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SPECTRAL PROPERTIES OF SOLUTIONS OF THE YANG–BAXTER-LIKE MATRIX EQUATION

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Abstract. We analyse the spectral properties of solutions of the Yang–Baxter-like matrix equation. We explore the solution set when A is nonsingular, give partial results for nilpotent matrices, and construct elementary solutions to the problem.

1. Introduction

Let A be a given $n \times n$ complex matrix. The following equation

$$AXA = XAX \tag{1.1}$$

is called the *Yang–Baxter-like* matrix equation, for some unknown matrix X. This equation comes from the more general Yang–Baxter equation first mentioned in the papers of Yang [**14**] and Baxter [**1**]. The Yang–Baxter equation comes up in areas such as statistical mechanics, knot theory, braid theory and quantum group theory [**15**], and because of that, it is important to have a deeper understanding of this problem. Although the easier matrix equation version (1.1) seems simple, currently, only some special cases have been solved. Some of the earlier works include results regarding stochastic matrices [**6**], spectral projector solutions [**7**–**9**, **18**], idempotent matrices [**4**, **11**], cases with A having few distinct eigenvalues [**2**,**19**], rank 1 and 2 matrices [**13**,**16**,**17**], commutative solutions [**10**,**12**], numerical algorithms and others. We will also mention a very recent work [**3**] which was an inspiration for this paper, and [**5**] that analysed isolated and connected solutions, which will be briefly discussed here.

In order to simplify the starting problem, we can transform the matrix A into its Jordan canonical form (JCF). Let $A, P \in \mathbb{C}^{m\times m}$ and $J = P^{-1}AP$ where J is JCF of A. Then the matrix equation $AXA = XAX$ is equivalent to $JZJ = ZJZ$, where $Z = P^{-1}XP$. We will mostly work with the latter form in this paper.

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2. Nonsingular matrix

First, we will give a generalisation of Lemma 1 from [**3**]:

THEOREM 2.1. *Let* $J = \operatorname{diag}(P_1(\lambda_1), P_2(\lambda_2), \dots, P_k(\lambda_k))$*, with* $\lambda_1\lambda_2\cdots\lambda_k \neq 0$ *and* $\lambda_i \neq \lambda_j$ *when* $i \neq j$*, where* $P_i(\lambda_i)$ *is in JCF and consists of Jordan blocks with eigenvalue* λ_i*. If* $JZJ = ZJZ$ *then:*

(1) *eigenvalue* λ *of* Z *satisfies* $\lambda \in \{\lambda_1, \dots, \lambda_k, 0\}$
(2) $a_J(\lambda_i) \geqslant a_Z(\lambda_i)$*,* $a_Z(0) = g_Z(0)$
(3) $a_J(\lambda_i) - g_J(\lambda_i) \geqslant a_Z(\lambda_i) - g_Z(\lambda_i)$

PROOF. Let J be of dimension $m \times m$ and define $s \coloneqq \operatorname{rank}(Z)$, $0 \leqslant s \leqslant m$, and Z_i, $i = 1, \dots, m$ as the i-th column vector of Z. Let Z_{p_i}, $i = 1, \dots, s$ be the linearly independent column vectors of Z, such that Z_{p_i} can not be written as a linear combination of $Z_1, \dots, Z_{p_i-1}$. We have

$$ZJZ_{p_i} = ZJZe_{p_i} = JZJe_{p_i} = \lambda_{q_i} JZ_{p_i} + \delta_i JZ_{p_i-1} \tag{2.1}$$

where $\lambda_{q_i} \in \{\lambda_1, \lambda_2, \dots, \lambda_k\}$ and $\delta_i \in \{0, 1\}$ ($\delta_i = 1$ iff p_i-th column of J has a 1 above its eigenvalue).

We will prove from induction that matrix Z has s linearly independent generalised eigenvectors which correspond to nonzero eigenvalues.

Base case: If $p_1 = 1$ then $\delta_1 = 0$ and we have $ZJZ_{p_1} = \lambda_{q_1} JZ_{p_1}$. Otherwise, $p_1 > 1$ and $Z_{p_1-1} = 0$ which also implies $ZJZ_{p_1} = \lambda_{q_1} JZ_{p_1}$. Therefore

$$(Z - \lambda_{q_1} I)JZ_{p_1} = 0$$

Induction hypothesis: For every $i = 1, \dots, n-1$ there exists a vector T_i, which is a linear combination of vectors $Z_{p_1}, \dots, Z_{p_i}$ with a nonzero coefficient next to Z_{p_i} such that $(Z - \lambda_{q_i} I)^{r_i} JT_i = 0$.

