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ON ĆIRIĆ TYPE THEOREMS IN b-METRIC SPACES

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Abstract. We show that the range of contraction constant in a result of Karapınar et al. can not be extended to $[0, 1)$. However, by using some additional conditions, we prove that the range can be extended to $[0, 1)$. The first result gives a negative answer to the open question on a Ćirić type theorem in b-metric spaces, and the next results are improvements of Ćirić type theorems in b-metric spaces in the literature.

1. Introduction and preliminaries

In [**4**], Bryant refined the Banach contraction map principle in the sense that the given map T does not have to be a contraction, but for some n, the map T^n is a contraction. The result of Bryant was improved by Sehgal [**15**] as follows.

Theorem 1.1. [**15**, p. 631] *Let (X, d) be a complete metric space, $\lambda \in [0, 1)$ and $T : X \to X$ be a continuous map satisfying for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$, $d(T^{n(x)}x, T^{n(x)}y) \leqslant \lambda\, d(x, y)$. Then T has a unique fixed point $x^* \in X$, and for all $x \in X$, $\lim_{m\to+\infty} T^m x = x^*$.*

In [**11**], Guseman refined Sehgal's result by removing the continuity of given map. After that, Ćirić [**5**] generalized the result of Sehgal as follows.

Theorem 1.2. [**5**, Theorem 1] *Let (X, d) be a complete metric space, $\lambda \in [0, 1)$ and $T : X \to X$ be a map satisfying for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$,*

$$d(T^{n(x)}x, T^{n(x)}y) \leqslant \lambda \max\{d(x, y), d(x, Ty), d(x, T^2y), \dots, d(x, T^{n(x)}y), d(x, T^{n(x)}x)\}.$$

Then T has a unique fixed point $x^* \in X$ and for all $x \in X$, $\lim_{m\to+\infty} T^m x = x^*$.

There have been many generalizations of metric spaces, one of them being the b-metric space by Bakhtin [**2**] and Czerwik [**7, 8**]. Compared to the metric, the b-metric is not necessarily continuous and the generalized inequality can not be

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applied to finite points in general. Many fixed point theorems in metric spaces have been studied and extended to b-metric spaces, for example, see [**3**, **10**] and the references therein. In particular, Karapinar et al. [**12**] extended Theorem 1.2 to b-metric spaces as follows.

THEOREM 1.3. [**12**, Theorem 2.1] *Let (X, d, κ) be a complete b-metric space, $\lambda \in [0, \frac{1}{\kappa})$ and $T : X \to X$ be a map satisfying for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$,*

$$d(T^{n(x)}x, T^{n(x)}y) \leqslant \lambda \max\left\{d(x,y), d(x,Ty), d(x,T^2y), \dots, d(x,T^{n(x)}y), d(x,T^{n(x)}x)\right\}.$$

Then T has a unique fixed point $x^* \in X$, and for all $x \in X$, $\lim_{m\to+\infty} T^m x = x^*$.

THEOREM 1.4. [**12**, Theorem 2.2] *Let (X, d, κ) be a complete b-metric space, $\lambda \in [0, \frac{1}{\kappa})$ and $f : X \to X$ be a continuous map satisfying for each $x \in X$, there exists a positive integer $n(x)$ such that for all $y \in X$,*

$$d(T^{n(x)}(x), T^{n(x)}(y)) \leqslant \lambda \max\left\{d(x,y), d(x,Ty), d(x,T^2y), \dots, d(x,T^{n(x)}y), d(x,Tx), d(x,T^2x), \dots, d(x,T^{n(x)}x)\right\}.$$

Then T has a unique fixed point x^*, and for all $x \in X$, $\lim_{m\to+\infty} T^m x = x^*$.

The proof of Theorem 1.3 in b-metric spaces follows that of Theorem 1.2 in metric spaces. However, the technique there is only available for $\lambda \in [0, \frac{1}{\kappa})$ where $[0, \frac{1}{\kappa}) \subset [0, 1)$. So, the authors posed the following question.

