_\alpha = (f_{nk}^{(\alpha)})$ are connected with the relation

$$f_{nk}^{(\alpha)} := \sum_{j=k}^{\infty} \binom{j-k-\alpha-1}{j-k}\binom{k+\alpha}{k} a_{nj} \tag{4.1}$$

for all $n,k \in \mathbb{N}$, where $\max\{0, j-\alpha\} \leqslant k$.

THEOREM 4.1. *The following statements hold:*

(i) *Let $0 < p_k \leqslant 1$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(C_\alpha,p):\ell_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}} \left|f_{nk}^{(\alpha)}\right|^{p_k} < \infty. \tag{4.2}$$

(ii) *Let $1 < p_k < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(C_\alpha,p):\ell_\infty)$ if and only if*

$$C(B) = \sup_{n\in\mathbb{N}} \sum_k \left|f_{nk}^{(\alpha)} B^{-1}\right|^{p_k'} < \infty \quad \text{for all} \quad B > 1. \tag{4.3}$$

PROOF. (i) Suppose that the condition (4.2) holds, and $x = (x_k) \in \ell(C_\alpha, p)$. This implies the fact that $A_n = (a_{nk})_{k\in\mathbb{N}} \in [\ell(C_\alpha,p)]^\beta$ for each $n \in \mathbb{N}$ and the product $F_\alpha C_\alpha$ exists. Hence, the A-transform Ax of x exists. Then, we derive the following relation from the m^{th} partial sum of the series $\sum_k a_{nk}x_k$ that

$$\begin{aligned}\sum_{k=0}^{m} a_{nk}x_k &= \sum_{k=0}^{m} a_{nk}\left[\sum_{j=0}^{k}\binom{k-j-\alpha-1}{k-j}\binom{j+\alpha}{j}y_j\right] \\ &= \sum_{k=0}^{m}\sum_{j=k}^{m}\binom{j-k-\alpha-1}{j-k}\binom{k+\alpha}{k}a_{nj}y_k\end{aligned} \tag{4.4}$$

for all $m \in \mathbb{N}$. Therefore, by passing to limit as $m \to \infty$ in (4.4) we obtain the consequence that

$$(Ax)_n = \sum_k a_{nk}x_k = \sum_k \left[\sum_{j=k}^{\infty}\binom{j-k-\alpha-1}{j-k}\binom{k+\alpha}{k}a_{nj}\right]y_k \tag{4.5}$$

$$= \sum_k f_{nk}^{(\alpha)} y_k = \big(F_\alpha y\big)_n$$

for all $n \in \mathbb{N}$. In this situation, since condition (3.2) of part (ii) of Lemma 3.2 is fulfilled by the matrix F_α, we conclude that $Ax = F_\alpha y \in \ell_\infty$. Hence, the condition is sufficient.

Conversely, suppose that $A = (a_{nk}) \in (\ell(C_\alpha, p) : \ell_\infty)$. Then, Ax exists and is in the space ℓ_∞ for all $x \in \ell(C_\alpha, p)$. This gives that $A_n = (a_{nk})_{k\in\mathbb{N}} \in [\ell(C_\alpha, p)]^\beta$ for each $n \in \mathbb{N}$ which shows the necessity of (4.2).

(ii) Suppose that condition (4.3) holds, and $x = (x_k) \in \ell(C_\alpha, p)$. Then, Ax exists and we again have relation (4.5) by following the same way in proving part (i), above. Now, consider the following inequality (see [**15**]) which holds for any $B > 0$ and $\alpha, \beta \in \mathbb{C}$ that

$$|\alpha\beta| \leqslant B[|\alpha B^{-1}|^{p'} + |\beta|^p] \quad \text{with} \quad p > 1. \tag{4.6}$$

Therefore, we observe by combining (4.5) and inequality (4.6) that

$$\sup_{n\in\mathbb{N}} \left| \sum_k a_{nk} x_k \right| \leqslant \sup_{n\in\mathbb{N}} \sum_k \big|f_{nk}^{(\alpha)}\big| |y_k| \leqslant B[C(B) + g_1(y)] < \infty$$

which means that $A \in (\ell(C_\alpha, p) : \ell_\infty)$.