Induction step: If $\delta_n = 0$, we get $(Z - \lambda_{q_n} I)JT_n = 0$ for $T_n = Z_{p_n}$ and we find an eigenvector JT_n. Otherwise, we have $(Z - \lambda_{q_n} I)JZ_{p_n} = JZ_{p_n-1}$. Because $p_n - 1 < p_n$, it follows that $Z_{p_n-1} = \sum_{i=1}^{n-1} \alpha_i Z_{p_i}$, $\alpha_i \in \mathbb{C}$. Define $j \in \{1, \dots, n\}$ as the largest number such that JT_{j-1} isn't a generalised eigenvector with eigenvalue λ_{q_n}. If such generalised eigenvector doesn't exist, $j \coloneqq 1$. From (2.1) we have

$$\begin{aligned} JZ_{p_k} &= (1 - t_k)JZ_{p_k} + \frac{t_k}{\lambda_{q_k}} ZJZ_{p_k} - \frac{t_k}{\lambda_{q_k}} JZ_{p_k-1} \\ &= \frac{1}{\lambda_{q_k} - \lambda_{q_n}} (Z - \lambda_{q_n} I)JZ_{p_k} + \sum_{i=1}^{k-1} \zeta_i JZ_{p_i} \\ &= (Z - \lambda_{q_n} I)J\xi_k Z_{p_k} + \sum_{i=1}^{k-1} \zeta_i JZ_{p_i} \end{aligned}$$

where $t_k = \frac{\lambda_{q_k}}{\lambda_{q_k} - \lambda_{q_n}}$ and ζ_k and ξ_k are some constants for $k = 1, \dots, j-1$. Consequently, we get that $\sum_{i=1}^{j-1} \gamma_i JZ_{p_i} = (Z - \lambda_{q_n} I)JZ'$ for some Z' which is a linear combination of $Z_{p_1}, \dots, Z_{p_{j-1}}$. Using this, we can transform the starting equation to get

$$\begin{aligned}
(Z - \lambda_{q_n} I) J Z_{p_n} = J Z_{p_n - 1} &= J \sum_{i=1}^{n-1} \alpha_i Z_{p_i} = \sum_{i=1}^{n-1} \beta_i J T_i \\
&= \sum_{i=1}^{j-1} \beta_i J T_i + \sum_{i=j}^{n-1} \beta_i J T_i \\
&= \sum_{i=1}^{j-1} \gamma_i J Z_{p_i} + \sum_{i=j}^{n-1} \beta_i J T_i \\
&= (Z - \lambda_{q_n} I) J Z' + \sum_{i=j}^{n-1} \beta_i J T_i \\
(Z - \lambda_{q_n} I) J (Z_{p_n} - Z') &= \sum_{i=j}^{n-1} \beta_i J T_i \\
(Z - \lambda_{q_n} I)^{r+1} J (Z_{p_n} - Z') &= (Z - \lambda_{q_n} I)^r \sum_{i=j}^{n-1} \beta_i J T_i \\
&= \sum_{i=j}^{n-1} \beta_i (Z - \lambda_{q_n} I)^r J T_i = 0
\end{aligned}$$

The last step follows from the induction hypothesis where $r \coloneqq \max\{r_j, \dots, r_{n-1}\}$. Setting $T_n \coloneqq Z_{p_n} - Z'$ and $r_n \coloneqq r + 1$ we get the desired result.

Because J is nonsingular, we have that JZ_{p_i}, $i = 1, \dots, s$ are the linearly independent generalised eigenvectors of Z with eigenvalues λ_{q_i}, $i = 1, \dots, s$. Let

$$J_Z = \operatorname{diag}(J_{n_1}(\sigma_1), \dots, J_{n_j}(\sigma_j), J_{n_{j+1}}(0), \dots, J_{n_l}(0))$$

be the JCF of Z for some $j, l \in \mathbb{N}_0$, where $\sigma_i \neq 0$, $i = 1, \dots, j$. From

$$s = \operatorname{rank}(Z) = \operatorname{rank}(J_Z) = n_1 + \cdots + n_j + (n_{j+1} - 1) + \cdots + (n_l - 1)$$

and because there are at least s linearly independent eigenvectors corresponding to nonzero eigenvalues, we get $s = n_1 + \cdots + n_j$ and $n_{j+1} = \cdots = n_l = 1$. We conclude that $J_Z = \operatorname{diag}(J_{n_1}(\sigma_1), \dots, J_{n_j}(\sigma_j), 0_{m-s \times m-s})$, where $\sigma_i \in \{\lambda_1, \dots, \lambda_k\}$. From construction of generalised eigenvectors, statements 2 and 3 of the theorem follow immediately. □

Let us look at a couple of direct implications of Theorem 2.1. Because (1.1) is symmetric in A and X, the first one easily follows.