QUESTION 1.1. [**12**, p. 9] *In Theorem* 1.3*, can we extend the range of λ to the case $\frac{1}{\kappa} \leqslant \lambda < 1$?*

We must say that not every fixed point theorem in metric spaces can be extended fully to b-metric spaces, for example, see [**10**, Examples 20 and 21]. Recently, Lu et al. [**13**] proved an extension of the Ciric fixed point theorem in metric spaces [**6**, Theorem 1] to b-metric spaces with certain additional assumptions as follows.

THEOREM 1.5. [**14**, Theorem 3] *Let (X, d, κ) be a complete b-metric space, $\lambda \in [0, 1)$ and $T : X \to X$ be a map such that for all $x, y \in X$,*

$$d(Tx, Ty) \leqslant \lambda \max\{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$$

and let one of the following conditions hold.

(1) *T is continuous.*
(2) *d satisfies the* Fatou property*, that is, for all $x, y \in X$ and $\lim_{n\to+\infty} x_n = x$, we have $d(x,y) \leqslant \liminf_{n\to+\infty} d(x_n, y)$.*
(3) *$\lambda \in [0, \frac{1}{\kappa})$.*

Then T has a unique fixed point x^*, and for all $x \in X$, $\lim_{n\to+\infty} T^n x = x^*$.

In this paper, we give an example to show that the range of contraction constant in [**12**, Theorem 2.1] can not be extended to $[0, 1)$ which is a negative answer to Question 1.1. By using some suitable conditions, we also prove that the range can be extended to $[0,1)$ which are improvements of Theorems 1.3 and 1.4 in b-metric spaces.

2. Main results

First, we give an example to show that the answer to Question 1.1 is negative.

EXAMPLE 2.1. Let $X = \{0, 1, \frac{1}{2}, \dots, \frac{1}{n}, \dots\}$, and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \in \{0,1\} \\ |x-y| & \text{if } x \neq y \in \{0\} \cup \{\frac{1}{2n} : n \in \mathbb{N}\} \\ \frac{1}{4} & \text{otherwise,} \end{cases}$$

and let $T : X \to X$ be defined by

$$Tx = \begin{cases} 1, & \text{if } x = 0 \\ \frac{1}{10}x & \text{if } x = \frac{1}{n}, n \in \mathbb{N}. \end{cases}$$

Then

(1) (X, d, κ) is a complete b-metric space with $\kappa = 4$.

(2) There exists $\lambda \in [\frac{1}{\kappa}, 1)$ satisfying for each $x \in X$, there is a positive integer $n(x)$ such that for all $y \in X$,

$$d(T^{n(x)}x, T^{n(x)}y) \leqslant \lambda \max\{d(x,y), d(x,Ty), d(x,T^2y), \dots, d(x,T^{n(x)}y), d(x,T^{n(x)}x)\}. \tag{2.1}$$

(3) T is fixed point free.

PROOF. (1). See [**9**, Example 2.6.(1)].

(2) We find that $\frac{1}{\kappa} = \frac{1}{4}$. For $\lambda = \frac{1}{4}$, we consider the following cases.

Case 1. $x = 0$. Let $n(x) = 1$. If $y = 0$, then $d(T^{n(x)}x, T^{n(x)}y) = 0$. If $y = \frac{1}{n}$, then

$$d(T^{n(x)}x, T^{n(x)}y) = d\Big(T0, T\frac{1}{n}\Big) = d\Big(1, \frac{1}{10n}\Big) = \frac{1}{4},$$
$$d(x, T^{n(x)}x) = d(0, T0) = d(0,1) = 1.$$

Therefore $d(T^{n(x)}x, T^{n(x)}y) \leqslant \lambda\, d(x, T^{n(x)}x)$.

The above calculations show that, for $n(x) = 1$, (2.1) holds for all $y \in X$.

Case 2. $x = \frac{1}{2n}$. Let $n(x) = 2n$. If $y = \frac{1}{2m}$, then

$$\begin{aligned} d(T^{n(x)}x, T^{n(x)}y) &= d\Big(T^{2n}\frac{1}{2n}, T^{2n}\frac{1}{2m}\Big) \\ &= d\Big(\frac{1}{10^{2n}\,2n}, \frac{1}{10^{2n}\,2m}\Big) = \Big|\frac{1}{10^{2n}\,2n} - \frac{1}{10^{2n}\,2m}\Big|, \end{aligned}$$
$$d(x,y) = d\Big(\frac{1}{2n}, \frac{1}{2m}\Big) = \Big|\frac{1}{2n} - \frac{1}{2m}\Big|.$$

Therefore $d(T^{n(x)}x, T^{n(x)}y) \leqslant \frac{1}{100^{2n}} d(x,y) \leqslant \lambda\, d(x,y)$.