Conversely, let us suppose that $A = (a_{nk}) \in (\ell(C_\alpha, p) : \ell_\infty)$. Then, Ax exists and belongs to the space ℓ_∞ for all $x \in \ell(C_\alpha, p)$. This yields that $A_n = (a_{nk})_{k\in\mathbb{N}} \in [\ell(C_\alpha, p)]^\beta$ for each $n \in \mathbb{N}$ which shows the necessity of (4.3). □

THEOREM 4.2. *$A = (a_{nk}) \in (\ell(C_\alpha, p) : f)$ if and only if conditions (4.2) and (4.3) hold, and*

$$\alpha_k \in \mathbb{C} \ni \lim_{m\to\infty} \frac{1}{m+1} \sum_{r=0}^{m} \sum_{j=k}^{\infty} \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_{n+r,j} = \alpha_k \tag{4.7}$$

uniformly in n, for all $k \in \mathbb{N}$.

PROOF. Since the theorem can be proved for $0 < p_k \leqslant 1$ by a similar way, to avoid the repetition of the similar statements, we only consider the case $1 < p_k < \infty$.

Let $A = (a_{nk}) \in \ell(C_\alpha, p) : f)$ with $1 < p_k < \infty$. Then, Ax exists and is in the space f for all $x \in \ell(C_\alpha, p)$. Since the inclusion $f \subset \ell_\infty$ holds, the necessity of condition (4.3) follows from Theorem 4.1.

Besides, one can conclude for $x = b^{(k)}(\alpha) = \{b_n^{(k)}(\alpha)\} \in \ell(C_\alpha, p)$ defined by (2.4) that

$$Ax = \left\{ \sum_{j=k}^{\infty} \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_{nj} \right\}_{n\in\mathbb{N}}$$

belongs to the space f for each $k \in \mathbb{N}$. This gives the necessity of condition (4.7).

Conversely, suppose that conditions (4.3) and (4.7) hold, and take any $x = (x_k) \in \ell(C_\alpha, p)$. Then, since $A_n = (a_{nk})_{k\in\mathbb{N}} \in [\ell(C_\alpha, p)]^\beta$ for each $n \in \mathbb{N}$, Ax exists. Therefore, we again have relation (4.5) by following the same way in proving part (i), above. Since the series $\sum_{k=0}^{\infty} a_{nk} x_k$ is convergent by the hypothesis, the series $\sum_{k=0}^{\infty} \big[\sum_{j=k}^{\infty} \binom{j-k-\alpha-1}{j-k} \binom{k+\alpha}{k} a_{nj}\big] y_k$ is also convergent. Therefore, we have from

(4.7) that $|f^{(\alpha)}(n,k,m)|^{p'_k} \to |\alpha_k|^{p'_k}$, as $m \to \infty$, uniformly in n for each $k \in \mathbb{N}$ which leads with (4.3) that the inequality

$$\sum_{k=0}^{i} |\alpha_k|^{p'_k} \leqslant \sup_{n,m\in\mathbb{N}} \sum_{k=0}^{\infty} |f^{(\alpha)}(n,k,m)|^{p'_k} = C < \infty$$

holds for every $i \in \mathbb{N}$. That is, $(\alpha_k) \in \ell(p')$. Since $x \in \ell(C_\alpha, p)$ by the hypothesis and $\ell(C_\alpha, p) \cong \ell(p)$, $y = (y_k) \in \ell(p)$. Therefore, we see by applying Hölder's inequality that $\sum_{k=0}^{\infty} |\alpha_k y_k| < \infty$ for all $y \in \ell(p)$. For any given $\varepsilon > 0$, choose a fixed $k_0 \in \mathbb{N}$ such that

$$\left(\sum_{k=k_0+1}^{\infty} |y_k|^{p_k} \right)^{1/p_k} < \frac{\varepsilon}{4C^{1/q}}.$$

Then, there is some $m_0 \in \mathbb{N}$ by (4.7) such that $\left| \sum_{k=0}^{k_0} [f^{(\alpha)}(n,k,m) - \alpha_k] y_k \right| < \varepsilon/2$ for every $m \geqslant m_0$, uniformly in n. Therefore, we see by applying Hölder's inequality that

$$\begin{aligned}
&\left| \frac{1}{m+1} \sum_{i=0}^{m} (F_\alpha y)_{n+i} - \sum_{k=0}^{\infty} \alpha_k y_k \right| \\
&\qquad \leqslant \left| \sum_{k=0}^{k_0} [f^{(\alpha)}(n,k,m) - \alpha_k] y_k \right| + \left| \sum_{k=k_0+1}^{\infty} [f^{(\alpha)}(n,k,m) - \alpha_k] y_k \right| \\
&\qquad < \frac{\varepsilon}{2} + \left\{ \sum_{k=k_0+1}^{\infty} [|f^{(\alpha)}(n,k,m)| + |\alpha_k|]^{p'_k} \right\}^{1/p'_k} \left(\sum_{k=k_0+1}^{\infty} |y_k|^{p_k} \right)^{1/p_k} \\
&\qquad < \frac{\varepsilon}{2} + 2C^{1/p'_k} \frac{\varepsilon}{4C^{1/p'_k}} = \varepsilon
\end{aligned}$$

for all sufficiently large m uniformly in n. Hence, $F_\alpha y \in f$ which leads to the fact that $Ax \in f$, as desired. That is to say that the conditions (4.3) and (4.7) are sufficient.