Corollary 2.1. *If $a_A(0) > g_A(0)$, then all solutions of* (1.1) *are singular. Specifically, nonzero nilpotent matrices don't pair up with nonsingular matrices.*

For the other one we need a notion of connectedness which is in the spirit of [**5**].

Definition 2.1. Let $A \in \mathbb{C}^{m\times m}$ and X_1, X_2 be solutions of (1.1). We say that X_1 and X_2 are *connected solutions* if there exists a continuous transformation from X_1 to X_2 such that all matrices on that path are solutions of (1.1).

Corollary 2.2. *Let $A \in \mathbb{C}^{m\times m}$ be nonsingular and X_1, X_2 be solutions of (1.1). X_1 and X_2 are connected solutions only if they have the same spectrum (counting multiplicity).*

Proof. Observe that from Theorem 2.1 it follows that there are only finitely many possible spectra (counting multiplicity), and also finitely many possible characteristic polynomials a solution X may have. Suppose that there are solutions of (1.1) which are connected and which do not share the same spectrum. This implies that there are two different characteristic polynomials $p_1(t)$ and $p_2(t)$ such that for every $\epsilon > 0$ there are solutions X_1, X_2 with characteristic polynomials $p_1(t), p_2(t)$ respectively, such that $|[X_2]_{i,j} - [X_1]_{i,j}| < \epsilon$, for every $i, j \in \{1, \dots, m\}$, which are connected.

We will use $\|\cdot\|_1$ to denote 1-norm for polynomials. Define $\delta \coloneqq \|p_1(t) - p_2(t)\|_1 > 0$, $\epsilon \coloneqq \frac{\delta}{(1+M+\delta)^m 2^m m!} < \delta$, where $M \coloneqq \max_{i,j\in\{1,\dots,m\}} |[X_1]_{i,j}|$, and $\epsilon_{i,j} \coloneqq [X_1]_{i,j} - [X_2]_{i,j}$.

$$\begin{aligned}
\|p_2(t) - p_1(t)\|_1 &= \left\|\det(tI - X_2) - \det(tI - X_1)\right\|_1 \\
&= \left\| \sum_{\pi\in S_m} \operatorname{sgn}(\pi)\left(\prod_{i=1}^m [tI - X_2]_{i,\pi(i)} - \prod_{i=1}^m [tI - X_1]_{i,\pi(i)}\right)\right\|_1 \\
&\leqslant \sum_{\pi\in S_m} \left\| \prod_{i=1}^m ([tI - X_1]_{i,\pi(i)} + \epsilon_{i,\pi(i)}) - \prod_{i=1}^m [tI - X_1]_{i,\pi(i)}\right\|_1 \\
&\leqslant m! \max_{\pi\in S_m} \Bigg\| \underbrace{\prod_{i=1}^m ([tI - X_1]_{i,\pi(i)} + \epsilon_{i,\pi(i)}) - \prod_{i=1}^m [tI - X_1]_{i,\pi(i)}}_{2^m - 1 \text{ summands, all containing some factor } \epsilon_{i,j}} \Bigg\|_1 \\
&< m! \max_{\pi\in S_m} \left\|(2^m - 1)\epsilon(1 + M + \delta)^{m-1}\right\|_1 \\
&= \epsilon(1 + M + \delta)^{m-1}(2^m - 1)m! < \delta
\end{aligned}$$

This leads to a contradiction, from which the statement follows. □

3. Nilpotent matrix

The idea from Theorem 2.1 can be also used to give partial results for nilpotent matrix case.

Theorem 3.1. *Let $J = \operatorname{diag}(J_{n_1}(0), J_{n_2}(0), \dots, J_{n_k}(0))$ and $n_1 + \cdots + n_k = n$. If $JZJ = ZJZ$, then $a_Z(0) + g_Z(0) + g_J(0) \geqslant n$.*

Proof. Define $s \coloneqq \operatorname{rank}(Z)$, $0 \leqslant s \leqslant n$, and Z_i, $i = 1, \dots, n$ as the i-th column vector of Z. Let Z_{p_i}, $i = 1, \dots, s$ be the linearly independent column vectors of Z, such that Z_{p_i} can not be written as a linear combination of $Z_1, \dots, Z_{p_i-1}$. We have

$$ZJZ_{p_i} = ZJZe_{p_i} = JZJe_{p_i} = \delta_i JZ_{p_i-1}$$

where $\delta_i \in \{0,1\}$ ($\delta_i = 1$ iff p_i-th column of J has a 1).