If $y = \frac{1}{2m-1}$, then

$$d(T^{n(x)}x, T^{n(x)}y) = d\Big(T^{n(x)}\frac{1}{n}, T^{n(x)}\frac{1}{2m-1}\Big) = d\Big(\frac{1}{10^{2n}n}, \frac{1}{10^{2n}(2m-1)}\Big)$$
$$= \Big|\frac{1}{10^{2n}n} - \frac{1}{10^{2n}(2m-1)}\Big| < \frac{1}{100},$$
$$d(x,y) = d\Big(\frac{1}{n}, \frac{1}{2m-1}\Big) = \frac{1}{4}.$$

Therefore $d(T^{n(x)}x, T^{n(x)}y) < \frac{1}{25}d(x,y) \leqslant \lambda\, d(x,y)$.

If $y = 0$, then

$$d(T^{n(x)}x, T^{n(x)}y) = d\Big(T^{2n}\frac{1}{2n}, T^{2n}0\Big) = d\Big(\frac{1}{10^{2n}\,2n}, \frac{1}{10^{2n-1}\,2}\Big)$$
$$= \Big|\frac{1}{10^{2n}\,2n} - \frac{1}{10^{2n-1}\,2}\Big| < \frac{1}{2\,10^{2n-1}},$$
$$d(x,y) = d\Big(\frac{1}{2n}, 0\Big) = |\frac{1}{2n} - 0| = \frac{1}{2n}.$$

Therefore $d(T^{n(x)}x, T^{n(x)}y) \leqslant \frac{1}{10}d(x,y) \leqslant \lambda\, d(x,y)$.

The above calculations show that, for $n(x) = 2n$, (2.1) holds for all $y \in X$.

Case 3. $x = \frac{1}{2n-1}$. Let $n(x) = 2n$. If $y = \frac{1}{2m}$, then

$$d(T^{n(x)}x, T^{n(x)}y) = d\Big(T^{2n}\frac{1}{2n-1}, T^{2n}\frac{1}{2m}\Big) = d\Big(\frac{1}{10^{2n}(2n-1)}, \frac{1}{10^{2n}\,2m}\Big)$$
$$= \Big|\frac{1}{10^{2n}(2n-1)} - \frac{1}{10^{2n}\,2m}\Big| < \frac{1}{100},$$
$$d(x,y) = d\Big(\frac{1}{2n-1}, \frac{1}{2m}\Big) = \frac{1}{4}.$$

Therefore $d(T^{n(x)}x, T^{n(x)}y) < \frac{1}{25}d(x,y) \leqslant \lambda\, d(x,y)$.

If $y = \frac{1}{2m-1}$, then

$$d(T^{n(x)}x, T^{n(x)}y) = d\Big(T^{2n}\frac{1}{2n-1}, T^{2n}\frac{1}{2m-1}\Big)$$
$$= d\Big(\frac{1}{10^{2n}(2n-1)}, \frac{1}{10^{2n}(2m-1)}\Big) = \Big|\frac{1}{10^{2n}(2n-1)} - \frac{1}{10^{2n}(2m-1)}\Big| < \frac{1}{100},$$
$$d(x,y) = d\Big(\frac{1}{2n-1}, \frac{1}{2m-1}\Big) = \frac{1}{4}.$$

Therefore $d(Tx, Ty) < \frac{1}{25}d(x,y) \leqslant \lambda\, d(x,y)$.

If $y = 0$, then

$$d(T^{n(x)}x, T^{n(x)}y) = d\Big(T^{2n}\frac{1}{2n-1}, T^{2n}0\Big) = d\Big(\frac{1}{10^{2n}(2n-1)}, \frac{1}{10^{2n-1}}\Big)$$
$$= \Big|\frac{1}{10^{2n}(2n-1)} - \frac{1}{10^{2n-1}}\Big| < \frac{1}{10^{2n-1}},$$
$$d(x,y) = d\Big(\frac{1}{2n-1}, 0\Big) = |\frac{1}{2n-1} - 0| = \frac{1}{2n-1}.$$

Therefore $d(T^{n(x)}x, T^{n(x)}y) \leqslant \frac{1}{10}d(x,y) \leqslant \lambda\, d(x,y)$.