This step completes the proof of the theorem for the case $1 < p_k < \infty$. □

If we replace the space f_0 with the space f, then Theorem 4.2 is reduced to the following:

COROLLARY 4.1. *Let $A = (a_{nk})$ be an infinite matrix. Then, $A \in (\ell(C_\alpha, p) : f_0)$ if and only if* (4.2) *and* (4.3) *hold, and* (4.7) *also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

If we replace the spaces c and c_0 with the spaces f and f_0, then Theorem 4.2 and Corollary 4.1 are respectively reduced to the following:

COROLLARY 4.2. *Let $A = (a_{nk})$ be an infinite matrix. Then, the following statements hold:*

(i) $A \in (\ell(C_\alpha, p) : c)$ *if and only if* (4.2) *and* (4.3) *hold, and*

$$\exists \alpha_k \in \mathbb{C} \quad \text{such that} \quad \lim_{n\to\infty} f_{nk}^{(\alpha)} = \alpha_k \quad \text{for each fixed} \quad k \in \mathbb{N}. \tag{4.8}$$

(ii) *$A \in (\ell(C_\alpha, p) : c_0)$ if and only if (4.2) and (4.3) hold, and (4.8) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

By combining Theorems 4.1 and 4.2 with Corollaries 4.1 and 4.2, the following results are derived for the characterization of some matrix classes concerning with the Cesàro sequence spaces $\ell(C_\alpha, p)$ of order α:

Corollary 4.3. *Let the entries of the infinite matrices $A = (a_{nk})$ and $F_\alpha = (f_{nk}^{(\alpha)})$ are connected with the relation (4.1), and $a(n,k) = \sum_{i=0}^{n} a_{ik}$ for all $n, k \in \mathbb{N}$. Then, the following statements hold:*

(i) *Let $0 < p_k \leqslant 1$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(C_\alpha, p) : bs)$ if and only if (4.2) holds with $a(n,k)$ instead of a_{nk}.*
(ii) *Let $1 < p_k < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(C_\alpha, p) : bs)$ if and only if (4.3) holds with $a(n,k)$ instead of a_{nk}.*
(iii) *Let $0 < p_k \leqslant 1$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(C_\alpha, p) : fs)$ if and only if (4.2) and (4.7) hold with $a(n,k)$ instead of a_{nk}, where fs denotes the space of all series whose sequence of partial sums are in the space f.*
(iv) *Let $1 < p_k < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(C_\alpha, p) : fs)$ if and only if (4.3) and (4.7) hold with $a(n,k)$ instead of a_{nk}.*
(v) *Let $0 < p_k \leqslant 1$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(C_\alpha, p) : fs_0)$ if and only if (4.2) holds and (4.7) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$ with $a(n,k)$ instead of a_{nk}, where fs_0 denotes the space of all series whose sequence of partial sums are in the space f_0.*
(vi) *Let $1 < p_k < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(C_\alpha, p) : fs_0)$ if and only if (4.3) holds and (4.7) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$ with $a(n,k)$ instead of a_{nk}.*
(vii) *Let $0 < p_k \leqslant 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(C_\alpha, p) : cs)$ if and only if (4.2) and (4.8) hold with $a(n,k)$ instead of a_{nk}.*
(viii) *Let $1 < p_k < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(C_\alpha, p) : cs)$ if and only if (4.3) and (4.8) hold with $a(n,k)$ instead of a_{nk}.*
(ix) *Let $0 < p_k \leqslant 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(C_\alpha, p) : cs_0)$ if and only if (4.2) holds and (4.8) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$ with $a(n,k)$ instead of a_{nk}, where cs_0 denotes the space of all series whose sequence of partial sums are in the space c_0.*
(x) *Let $1 < p_k < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(C_\alpha, p) : cs_0)$ if and only if (4.3) holds and (4.8) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$ with $\alpha_k = 0$ for all $k \in \mathbb{N}$ with $a(n,k)$ instead of a_{nk}.*

