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DETERMINATION OF A JUMP BY (E, q) MEANS OF FOURIER–STIELTJES SERIES

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Abstract. We generalize Fejer's theorem for Fourier–Stieltjes series of functions of bounded variation.

1. Introduction

Let f be a real-valued function on the closed and bounded interval $[a, b]$ and let $P = \{x_0, x_1, \dots, x_k\}$ be a partition of $[a, b]$. Then the variation of f with respect to P is

$$V(f; P) = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

and the total variation of f on $[a, b]$ is $TV(f) = \sup V(f; P)$ for all partition P of $[a, b]$. A real-valued function f on the closed and bounded interval $[a, b]$ is said to be a function of bounded variation if $TV(f)$ is finite. From now on let f be a function of bounded variation on $[0, 2\pi]$. It is well known that such an f may have only discontinuities of the first kind, i.e., the left-hand limit $f(x^-)$ and the right-hand limit $f(x^+)$ exist. Throughout this paper, a function f of bounded variation is normalized by the condition

$$f(x) = \frac{1}{2}(f(x^+) + f(x^-)).$$

The Fourier–Stieltjes coefficients of f (equivalently, the Fourier coefficients of df) are defined by

$$\hat{df}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx}\, df(x),$$

where $k \in Z$ and the integral is Riemann–Stieltjes integral. We write

$$df(x) \sim \sum_{k\in Z} \hat{df}(k) e^{ikx} \tag{1.1}$$

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and call this series the Fourier–Stieltjes series of f (equivalently, the Fourier series of df). The n-th symmetric partial sum of series in (1.1) is defined as

$$s_n(df, x) = \sum_{|k| \leqslant n} \hat{df}(k) e^{ikx}.$$

The following result is attributed to Fejer [1] (see the details in [3]): If f is a periodic function of bounded variation on $[0, 2\pi]$, then for every $0 < x < 2\pi$, we have

$$\lim_{n\to\infty} \frac{1}{n} s_n(df, x) = \frac{1}{\pi}(f(x^+) - f(x^-)),$$

while for $x = 0$ or $x = 2\pi$, we have

$$\lim_{n\to\infty} \frac{1}{n} s_n(df, x) = \frac{1}{\pi}(f(0^+) - f(2\pi^-) + c(f)),$$

where $c(f) = 2\pi \hat{df}(0) = f(2\pi) - f(0)$.

Let $\sum_{k=0}^{\infty} u_k$ be a given infinite series with the sequence of its nth partial sum s_n. The sequence to sequence transformation

$$E_n^q = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} s_k \tag{1.2}$$

defines the sequence E_n^q of the Euler means (E, q) of order $q > 0$ [2] of the sequence s_n. The series $\sum_{k=0}^{\infty} u_k$ is said to be (E, q) summable to the sum s if $\lim_{n\to\infty} E_n^q$ exists and is equal to s. The purpose of the present paper is to extend the Fejer theorem for Fourier–Stieltjes series to (E, q) means of Fourier–Stieltjes series.

2. Main results

We recall the representation [4]

$$s_n(df, x) = \frac{1}{\pi} \int_0^{2\pi} D_n(x - t) df(t), \tag{2.1}$$

where

$$D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} \cos ku = \frac{\sin(n + \frac{1}{2})u}{2 \sin \frac{u}{2}}. \tag{2.2}$$

It follows from (1.2) and (2.3) that

$$E_n^q(df, x) = \frac{1}{\pi} \int_0^{2\pi} M_n^q(x - t) df(t), \tag{2.3}$$

where

$$M_n^q(u) = \frac{1}{(1+q)^n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} D_k(u). \tag{2.4}$$

We need the following lemmas for the proof of our theorem.

Lemma 2.1. *For any $n \in N$, $\sum_{k=0}^{n} \binom{n}{k} q^{n-k} k = n(1+q)^{n-1}$.*

Proof. Let $P(n)$ be the statement $\sum_{k=0}^{n}\binom{n}{k}q^{n-k}k = n(1+q)^{n-1}$. We give a proof by induction on n

Base case: For $n=1$, $\binom{1}{0}q^{1-0}0+\binom{1}{1}q^{1-1}1 = 1(1+q)^{1-1}$. Hence the statement $P(1)$ holds true.