We will prove from induction that there are at least $s - g_J(0)$ linearly independent generalised eigenvectors which correspond to eigenvalue 0.

Base case: If $p_1 = 1$ then $\delta_1 = 0$ and we have $ZJZ_{p_1} = 0$. Otherwise, $p_1 > 1$ and $Z_{p_1-1} = 0$ which implies $ZJZ_{p_1} = 0$. Therefore $(Z - 0I)JZ_{p_1} = 0$.

Induction hypothesis: For every $i = 1, \dots, m-1$: $(Z - 0I)^{r_i} JZ_{p_i} = 0$.

Induction step:

$$\begin{aligned}
(Z - 0I)JZ_{p_m} &= \delta_m JZ_{p_m-1} = \delta_m J \sum_{i=1}^{m-1} \alpha_i Z_{p_i} \\
(Z - 0I)^{r+1} JZ_{p_m} &= (Z - 0I)^r \delta_m J \sum_{i=1}^{m-1} \alpha_i Z_{p_i} \\
&= \delta_m \sum_{i=1}^{m-1} \alpha_i (Z - 0I)^r JZ_{p_i} = 0
\end{aligned}$$

The last step follows from the induction hypothesis where $r \coloneqq \max\{r_1, \dots, r_{m-1}\}$.

Because $\operatorname{rank}(J) = n - g_J(0)$ and JZ_{p_i} is a generalised eigenvector for eigenvalue 0 ($JZ_{p_i} = 0$ can also happen), for $i \in \{1, \dots, s\}$, there must be at least $s - g_J(0)$ linearly independent generalised eigenvectors corresponding to eigenvalue 0. From $a_Z(0) \geqslant s - g_J(0)$ and $s = n - g_Z(0)$ we get $a_Z(0) + g_Z(0) + g_J(0) \geqslant n$. □

We can even extend this to all possible matrices. The proof is left as an exercise to the reader.

THEOREM 3.2. *Let $J = \operatorname{diag}(P_m, N_n)$, where P_m is $m \times m$ nonsingular matrix with eigenvalues $\{\lambda_1, \dots, \lambda_k\}$, and N_n is $n \times n$ nilpotent matrix, both in JCF. If $JZJ = ZJZ$ then:*

(1) $\min\{a_Z(\lambda_1), a_J(\lambda_1)\} + \cdots + \min\{a_Z(\lambda_k), a_J(\lambda_k)\} \geqslant s_1$
(2) $a_Z(0) \geqslant \max\{g_Z(0), s_2 - g_J(0)\}$, $g_Z(0) = m + n - s$

where $s = \operatorname{rank}(Z)$, s_1 rank of the first m columns of Z and s_2 rank of the last n columns of Z.

4. Construction of solutions

Taking a look for the first time at equation (1.1), one can immediately see the trivial solutions $X = 0$ and $X = A$. However, a sharp-eyed one can notice the following:

LEMMA 4.1. *Let $J = \operatorname{diag}(J_{n_1}(\lambda_1), \dots, J_{n_k}(\lambda_k), 0_{m\times m})$, which is in JCF. All matrices of the form $Z = \operatorname{diag}(Z_1, \dots, Z_k, P_0)$, where $Z_i \in \{J_{n_i}(\lambda_i), 0_{n_i \times n_i}\}$ and P_0 is any $m \times m$ matrix (essentially, it is enough that P_0 is any matrix in JCF), are solutions to the Yang–Baxter-like matrix equation $JZJ = ZJZ$.*

Proof. Trivial. $\square$

There are other ways to construct solutions to $JZJ = ZJZ$. Notice that $J = J_{n+1}(0)$ has solutions $Z = \left(\begin{smallmatrix} P_{1\times n} \\ 0_{n-1\times n} \end{smallmatrix}\right)$ and $Z = \begin{pmatrix} 0_{n\times n-1} & P_{n\times 1} \end{pmatrix}$, where $P_{1\times n}$ and $P_{n\times 1}$ are arbitrary. These solutions have JCF of the type $\operatorname{diag}(\lambda I_{1\times 1}, 0)$ or $\operatorname{diag}(J_2(0), 0)$. With this we can generalise our earlier result.