In order to be able to characterize the classes of matrix transformations from the space $\ell(C_\alpha, p)$ to the any given sequence space Y and conversely from the any given sequence space Y to the space $\ell(C_\alpha, p)$, we give the following two theorems:

Theorem 4.3. *Suppose that the entries of the infinite matrices $A = (a_{nk})$ and $F_\alpha = (f_{nk}^{(\alpha)})$ are connected with the relation (4.1) for all $k, n \in \mathbb{N}$ and Y be any given sequence space. Then, $A \in (\ell(C_\alpha, p) : Y)$ if and only if $A_n \in \{\ell(C_\alpha, p)\}^\beta$ for all $n \in \mathbb{N}$ and $F_\alpha \in (\ell(p) : Y)$.*

PROOF. Let Y be any given sequence space. Suppose that (4.1) holds between the entries of the matrices $A = (a_{nk})$ and $F_\alpha = (f_{nk}^{(\alpha)})$, and take into account that the spaces $\ell(C_\alpha, p)$ and $\ell(p)$ are linearly paranorm isomorphic.

Let $A \in (\ell(C_\alpha, p) : Y)$ and take any $y \in \ell(p)$. Then,

$$(F_\alpha C_\alpha)_{nk} = \sum_{i=k}^{\infty} f_{ni}^{(\alpha)} c_{ik}^{(\alpha)} = \sum_{i=k}^{\infty}\sum_{j=i}^{\infty} \binom{j-i-\alpha-1}{j-i}\binom{i-k+\alpha-1}{i-k} a_{nj} = a_{nk},$$

i.e., $F_\alpha C_\alpha$ exists and $A_n \in \{\ell(C_\alpha, p)\}^\beta$ which yields that $(F_\alpha)_n \in \ell_1$ for each $n \in \mathbb{N}$. Hence, $F_\alpha y$ exists and thus

$$\begin{aligned}\sum_k f_{nk}^{(\alpha)} y_k &= \sum_k \sum_{j=k}^{\infty} \binom{j-k-\alpha-1}{j-k}\binom{k+\alpha}{k} a_{nj} \left[\frac{1}{\binom{k+\alpha}{k}} \sum_{i=0}^{k} \binom{k-i+\alpha-1}{k-i} x_i\right] \\ &= \sum_k \sum_{j=k}^{\infty} \binom{j-k-\alpha-1}{j-k} a_{nj} \sum_{i=0}^{k} \binom{k-i+\alpha-1}{k-i} x_i = \sum_k a_{nk} x_k\end{aligned}$$

for all $n \in \mathbb{N}$. So, we derive that $F_\alpha y = Ax$, which leads us to the consequence $F_\alpha \in (\ell(p) : Y)$.

Conversely, let $A_n \in \{\ell(C_\alpha, p)\}^\beta$ for each $n \in \mathbb{N}$ and $F_\alpha \in (\ell(p) : Y)$, and take $x = (x_k) \in \ell(C_\alpha, p)$. Then, Ax exists. Therefore, we again obtain the relation (4.5) by following the same way used in the proof of part (i) of Theorem 4.1 for all $n \in \mathbb{N}$, i.e., $Ax = F_\alpha y$ and this shows that $A \in (\ell(C_\alpha, p) : Y)$. □

By changing the roles of the spaces $\ell(C_\alpha, p)$ with Y in Theorem 4.3, we have:

THEOREM 4.4. *Suppose that Y be any given sequence space and the entries of the infinite matrices $A = (a_{nk})$ and $G_\alpha = (g_{nk}^{(\alpha)})$ are connected with the relation*

$$g_{nk}^{(\alpha)} = \frac{1}{\binom{n+\alpha}{n}} \sum_{j=0}^{n} \binom{n-j+\alpha-1}{n-j} a_{jk}$$

for all $n, k \in \mathbb{N}$. Then, $A \in (Y : \ell(C_\alpha, p))$ if and only if $G_\alpha \in (Y : \ell(p))$.

PROOF. Let $s = (s_k) \in Y$ and consider the following equality

$$\begin{aligned}&\frac{1}{\binom{n+\alpha}{n}} \sum_{j=0}^{n} \binom{n-j+\alpha-1}{n-j} \sum_{k=0}^{m} a_{jk} s_k \\ &\quad = \sum_{k=0}^{m} \frac{1}{\binom{n+\alpha}{n}} \sum_{j=0}^{n} \binom{n-j+\alpha-1}{n-j} a_{jk} s_k \\ &\quad = \sum_{k=0}^{m} g_{nk}^{(\alpha)} s_k \quad \text{for all} \quad n \in \mathbb{N}.\end{aligned} \tag{4.9}$$

Then, by letting $m \to \infty$ in (4.9) we have $\{C_\alpha(As)\}_n = (G_\alpha s)_n$ for all $n \in \mathbb{N}$. Since $As \in \ell(C_\alpha, p)$, $C_\alpha(As) = G_\alpha s \in \ell(p)$. □

5. The rotundity of the space $\ell(C_\alpha, p)$

In functional analysis, the rotundity of Banach spaces is one of the most important geometric property. For details, the reader may refer to [**9, 13, 18**]. In this section, we give the necessary and sufficient condition in order to the space $\ell(C_\alpha, p)$ be rotund and present some results related to this concept.