The above calculations show that for $n(x) = 2n$, (2.1) holds for all $y \in X$.

The above three cases show that for each $x \in X$, there is a positive integer $n(x)$ such that (2.1) holds for all $y \in X$.

(3). It follows from the definition of T that T is fixed point free. □

Now, with some additional conditions, we show that the range $[0, \frac{1}{\kappa})$ of λ in Theorem 1.3 can be extended to $[0, 1)$.

THEOREM 2.1. *Let (X, d, κ) be a complete b-metric space, $\lambda \in [0, 1)$ and $f : X \to X$ be a map satisfying for each $x \in X$, there exists a positive integer $n(x)$ such that for all $y \in X$,*

$$d(T^{n(x)}(x), T^{n(x)}(y)) \leqslant \lambda \max\left\{d(x,y), d(x,Ty), d(x,T^2y), \dots, d(x,T^{n(x)}y), d(x,T^{n(x)}x)\right\} \tag{2.2}$$

and let one of the following conditions hold.

(1) *T is continuous.*
(2) *d has the Fatou property, in particular, d is continuous.*
(3) *$\lambda \in [0, \frac{1}{\kappa})$.*

Then T has a unique fixed point x^*, and for all $x \in X$, $\lim_{m\to+\infty} T^m x = x^*$.

PROOF. Since $\lambda \in [0, 1)$, there exists m_0 such that $\lambda^{m_0} < \frac{1}{\kappa}$. Let $x \in X$. For all $m \in \mathbb{N}$, put

$$n_0 = n(x), \quad n_1 = n(T^{n_0}x), \quad n_2 = n(T^{n_0+n_1}x), \dots, n_m = n(T^{n_0+n_1+\cdots+n_{m-1}}x)$$

$$s_m = \sum_{i=0}^{m} n_i. \tag{2.3}$$

Now, fixing $x = x_0$ in (2.3) and considering the sequence $\{T^m x_0\}$, we shall prove that

$$r(x_0) \leqslant \frac{\kappa}{1-\kappa\,\lambda^{m_0}} \max\{d(x_0, T^m x_0) : 0 \leqslant m \leqslant s_{m_0}\} \tag{2.4}$$

where $r(x_0) = \sup\{d(x_0, T^m x_0) : m \in \mathbb{N}\}$. For each $m > s_{m_0}$, there exists $p \in \{0, 1, \dots, m\}$ such that $d(x_0, T^p x_0) = \max\{d(x_0, T^i x_0) : 0 \leqslant i \leqslant m\}$. To prove (2.4), we need to show that

$$d(x_0, T^p x_0) \leqslant \frac{\kappa}{1-\kappa\,\lambda^{m_0}} \max\{d(x_0, T^i x_0) : 0 \leqslant i \leqslant s_{m_0}\}. \tag{2.5}$$

Indeed, for the case $p \leqslant s_{m_0}$, since $\frac{\kappa}{1-\kappa\,\lambda^{m_0}} \geqslant 1$, we find that (2.5) holds. We consider the case $p > s_{m_0}$. Note that

$$d(x_0, T^p x_0) \leqslant \kappa\, d(x_0, T^{s_{m_0}} x_0) + \kappa\, d(T^{s_{m_0}} x_0, T^p x_0). \tag{2.6}$$

By using (2.2), we have

$$\begin{aligned} d(T^{s_{m_0}}x_0, T^p x_0) &= d(T^{n_{m_0}}T^{s_{m_0-1}}x_0, T^{n_{m_0}}T^{p-n_{m_0}}x_0) \\ &\leqslant \lambda \max\{d(T^{s_{m_0-1}}x_0, T^{p-n_{m_0}}x_0), d(T^{s_{m_0-1}}x_0, T^{p-n_{m_0}+1}x_0), \\ &\qquad \ldots, d(T^{s_{m_0-1}}x_0, T^p x_0), d(T^{s_{m_0-1}}x_0, T^{s_{m_0}}x_0)\} \\ &= \lambda\, d(T^{s_{m_0-1}}x_0, T^{i_{m_0}}x_0) \end{aligned}$$

for some $i_{m_0} \in \{p - n_{m_0}, p - n_{m_0} + 1, \ldots, p, s_{m_0}\} \subset \{0, 1, \ldots, m\}$.