Theorem 4.1. *Let $A \in \mathbb{C}^{m\times m}$ and let $J = \operatorname{diag}(J_{n_1}(\lambda_1), \dots, J_{n_k}(\lambda_k))$ be its JCF. All matrices of the form $Y = \operatorname{diag}(J_{m_1}(\sigma_1), \dots, J_{m_l}(\sigma_l))$ that are equal to J with some exceptions:*

1. *some Jordan blocks with eigenvalue 0 were swapped with $\operatorname{diag}(\lambda I_{1\times 1}, 0)$ or $\operatorname{diag}(J_2(0), 0)$ of the same size, and vice versa*
2. *some 0 blocks where swapped with arbitrary matrices of the same size (again, it is enough to just look at JCF)*

are solutions to the Yang–Baxter-like matrix equation $JZJ = ZJZ$. All matrices of the form $Z = \operatorname{diag}(Z_1, \dots, Z_l)$, where $Z_i \in \{J_{m_i}(\sigma_i), 0_{m_i\times m_i}\}$, are solutions to the Yang–Baxter-like matrix equation $JZJ = ZJZ$. We will call these solutions elementary.

Proof. Trivial. $\square$

Note that to find all elementary solutions to equation (1.1) we potentially need to check several Jordan canonical form representations of A. For example $J_1 = \operatorname{diag}(2, 2, 0, 0)$ and $J_2 = \operatorname{diag}(2, 0, 2, 0)$ have $Z_1 = \operatorname{diag}(2, J_3(0))$ and $Z_2 = \operatorname{diag}(J_2(0), J_2(0))$ as solutions respectively.

Corollary 4.1. *If A is nonsingular and diagonalisable, then for every solution X of (1.1) there is a similar elementary solution.*

Proof. Theorem 2.1 gives us all possibilities for the JCF of the solution, and Theorem 4.1 an example for each of them. $\square$

Corollary 4.2. *If A is diagonalisable, then for every solution X of (1.1) there is an elementary solution with the same spectrum (counting multiplicity).*

Proof. We will use the same notation as in Theorem 3.2 where J is a diagonal matrix similar to A. From Theorem 3.2 a solution Z has at most s nonzero eigenvalues from where we can choose s_1 of them from $\{\lambda_1, \dots, \lambda_k\}$ with each one having multiplicity of at most $a_J(\lambda_1), \dots, a_J(\lambda_k)$ respectively (denote this multiset as S). Since $s \leqslant s_1 + n$, we know that there are at most n nonzero eigenvalues which we did not mention. Let Z_J be equal to J with some changes:

1. $0_{n\times n}$ block changes to a diagonal matrix with those $s - s_1$ not mentioned eigenvalues and the rest 0
2. P_m diagonal matrix changes some of its eigenvalues to 0 such that the multiset of its nonzero eigenvalues is the same as S.

From Theorem 4.1 we have that J_Z is an elementary solution. $\square$

Finally, from the known solutions, we will try to generate new solutions to (1.1). Let J be in JCF, and let X_0 be a solution such that $JX_0J = X_0JX_0$. One way to find a new solution X is to find a similar matrix to X_0 which is a solution. Let $X = P^{-1}X_0P$, where P is nonsingular, be a solution. Then, $JP^{-1}X_0PJ = P^{-1}X_0PJP^{-1}X_0P$. There are two types of matrices P for which it is easy to see that they generate new solutions, those are matrices which commute with J and matrices which commute with X_0. Unfortunately, for the latter it happens that $X_0 = P^{-1}X_0P$, and we generate the starting solution. However, we still have the following:

THEOREM 4.2. *Let J be in JCF. If $JX_0J = X_0JX_0$, then $X = P^{-1}X_0P$ is a solution of Yang–Baxter-like matrix equation $JXJ = XJX$, for every nonsingular P such that $JP = PJ$.*

PROOF. Trivial. □

From Corollary 4.1 we can see that the problem of finding solutions of (1.1) for nonsingular diagonalisable matrices A is now reduced to finding solutions as similar matrices from the already known elementary solutions, with Theorem 4.2 being one of the techniques.

5. Conclusions

We have talked about the spectrum, and more generally the Jordan canonical form (JCF) of solutions of the Yang–Baxter-like matrix equation (YBME). We generalised results for nonsingular matrices, gave new insight for the nilpotent case, and finally combined them. We also constructed new (elementary) solutions to the YBME and proved that when A is diagonalisable every solution of YBME has the same spectrum, or even the same JCF, as some elementary solution. Finally, we gave one technique for the construction of new solutions from the already known ones.

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