Definition 5.1. Let $S(X)$ be the unit sphere of a Banach space X. Then, a point $x \in S(X)$ is called an extreme point if $2x = y + z$ implies $y = z$ for every $y, z \in S(X)$. A Banach space X is said to be rotund (strictly convex) if every point of $S(X)$ is an extreme point.

Definition 5.2. A Banach space X is said to have Kadec–Klee property (or property (H)) if every weakly convergent sequence on the unit sphere is convergent in norm.

Definition 5.3. A Banach space X is said to have

(i) the Opial property if every sequence (x_n) weakly convergent to $x_0 \in X$ satisfies
$$\liminf_{n\to\infty} \|x_n - x_0\| < \liminf_{n\to\infty} \|x_n + x\|$$
for every $x \in X$ with $x \neq x_0$.

(ii) the uniform Opial property if for each $\varepsilon > 0$, there exists an $r > 0$ such that
$$1 + r \leqslant \liminf_{n\to\infty} \|x_n + x\|$$
for each $x \in X$ with $\|x\| \geqslant \varepsilon$ and each sequence (x_n) in X such that $x_n \xrightarrow{w} 0$ and $\liminf_{n\to\infty} \|x_n\| \geqslant 1$.

Definition 5.4. Let X be a real vector space. A functional $\sigma\colon X \to [0,\infty)$ is called a modular if

(i) $\sigma(x) = 0$ if and only if $x = \theta$;

(ii) $\sigma(\eta x) = \sigma(x)$ for all scalars η with $|\eta| = 1$;

(iii) $\sigma(\eta x + \beta y) \leqslant \sigma(x) + \sigma(y)$ for all $x, y \in X$ and $\eta, \beta \geqslant 0$ with $\eta + \beta = 1$;

(iv) the modular σ is called convex if $\sigma(\eta x + \beta y) \leqslant \eta\sigma(x) + \beta\sigma(y)$ for all $x, y \in X$ and $\eta, \beta > 0$ with $\eta + \beta = 1$;

A modular σ on X is called

(a) right continuous if $\lim_{\eta\to 1^+} \sigma(\eta x) = \sigma(x)$ for all $x \in X_\sigma$.

(b) left continuous if $\lim_{\eta\to 1^-} \sigma(\eta x) = \sigma(x)$ for all $x \in X_\sigma$.

(c) continuous if it is both right and left continuous, where
$$X_\sigma = \Big\{x \in X : \lim_{\eta\to 0^+} \sigma(\eta x) = 0\Big\}.$$

We define σ_p on $\ell(C_\alpha, p)$ by
$$\sigma_p(x) = \sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k}.$$

If $p_k \geqslant 1$ for all positive integer k, by the convexity of the function $t \mapsto |t|^{p_k}$ for each k, σ_p is a convex modular on $\ell(C_\alpha, p)$. We consider $\ell(C_\alpha, p)$ equipped with Luxemburg norm given by

$$\|x\| = \inf\{\eta > 0 : \sigma_p(x/\eta) \leqslant 1\}. \tag{5.1}$$

$\ell(C_\alpha, p)$ is a Banach space with this norm. This can be showed by the similar way used in the proof of Theorem 7 in [**22**].

We establish some basic properties for the modular σ_p.