By using (2.2) again, we also have

$$\begin{aligned} d(T^{s_{m_0-1}}x_0, T^{i_{m_0}}x_0) &= d(T^{n_{m_0-1}}T^{s_{m_0-2}}x_0, T^{n_{m_0-1}}T^{i_{m_0}-n_{m_0-1}}x_0) \\ &\leqslant \lambda \max\{d(T^{s_{m_0-2}}x_0, T^{i_{m_0}-n_{m_0-1}}x_0), d(T^{s_{m_0-2}}x_0, T^{i-n_{m_0-1}+1}x_0), \\ &\qquad \ldots, d(T^{s_{m_0-2}}x_0, T^{i}x_0), d(T^{s_{m_0-2}}x_0, T^{s_{m_0-1}}x_0)\} \\ &= \lambda\, d(T^{s_{m_0-2}}x_0, T^{i_{m_0-1}}x_0) \end{aligned}$$

for some $i_{m_0-1} \in \{i_{m_0} - n_{m_0-1}, i_{m_0} - n_{m_0-1} + 1, \ldots, i_{m_0}, s_{m_0-1}\} \subset \{0, 1, \ldots, m\}$.

Continuing the process, we find that

$$\begin{aligned} d(T^{s_{m_0}}x_0, T^p x_0) &\leqslant \lambda\, d(T^{s_{m_0-1}}x_0, T^{i_{m_0}}x_0) \\ &\leqslant \lambda^2\, d(T^{s_{m_0-2}}x_0, T^{i_{m_0-1}}x_0) \leqslant \cdots \leqslant\leqslant \lambda^{m_0}\, d(x_0, T^{i_1}x_0) \end{aligned}$$

for some $i_1 \in \{0, 1, \ldots, m\}$. It implies that

$$d(T^{s_{m_0}}x_0, T^p x_0) \leqslant \lambda^{m_0}\, d(x_0, T^{i_1}x_0) \leqslant \lambda^{m_0}\, d(x_0, T^p x_0). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$d(x_0, T^p x_0) \leqslant \kappa\, d(x_0, T^{s_{m_0}}x_0) + \kappa\, \lambda^{m_0}\, d(x_0, T^p x_0).$$

Note that $1 - \kappa\, \lambda^{m_0} > 0$. So we get

$$d(x_0, T^p x_0) \leqslant \frac{\kappa}{1 - \kappa\, \lambda^{m_0}} d(x_0, T^{s_{m_0}}x_0).$$

This proves that (2.5) holds.

Now, we prove that the sequence $\{T^m x_0\}$ is Cauchy. Let $k, l \geqslant s_m$. By using (2.2), we have

$$\begin{aligned} d(T^{s_m}x_0, T^l x_0) &= d(T^{n_m}T^{s_{m-1}}x_0, T^{n_m}T^{l-n_m}x_0) \\ &\leqslant \lambda \max\{d(T^{s_{m-1}}x_0, T^{l-n_m}x_0), d(T^{s_{m-1}}x_0, T^{l-n_m+1}x_0), \\ &\qquad \ldots, d(T^{s_{m-1}}x_0, T^l x_0), d(T^{s_{m-1}}x_0, T^{s_m}x_0)\} \\ &= \lambda\, d(T^{s_{m-1}}x_0, T^{i_m}x_0) \end{aligned}$$

for some $i_m \in \{l - n_m, l - n_m + 1, \ldots, l, s_m\}$. By using (2.2) again, we also have

$$\begin{aligned} d(T^{s_{m-1}}x_0, T^{i_m}x_0) &= d(T^{n_{m-1}}T^{s_{m-2}}x_0, T^{n_{m-1}}T^{i_m-n_{m-1}}x_0) \\ &\leqslant \lambda \max\{d(T^{s_{m-2}}x_0, T^{i_m-n_{m-1}}x_0), d(T^{s_{m-2}}x_0, T^{i_m-n_{m-1}+1}x_0), \\ &\qquad \ldots, d(T^{s_{m-2}}x_0, T^{i_m}x_0), d(T^{s_{m-2}}x_0, T^{s_{m-1}}x_0)\} \\ &= \lambda\, d(T^{s_{m-2}}x_0, T^{i_{m-1}}x_0) \end{aligned}$$

for some $i_{m-1} \in \{i_m - n_{m-1}, i_m - n_{m-1} + 1, \ldots, i_m, s_{m-1}\}$.