Induction step: Assume that for $n=m$, the statement $P(m)$ holds true:

$$\sum_{k=0}^{m}\binom{m}{k}q^{m-k}k = m(1+q)^{m-1}.$$

It follows that

$$\begin{aligned}(m+1)(1+q)^m &= m(1+q)^{m-1}(1+q)+(1+q)^m\\ &= \left(\sum_{k=0}^{m}\binom{m}{k}q^{m-k}k\right)(1+q)+(1+q)^m\\ &= \left(\sum_{k=0}^{m}\binom{m}{k}q^{m-k}k\right)(1+q)+\sum_{k=0}^{m}\binom{m}{k}q^{m-k}\\ &= \sum_{k=0}^{m}\binom{m}{k}q^{m-k}(k+1)+\sum_{k=0}^{m}\binom{m}{k}q^{m+1-k}k\\ &= \sum_{k=0}^{m+1}\binom{m+1}{k}q^{m+1-k}k\end{aligned}$$

because of $\binom{m}{k}+\binom{m}{k-1}=\binom{m+1}{k}$. Hence $P(m+1)$ holds true, establishing the induction step. Therefore $P(n)$ holds true for every natural number n. □

Lemma 2.2. (i) *For all n and x,*

$$|M_n^q(x)| \leqslant n+\frac{1}{2}. \tag{2.5}$$

(ii) *For all n and $0<x<2\pi$,*

$$|M_n^q(x)| \leqslant \frac{\pi}{2\min\{x,2\pi-x\}}. \tag{2.6}$$

Proof. From (2.2) it follows that for all n and x, $|D_n(x)| \leqslant n+\frac{1}{2}$ and for all n and $0<x<2\pi$,

$$|D_n(x)| \leqslant \frac{\pi}{2\min\{x,2\pi-x\}}.$$

Since all numbers $\binom{n}{k}q^{n-k}$ are nonnegative, inequalities (2.5) and (2.6) follows immediately from (2.4) and Lemma 2.1. □

Now we generalize a theorem of Fejer by establishing the following theorem:

Theorem 2.1. *Let f be a periodic function of bounded variation on $[0,2\pi]$. Then for $0<x<2\pi$, we have*

$$\lim_{n\to\infty}\frac{1+q}{n}E_n^q(df,x) = \frac{1}{\pi}(f(x^+)-f(x^-)), \tag{2.7}$$

while for $x = 0$ or $x = 2\pi$, we have

$$\lim_{n\to\infty} \frac{1+q}{n} E_n^q(df, x) = \frac{1}{\pi}(f(0^+) - f(2\pi^-) + c(f)), \tag{2.8}$$

where $c(f) = 2\pi\hat{df}(0) = f(2\pi) - f(0)$.

Proof. We shall carry out the proof in four steps.

(i) We consider the particular case when f is continuous at an inner point x (i.e., $0 < x < 2\pi$). As it is well known, then the total variation of f is also continuous at x [3]. Therefore, given any $\varepsilon > 0$, we can choose $\delta = \frac{TV(f)+1}{\sqrt{n}}$ for sufficiently large n so that $0 < x - \delta < x + \delta < 2\pi$ and the total variation of f over the interval $[x - \delta, x + \delta]$ does not exceed ε. Then we decompose the integral in (2.3) as follows:

$$E_n^q(df, x) = \frac{1}{\pi}\Big(\int_0^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{2\pi}\Big) M_n^q(x-t)df(t) = A + B + C.$$

Taking (2.5) and (2.6) into account, we get

$$|B| \leqslant \frac{1}{\pi}\Big(n + \frac{1}{2}\Big)\int_{x-\delta}^{x+\delta} |df(t)| \leqslant \frac{1}{\pi}(n+1)\varepsilon < \varepsilon n,$$

$$|A| + |C| \leqslant \frac{1}{2\delta}\Big(\int_0^{x-\delta} + \int_{x+\delta}^{2\pi}\Big)|df(t)| \leqslant \frac{\sqrt{n}}{2TV(f) + 2} 2TV(f) \leqslant \sqrt{n},$$

which implies $|A| + |C| \leqslant O(\sqrt{n})$. Hence $A + B + C = o(n)$ and this proves (2.7) with $f(x^+) - f(x^-) = 0$.