Proposition 5.1. *The modular σ_p on $\ell(C_\alpha, p)$ satisfies the following properties with $p_k \geqslant 1$ for all positive integer k:*

(i) *If $0 < \eta \leqslant 1$, then $\eta^M \sigma_p(x/\eta) \leqslant \sigma_p(x)$ and $\sigma_p(\eta x) \leqslant \eta\sigma_p(x)$.*
(ii) *If $\eta \geqslant 1$, then $\sigma_p(x) \leqslant \eta^M \sigma_p(x/\eta)$.*
(iii) *If $\eta \geqslant 1$, then $\sigma_p(x) \geqslant \eta\sigma_p(x/\eta)$.*
(iv) *The modular σ_p is continuous.*

Proof. (i) Let $0 < \eta \leqslant 1$. Then $\eta^M/\eta^{p_k} \leqslant 1$ for all $p_k \geqslant 1$. So, we have

$$\begin{aligned}\eta^M \sigma_p(x/\eta) &= \sum_k \frac{\eta^M}{\eta^{p_k}} \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k} \\ &\leqslant \sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k} = \sigma_p(x), \\ \sigma_p(\eta x) &= \sum_k \eta^{p_k} \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k} \\ &\leqslant \eta \sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k} = \eta\sigma_p(x).\end{aligned}$$

(ii) Let $\eta \geqslant 1$. Then $1 \leqslant \eta^M/\eta^{p_k}$ for all $p_k \geqslant 1$. So, we have

$$\sigma_p(x) \leqslant \frac{\eta^M}{\eta^{p_k}} \sigma_p(x) = \eta^M \sigma_p(x/\eta).$$

(iii) Let $\eta \geqslant 1$. Then $\eta/\eta^{p_k} \leqslant 1$ for all $p_k \geqslant 1$. Therefore, one can easily see that

$$\begin{aligned}\sigma_p(x) &= \sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k} \\ &\geqslant \sum_k \frac{\eta}{\eta^{p_k}} \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k} \\ &= \eta\sigma_p\big(x/\eta\big).\end{aligned}$$

(iv) If $\eta > 1$, then we have

$$\sum_k \eta \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k} \leqslant \sum_k \eta^{p_k} \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \right|^{p_k}$$

$$\leqslant \sum_k \eta^M \bigg| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \bigg|^{p_k},$$

that is to say that

$$\eta\sigma_p(x) \leqslant \sigma_p(\eta x) \leqslant \eta^M \sigma_p(x). \tag{5.2}$$

By passing to limit as $\eta \to 1^+$ in (5.2), we have $\lim_{\eta\to 1^+} \sigma_p(\eta x) = \sigma_p(x)$. Hence, σ_p is right continuous.

If $0 < \eta < 1$, we have

$$\sum_k \eta^M \bigg| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \bigg|^{p_k} \leqslant \sum_k \eta^{p_k} \bigg| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \bigg|^{p_k}$$
$$\leqslant \sum_k \eta \bigg| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \bigg|^{p_k},$$

that is

$$\eta^M \sigma_p(x) \leqslant \sigma_p(\eta x) \leqslant \eta\sigma_p(x). \tag{5.3}$$

By letting $\eta \to 1^-$ in (5.3), we have $\lim_{\eta\to 1^-} \sigma_p(\eta x) = \sigma_p(x)$. Hence, σ_p is left continuous. Since σ_p is both right and left continuous, it is continuous. □

Now, we give some relationships between the modular σ_p and the Luxemburg norm on $\ell(C_\alpha, p)$.

Proposition 5.2. *For any $x \in \ell(C_\alpha, p)$, the following statements hold:*

(i) *If $\|x\| < 1$, then $\sigma_p(x) \leqslant \|x\|$.*
(ii) *If $\|x\| > 1$, then $\sigma_p(x) \geqslant \|x\|$.*
(iii) *$\|x\| = 1$ if and only if $\sigma_p(x) = 1$.*
(iv) *$\|x\| < 1$ if and only if $\sigma_p(x) < 1$.*
(v) *$\|x\| > 1$ if and only if $\sigma_p(x) > 1$.*
(vi) *If $0 < \eta < 1$ and $\|x\| > \eta$, then $\sigma_p(x) > \eta^M$.*
(vii) *If $\eta \geqslant 1$ and $\|x\| < \eta$, then $\sigma_p(x) < \eta^M$.*

Proof. Let $x \in \ell(C_\alpha, p)$.

(i) Let $\varepsilon > 0$ such that $0 < \varepsilon < 1 - \|x\|$. By the definition of $\|\cdot\|$ in (5.1), there exists an $\eta > 0$ such that $\|x\| + \varepsilon > \eta$ and $\sigma_p(x/\eta) \leqslant 1$. So, we have

$$\begin{aligned}\sigma_p(x) &\leqslant \sum_k \Big(\frac{\|x\|+\varepsilon}{\eta}\Big)^{p_k} \bigg| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} x_j \bigg|^{p_k} \\ &\leqslant (\|x\|+\varepsilon)\sigma_p(x/\eta) \leqslant \|x\| + \varepsilon\end{aligned} \tag{5.4}$$

Since ε is arbitrary, we have $\sigma_p(x) \leqslant \|x\|$ from (5.4).