Continuing the process, we have

$$d(T^{s_m}x_0, T^l x_0) \leqslant \lambda\, d(T^{s_{m-1}}x_0, T^{i_m}x_0) \leqslant \lambda^2 d(T^{s_{m-2}}x_0, T^{i_{m-1}}x_0) \leqslant \cdots \leqslant \lambda^m d(x_0, T^{i_0}x_0)$$

for some $i_0 \in \{i_1 - n_1, i_m - n_{m-1} + 1, \dots, i_1, s_1\}$. It implies that

$$d(T^{s_m}x_0, T^l x_0) \leqslant \lambda^m d(x_0, T^{i_0}x_0) \leqslant \lambda^m r(x_0). \tag{2.8}$$

Similarly, we also have

$$d(T^{s_m}x_0, T^k x_0) \leqslant \lambda^m\, r(x_0). \tag{2.9}$$

It follows from (2.8) and (2.9) that for all $k, l \geqslant s_m$,

$$d(T^l x_0, T^k x_0) \leqslant \kappa[d(T^l x_0, T^{s_m}x_0) + d(T^{s_m}x_0, T^k x_0)] \leqslant 2\kappa\lambda^m\, r(x_0).$$

Combining this with (2.4), we find that the sequence $\{T^m x_0\}$ is Cauchy. Since (X, d, κ) is complete, there exists $x^* \in X$ such that

$$\lim_{m\to+\infty} T^m x_0 = x^*. \tag{2.10}$$

By using (2.2), we have for all $m \in \mathbb{N}$,

$$\begin{aligned} (T^{n(x^*)}x^*, T^{n(x^*)+m}(x_0)) &= d(T^{n(x^*)}x^*, T^{n(x^*)}(T^m x_0)) \\ &\leqslant \lambda \max\big\{d(x^*, T^m x_0), d(x^*, TT^m x_0), \dots, d(x^*, T^{n(x^*)}T^m x_0), d(x^*, T^{n(x^*)}x^*)\big\} \\ &= \lambda \max\big\{d(x^*, T^m x_0), d(x^*, T^{m+1}x_0), \dots, d(x^*, T^{n(x^*)+m}x_0), d(x^*, T^{n(x^*)}x^*)\big\}. \end{aligned} \tag{2.11}$$

Now, if T is continuous, then

$$Tx^* = T\big(\lim_{m\to+\infty} T^m x\big) = \lim_{m\to+\infty} T^{m+1}x = x^*.$$

This proves that x^* is a fixed point of T.

If d has the Fatou property, then letting $m \to +\infty$ in (2.11) and using (2.10) we get

$$\begin{aligned} d(T^{n(x^*)}x^*, x^*) &\leqslant \liminf_{m\to+\infty} d(T^{n(x^*)}x^*, T^{n(x^*)+m}(x_0)) \\ &\leqslant \lambda \liminf_{m\to+\infty} \max\big\{d(x^*, T^m x_0), d(x^*, T^{m+1}x_0), \dots, \\ &\qquad d(x^*, T^{n(x^*)+m}x_0), d(x^*, T^{n(x^*)}x^*)\big\} \end{aligned} \tag{2.12}$$

$$= \lambda\, d(x^*, T^{n(x^*)}x^*). \tag{2.13}$$

Since $0 \leqslant \lambda < 1$, we find that $d(T^{n(x^*)}x^*, x^*) = 0$, that is, x^* is a fixed point of $T^{n(x^*)}$.

If $\lambda \in [0, \frac{1}{\kappa})$, then as in the proof of [**12**, Theorem 2.1], x^* is a fixed point of $T^{n(x^*)}$.