(ii) From (2.3) it follows that $E_n^q(df, 0) = E_n^q(df, 2\pi)$. Hence it is enough to prove (2.8) for $x = 0$. In this step, we consider the special case when

$$f(0^+) - f(2\pi^-) + c(f) = 0, \tag{2.9}$$

which means that the function $f(t) - f(2\pi - t)$ is continuous at $t = 0$ from the right. Therefore, given any $\varepsilon > 0$, we can choose $\delta = \frac{TV(f)+1}{2\sqrt{n}}$ for sufficiently large n so that the total variation of $f(t) - f(2\pi - t)$ over the interval $[0, \delta]$ does not exceed ε. Now we decompose the integral in (2.3) as follows:

$$\begin{aligned} E_n^q(df, 0) &= \frac{1}{\pi}\Big(\int_0^{\delta} + \int_{\delta}^{2\pi-\delta} + \int_{2\pi-\delta}^{2\pi}\Big) M_n^q(t)df(t) \\ &= \frac{1}{\pi}\int_0^{\delta} M_n^q(t)d(f(t) - f(2\pi - t)) + \frac{1}{\pi}\int_{\delta}^{2\pi-\delta} M_n^q(t)df(t) = A + B, \end{aligned}$$

where we made use of the evenness of the kernel $M_n^q(t)$. By Lemma 2.2, we have

$$|A| \leqslant \frac{1}{\pi}\Big(n + \frac{1}{2}\Big)\int_0^{\delta} |d(f(t) - f(2\pi - t))| \leqslant \frac{1}{\pi}(n + \frac{1}{2})\varepsilon < \varepsilon n,$$

$$|B| \leqslant \frac{1}{2\delta}\int_{\delta}^{2\pi-\delta} |df(t)| \leqslant \frac{\sqrt{n}}{TV(f) + 1} TV(f) \leqslant \sqrt{n},$$

which implies $|B| \leqslant O(\sqrt{n})$. Hence $A+B=o(n)$ and this proves (2.8) at $x=0$ in the special case (2.9).

(iii) We shall prove (2.7) at an inner point x in the general case when f is discontinuous. Now we introduce a new function g as follows:

$$g(t)=f(t)-\frac{1}{\pi}(f(x^+)-f(x^-))\phi(t-x), \tag{2.10}$$

where ϕ is defined by $\phi(t)=\frac{1}{2}(\pi-t)$ for $0<t<2\pi$, $\phi(0)=\phi(2\pi)=0$, and continued periodically.

Observe that g is of bounded variation on $[0,2\pi]$ and g is continuous at $t=x$. Hence the argument in step (i) applies to g in place of f and yields

$$\lim_{n\to\infty}\frac{1+q}{n}E_n^q(dg,x)=0. \tag{2.11}$$

On the other hand, from (1.2) and (2.10) it follows that

$$E_n^q(dg,x)=E_n^q(df,x)-\frac{1}{\pi}(f(x^+)-f(x^-))E_n^q(d\phi,0). \tag{2.12}$$

We recall that for $0<x<2\pi$, the Fourier–Stieltjes series of ϕ is given by

$$d\phi(x)\sim\frac{1}{2}\sum_{k\in Z-\{0\}}e^{ikx}=\sum_{k=1}^{\infty}\cos kx. \tag{2.13}$$

From (1.2), (2.13) and Lemma 2.1 it follows that

$$\begin{aligned}E_n^q(d\phi,0)=\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}s_k(d\phi,0)&=\frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}k\\&=\frac{1}{(1+q)^n}n(1+q)^{n-1}=\frac{n}{(1+q)}.\end{aligned}$$

Now by virtue of (2.11), (2.12) and the last equality, we obtain (2.7).

(iv) We shall prove (2.8) at the endpoint $x=0$ (equivalently, at $x=2\pi$) in the general case when condition (2.9) is not satisfied. We define

$$g(t)=f(t)-\frac{1}{\pi}(f(0^+)-f(2\pi^-)+c(f))\phi(t),\quad \text{where } \phi(t)=\frac{1}{2}(\pi-t).$$

We see that g is of bounded variation on $[0,2\pi]$ and condition (2.9) is satisfied with g in place of f. The rest of the proof is the same as in step (iii) above. □

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