(ii) If we choose $\varepsilon > 0$ such that $0 < \varepsilon < 1 - 1/\|x\|$, then $1 < (1-\varepsilon)\|x\| < \|x\|$. By the definition of $\|\cdot\|$ in (5.1) and part (iii) of Proposition 5.1, we have

$$1 < \sigma_p\Big[\frac{x}{(1-\varepsilon)\|x\|}\Big] \leqslant \frac{1}{(1-\varepsilon)\|x\|}\sigma_p(x).$$

So, $(1-\varepsilon)\|x\| < \|x\|$ for all $\varepsilon \in (0, 1-(1/\|x\|))$. This implies that $\|x\| < \sigma_p(x)$
(iii) Since σ_p is continuous by Theorem 1.4 of [**18**], we directly have (iii).
(iv) This follows from parts (i) and (iii).
(v) This follows from parts (ii) and (iii). (vi) This follows from part (ii) and part (i) of Proposition 5.1.
(vii) This follows from part (i) and part (ii) of Proposition 5.1. □

THEOREM 5.1. *The space $\ell(C_\alpha, p)$ is rotund if only if $p_k > 1$ for all $k \in \mathbb{N}$.*

PROOF. Let $\ell(C_\alpha, p)$ be rotund and choose $k \in \mathbb{N}$ such that $p_k = 1$ for all $k < 3$. Consider the sequences $x = (x_k)$ and $u = (u_k)$ given by

$$x_k := (-1)^k \binom{\alpha}{k} \quad \text{and} \quad u_k := \begin{cases} (-1)^{k+1}\binom{\alpha+1}{1}\binom{\alpha}{k-1}, & k \geqslant 1, \\ 0, & k = 0 \end{cases}$$

Then, obviously $x \neq u$ and

$$\sigma_p(x) = \sigma_p(u) = \sigma_p\Big(\frac{x+u}{2}\Big) = 1$$

By part (iii) of Proposition 5.2, $x, u, (x+u)/2 \in S[\ell(C_\alpha, p)]$ which leads us to the contradiction that the sequence space $\ell(C_\alpha, p)$ is not rotund. Hence, $p_k > 1$ for all $k \in \mathbb{N}$.

Conversely, let $x \in S[\ell(C_\alpha, p)]$ and $v, z \in S[\ell(C_\alpha, p)]$ with $x = (v+z)/2$. By convexity of σ_p and part (iii) of Proposition 5.2, we have

$$1 = \sigma_p(x) \leqslant \frac{\sigma_p(v) + \sigma_p(z)}{2} = 1,$$

which gives that

$$\sigma_p(x) = \frac{\sigma_p(v) + \sigma_p(z)}{2}. \tag{5.5}$$

Also, since $x = (v+z)/2$ and from (5.5) we obtain that

$$\begin{aligned} \sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} \frac{(v_j+z_j)}{2} \right|^{p_k} &= \frac{1}{2} \sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} v_j \right|^{p_k} \\ &\quad + \frac{1}{2} \sum_k \left| \frac{1}{\binom{k+\alpha}{k}} \sum_{j=0}^{k} \binom{k-j+\alpha-1}{k-j} z_j \right|^{p_k}. \end{aligned}$$

This implies that

$$\left| \frac{v_j + z_j}{2} \right|^{p_k} = \frac{|v_j|^{p_k} + |z_j|^{p_k}}{2} \tag{5.6}$$

for all $k \in \mathbb{N}$. Since the function $t \to |t|^{p_k}$ is strictly convex for all $k \in \mathbb{N}$, it follows by (5.6) that $v_k = z_k$ for all $k \in \mathbb{N}$. Hence, $v = z$. That is, $\ell(C_\alpha, p)$ is rotund. □

THEOREM 5.2. *Let (x_k) be a sequence in $\ell(C_\alpha, p)$. Then, the following statements hold:*

(i) $\lim_{k\to\infty} \|x_k\| = 1$ *implies* $\lim_{k\to\infty} \sigma_p(x_k) = 1$.
(ii) $\lim_{k\to\infty} \sigma_p(x_k) = 0$ *implies* $\lim_{k\to\infty} \|x_k\| = 0$.