Now, for a fixed point x^* of $T^{n(x^*)}$, we find that for all $m \in \mathbb{N}$,

$$T^m x^* = T^m T^{n(x^*)}x^* = T^{n(x^*)}T^m x^*.$$

This proves that $T^m x^*$ is a fixed point of $T^{n(x^*)}$ for all $m \in \mathbb{N}$. So, if

$$d(x^*, T^q x^*) = \sup\{d(x^*, T^m x^*) : m \in \mathbb{N}\}$$

then $d(x^*, T^q x^*) = \max\{d(x^*, T^m x^*) : 1 \leqslant m < n(x^*)\}$. Therefore, we have

$$\begin{aligned}
& d(x^*, T^q x^*) \\
= \;& d(T^{n(x^*)} x^*, T^{n(x^*)+q}(x^*)) \\
\leqslant \;& \lambda \max\big\{d(x^*, T^q x^*), d(x^*, TT^q x^*), \ldots, d(x^*, T^{n(x^*)} T^q x^*), d(x^*, T^{n(x^*)} x^*)\big\} \\
= \;& \lambda \max\big\{d(x^*, T^q x^*), d(x^*, T^{1+q} x^*), \ldots, d(x^*, T^{n(x^*)+q} x^*), d(x^*, T^{n(x^*)} x^*)\big\} \\
\leqslant \;& \lambda d(x^*, T^q x^*).
\end{aligned}$$

Since $0 \leqslant \lambda < 1$, we find that $d(x^*, T^q x^*) = 0$. This proves that $d(x^*, T^m x^*) = 0$ for all $m \in \mathbb{N}$. So $d(x^*, Tx^*) = 0$, and x^* is a fixed point of T.

The above arguments show that T has a fixed point x^*. Now, if y^* is also a fixed point of T, then we have

$$\begin{aligned}
d(x^*, y^*) &= d(T^{n(x^*)} x^*, T^{n(x^*)}(y^*)) \\
&\leqslant \lambda \max\big\{d(x^*, y^*), d(x^*, Ty^*), \ldots, d(x^*, T^{n(x^*)} y^*), d(x^*, T^{n(x^*)} x^*)\big\} \\
&= \lambda \max\big\{d(x^*, y^*), d(x^*, y^*), \ldots, d(x^*, y^*), d(x^*, x^*)\big\} = \lambda\, d(x^*, y^*).
\end{aligned} \tag{2.14}$$

Since $0 \leqslant \lambda < 1$, we get that $d(x^*, y^*) = 0$. This proves that the fixed point of T is unique.

Finally, since x_0 in (2.10) is arbitrary in X and x^* is unique. So for all $x \in X$, we have $\lim_{m\to+\infty} T^m x = x^*$. □

Next, we show that the range $[0, \frac{1}{\kappa})$ of λ in [**12**, Theorem 2.2] can be extended to $[0, 1)$.

THEOREM 2.2. *Let (X, d, κ) be a complete b-metric space, $\lambda \in [0, 1)$ and $f : X \to X$ be a continuous map satisfying for each $x \in X$, there exists a positive integer $n(x)$ such that for all $y \in X$,*

$$\begin{aligned}
d(T^{n(x)}(x), T^{n(x)}(y)) \leqslant \lambda \max\big\{&d(x, y), d(x, Ty), d(x, T^2 y), \ldots, d(x, T^{n(x)} y), \\
&d(x, Tx), d(x, T^2 x), \ldots, d(x, T^{n(x)} x)\big\}.
\end{aligned} \tag{2.15}$$

Then T has a unique fixed point x^*, and for all $x \in X$, $\lim_{m\to+\infty} T^m x = x^*$.

PROOF. By using notations and arguments in the proof of Theorem 2.1, we also get (2.6), that is $d(x_0, T^p x_0) \leqslant \kappa\, d(x_0, T^{s_{m_0}} x_0) + \kappa\, d(T^{s_{m_0}} x_0, T^p x_0)$. By using (2.15), we have