PROOF. This is easily obtained by following the proof of Theorem 10 in [**22**]. □

THEOREM 5.3. *Let $x \in \ell(C_\alpha, p)$ and $(x^{(j)}) \subset \ell(C_\alpha, p)$. If $\sigma_p(x^{(j)}) \to \sigma_p(x)$ as $n \to \infty$ and $x_k^{(j)} \to x_k$ as $n \to \infty$ for all $k \in \mathbb{N}$, then $x^{(j)} \to x$ as $j \to \infty$.*

PROOF. Let $\varepsilon > 0$ be given. Since $x \in \ell(C_\alpha, p)$ and $(x^{(j)}) \subset \ell(C_\alpha, p)$, $\sigma_p(x^{(j)} - x) = \sum_k |\{C_\alpha(x^{(j)} - x)\}_k|^{p_k} < \infty$. So, there exists an $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0+1}^{\infty} \left|\left\{C_\alpha(x^{(j)} - x)\right\}_k\right|^{p_k} < \frac{\varepsilon}{2}. \tag{5.7}$$

Also, since $x_k^{(j)} \to x_k$, we have

$$\sum_{k=1}^{k_0} \left|\left\{C_\alpha(x^{(j)} - x)\right\}_k\right|^{p_k} < \frac{\varepsilon}{2}. \tag{5.8}$$

Therefore, we obtain from (5.7) and (5.8) that $\sigma_p(x^{(j)} - x) < \varepsilon$. This means that $\sigma_p(x^{(j)} - x) \to 0$, as $j \to \infty$. This result implies $\|x^{(j)} - x\| \to 0$, as $j \to \infty$ from part (ii) of Theorem 5.2. Hence, $x_k \to x$ as $k \to \infty$. □

THEOREM 5.4. *The sequence space $\ell(C_\alpha, p)$ has the Kadec–Klee property.*

PROOF. Let $x \in S[\ell(C_\alpha, p)]$ and $(x^{(j)}) \subset \ell(C_\alpha, p)$ such that $\|x^{(j)}\| \to 1$ and $x^{(j)} \xrightarrow{w} x$ are given. By part (i) of Theorem 5.2, we have $\sigma_p(x^{(j)}) \to 1$, as $n \to \infty$. Also, $x \in S[\ell(C_\alpha, p)]$ implies $\|x\| = 1$. By part (iii) of Proposition 5.2, we obtain $\sigma_p(x) = 1$. Therefore, we have $\sigma_p(x^{(j)}) \to \sigma_p(x)$, as $n \to \infty$.

Since $x^{(j)} \xrightarrow{w} x$ and $q_k\colon \ell(C_\alpha, p) \to \mathbb{R}$ or $\mathbb{C}$) defined by $q_k(x) = x_k$ is continuous, $x_k^{(j)} \to x_k$, as $j \to \infty$. Therefore, $x^{(j)} \to x$, as $j \to \infty$. □

THEOREM 5.5. *For any $1 < p < \infty$, the space $X_{a(p)}$ has the uniform Opial property.*

PROOF. Since the proof can be given by the similar way used in proving Theorem 13 of Nergiz and Başar [**22**], we omit details. □

Conclusion

Wang introduced the sequence space $X_{a(p)}$, in [**25**]. Although the domain of several triangle matrices in the classical sequence spaces ℓ_p, c_0, c and ℓ_∞ and in the Maddox spaces $\ell(p)$, $c_0(p)$, $c(p)$ and $\ell_\infty(p)$ were investigated by researchers, we introduce the Cesàro sequence space $\ell(C_\alpha, p)$ of order α and prove that the spaces $\ell(C_\alpha, p)$ and $\ell(p)$ are linearly paranorm isomorphic. Furthermore, we give the α-, β-and γ-duals of the space $\ell(C_\alpha, p)$ and characterize the classes $(\ell(C_\alpha, p) : \ell_\infty)$,

$(\ell(C_\alpha, p) : f)$, $(\ell(C_\alpha, p) : Y)$ and $(Y : \ell(C_\alpha, p))$ of infinite matrices, where Y is any given sequence space. Finally, we investigate some geometric properties of the space $\ell(C_\alpha, p)$.

It is clear that by depending the choice of the sequence space Y, the characterization of several classes of matrix transformations from the space $\ell(C_\alpha, p)$ and into the space $\ell(C_\alpha, p)$ can be obtained from Theorems 4.3 and 4.4, respectively. Since $p_k = p$ for all $k \in \mathbb{N}$ our space $\ell(C_\alpha, p)$ is reduced to the space $\ell_p(C_\alpha)$, our results are more general and more comprehensive than the corresponding results given by Roopaei and Başar [**23**]. As a natural continuation of this paper, one can study the domains $\ell_\infty(C_\alpha, p)$, $c(C_\alpha, p)$ and $c_0(C_\alpha, p)$ of the Cesàro mean of order α in the Maddox's spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$, respectively.

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