$$\begin{aligned}
d(T^{s_{m_0}} x_0, T^p x_0) &= d(T^{n_{m_0}} T^{s_{m_0-1}} x_0, T^{n_{m_0}} T^{p-n_{m_0}} x_0) \\
&\leqslant \lambda \max\{d(T^{s_{m_0-1}} x_0, T^{p-n_{m_0}} x_0), d(T^{s_{m_0-1}} x_0, T^{p-n_{m_0}+1} x_0), \\
&\qquad \ldots, d(T^{s_{m_0-1}} x_0, T^p x_0), d(T^{s_{m_0-1}} x_0, T^{s_{m_0-1}+1} x_0), \\
&\qquad \ldots, d(T^{s_{m_0-1}} x_0, T^{s_{m_0}} x_0)\} \\
&= \lambda\, d(T^{s_{m_0-1}} x_0, T^{i_{m_0}} x_0)
\end{aligned}$$

for some i_{m_0} where

$$i_{m_0} \in \{p - n_{m_0}, p - n_{m_0} + 1, \ldots, p, s_{m_0-1}, s_{m_0-1} + 1, \ldots, s_{m_0}\} \subset \{0, 1, \ldots, m\}.$$

Then, by doing similar as in the proof of Theorem 2.1, there exists $x^* \in X$ such that $\lim_{m\to+\infty} T^m x_0 = x^*$. Since T is continuous, we have

$$Tx^* = T(\lim_{m\to+\infty} T^m x) = \lim_{m\to+\infty} TT^m x = \lim_{m\to+\infty} T^{m+1}x = x^*.$$

Then x^* is a fixed point of T. By doing similar as in the proof of Theorem 2.1, x^* is the unique fixed point of T. □

The following example shows that the continuity of the map T in Theorem 2.2 is essential.

EXAMPLE 2.2. Let $X = [0, 1]$ and for all $x, y \in X$,

$$d(x,y) = \begin{cases} |x-y|, & \text{if } xy \neq 0 \\ 2|x-y|, & \text{if } xy = 0 \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in (0,1] \\ 1, & \text{if } x = 0. \end{cases}$$

Then we have

(1) All assumptions of Theorem 2.2 are satisfied except for the continuity of T.
(2) T is fixed point free.

PROOF. (1). It follows from the definition of T that T is not continuous. It follows from [**13**, Example 6] that (X, d, κ) is a complete b-metric space with $\kappa = 2$.

Let $\lambda = \frac{3}{4} \in [\frac{1}{\kappa}, 1)$ and $n(x) = 2$ for all x. For the case $x = 0$, if $y = 0$, then $d(T^{n(x)}x, T^{n(x)}y) = 0$. If $y \neq 0$, then

$$d(T^{n(x)}x, T^{n(x)}y) = d(T^2 0, T^2 y) = d\Big(\frac{1}{2}, \frac{y}{4}\Big) = \Big|\frac{1}{2} - \frac{y}{2}\Big| \leqslant \frac{1}{2} < \frac{3}{4}d(0,1) = \frac{3}{4}d(x, Tx).$$

For the case $x \neq 0$, if $y \neq 0$, then

$$d(T^{n(x)}x, T^{n(x)}y) = d(T^2x, T^2y) = d\Big(\frac{x}{4}, \frac{y}{4}\Big) = \Big|\frac{x}{4} - \frac{y}{4}\Big| \leqslant \frac{3}{4}d(x,y).$$

If $y = 0$, then

$$d(x,y) = d(x,0) = 2x$$
$$d(x,Ty) = d(x,T0) = d(x,1) = 1 - x.$$

So we have

$$d(T^{n(x)}x, T^{n(x)}y) = d(T^2x, T^2 0) = d\Big(\frac{x}{4}, \frac{1}{2}\Big) = \Big|\frac{x}{4} - \frac{1}{2}\Big| \leqslant \frac{3}{4}\max\{d(x,y), d(x,Ty)\}.$$

The above calculations show that (2.15) holds for all $y \in X$.

(2). It follows from the definition of T that T is fixed point free. □

In Theorem 2.1, if we replace the assumption b-metric with b-metric-like [**1**, Definition 2.3], then the proof is similar, except for the arguments in proving (2.12) and (2.14). Indeed, in proving (2.12) and (2.14), we need $d(x^*, x^*) = 0$ while, for a b-metric-like d, we only have $d(x, y) = 0 \Rightarrow x = y$. So, the following question remains open.

QUESTION 2.1. *Do the conclusions of Theorems* 2.1, *and* 2.2 *hold if the assumption b-metric is replaced by b-metric-like?*

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