PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 113 (127) (2023), 83–92 DOI: https://doi.org/10.2298/PIM2327083K

ON TWO COMMUTATIVITY CRITERIA FOR δ-PRIME RINGS

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Abstract. The note concerns the commutativity of associative rings (possibly nonunital) endowed with a derivation. Our focus is on δ-prime rings. We give a new proof of Hirano and Tominaga's result that a δ-prime ring is commutative whenever the derivation δ is nonzero and commuting on a nonzero two-sided δ-ideal. We also provide some further generalizations of Herstein's classical theorem on a prime ring admitting a nonzero derivation with commutative range.

1. Preliminaries and introduction

Throughout the present note, R stands for an associative ring (possibly without identity) and $\mathbb{N}$ for the set of non-negative integers. Recall that the ring R is said to be 2-torsion free, if it has no element of additive order 2 (this is equivalent to saying that $2x \neq 0$ for any $x \in R \setminus \{0\}$). The commutator $[x, y]$ of elements $x, y \in R$ is defined by $[x, y] = xy - yx$. We use the standard notation $\mathrm{Z}(R)$ to represent the center of the ring R. It is worth noticing that

$$\mathrm{Z}(R) = \{x \in R : \; [x, y] = 0 \text{ for any } y \in R\}.$$

Consider now a set $E \subseteq R$ and a mapping $\varphi : R \to R$. The set E is called φ-invariant, if $\varphi(E) \subseteq E$. The mapping φ is said to be commuting on E, if $[\varphi(a), a] = 0$ for any $a \in E$ (in other words, a and $\varphi(a)$ commute whenever $a \in E$). The left and right annihilators of E are defined by

$$\mathrm{ann}^{\ell}_R(E) = \{x \in R : \; xa = 0 \text{ for any } a \in E\},$$
$$\mathrm{ann}^{r}_R(E) = \{x \in R : \; ax = 0 \text{ for any } a \in E\},$$

respectively.

2020 *Mathematics Subject Classification:* Primary 16W25; Secondary 16U99.

Key words and phrases: associative ring, commutativity criterion, derivation, δ-ideal, δ-primeness.

Communicated by Zoran Petrović.

The authors are sincerely grateful to Professor Orest D. Artemovych for his valuable comments and suggestions.

Let us next recall that an additive mapping $\delta : R \to R$ is said to be a derivation of the ring R, if it satisfies the Leibniz rule

$$\forall\, x, y \in R\ (\delta(xy) = \delta(x)y + x\delta(y)).$$

The constant mapping $R \ni x \mapsto 0 \in R$ is a derivation of R referred to as the zero derivation. More generally, $\partial_c : R \ni x \mapsto [c, x] \in R$ is a derivation of R for any element $c \in R$. This derivation is called the inner derivation induced by c. We denote the set of all derivations of the ring R by $\mathrm{Der}(R)$. For an arbitrary $\delta \in \mathrm{Der}(R)$ and an arbitrary $k \in \mathbb{N}$ we define δ^k to be the kth iterate of δ. In other words,

$$\delta^k = \begin{cases} \mathrm{id}_R, & \text{if } k = 0, \\ \underbrace{\delta \circ \cdots \circ \delta}_{k}, & \text{if } k \geqslant 1. \end{cases}$$

The following lemma gathers some well-known simple properties of derivations and commutators. We will use these properties in the next section.

Lemma 1.1. *Let $\delta \in \mathrm{Der}(R)$ and $x, y, c \in R$. Moreover, let $k \in \mathbb{N}$. Then*

(i) $[c, xy] = [c, x]y + x[c, y]$,
(ii) $\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$,
(iii) $\delta^k(xy) = \sum_{\ell=0}^{k} \binom{k}{\ell} \delta^{k-\ell}(x)\delta^{\ell}(y)$.

Consider once again a derivation $\delta \in \mathrm{Der}(R)$. A left, right or two-sided ideal of the ring R is said to be a δ-ideal, if it is δ-invariant. Let $\mathcal{F}$ be the family of all left δ-ideals of R which contain a set $E \subseteq R$. Then $\bigcap \mathcal{F}$ is also a left δ-ideal of R. This intersection is obviously called the left δ-ideal of R generated by the set E. In the same way one can define the right and two-sided δ-ideals generated by E.

The present note deals mainly with δ-prime rings. Recall that given a derivation $\delta \in \mathrm{Der}(R)$, the ring R is said to be δ-prime (or prime with respect to δ), if it is nonzero and the product of any two of its nonzero two-sided δ-ideals is again nonzero. Recall also that the ring is said to be δ-semiprime, if it has no nonzero nilpotent two-sided δ-ideals. Obviously, each δ-prime ring is δ-semiprime. It should be noticed that the word "two-sided" in the definitions of a δ-prime ring and a δ-semiprime ring can be replaced by "left" or by "right".

The δ-primeness extends the standard notion of a prime ring. To be more precise, the following conditions are equivalent for a ring R:

- it is prime in the standard sense,
- it is prime with respect to the zero derivation,
- it is prime with respect to any derivation $\delta \in \mathrm{Der}(R)$.

Notice that the above equivalence remains true, if the word "prime" is replaced by "semiprime". Let us also point out that there exist rings which are prime with respect to some nonzero derivation, although they are not even semiprime in the standard sense (see, for instance, [**4**, Example 3.5]).

The two results below play a very important role in the note. The first one is a part of the comprehensive characterization of δ-prime rings presented in [**4**]. The second one is taken from [**5**].

Theorem 1.1. *Let R be a nonzero ring and $\delta \in \mathrm{Der}(R)$. Then the following conditions are equivalent:*

(1) *R is δ-prime,*
(2) *for any elements $x, y \in R \setminus \{0\}$ there exists $k \in \mathbb{N}$ such that $xR\delta^k(y) \neq \{0\}$ (by $xR\delta^k(y)$ we simply mean $\{xz\delta^k(y) : z \in R\}$),*
(3) *$\mathrm{ann}^{\ell}_R(I) = \{0\}$ for any nonzero left δ-ideal I of the ring R,*
(4) *$\mathrm{ann}^{r}_R(J) = \{0\}$ for any nonzero right δ-ideal J of the ring R.*

Lemma 1.2. *Let $\delta, d \in \mathrm{Der}(R)$ and I be a nonzero left or right δ-ideal of the ring R. Suppose that R is δ-prime. Then*

(i) *d is the zero derivation whenever it vanishes on I,*
(ii) *R is commutative whenever so is I.*

For further information about noncommutative rings, ideals and derivations we refer to [**2**].

Various criteria for commutativity of associative rings have been attracting the attention of mathematicians since the 1950s. Pinter-Lucke's survey article [**7**] offers a valuable insight into the topic.

There are a lot of interesting commutativity criteria which involve derivations. A nice result of this type can be found in [**3**] (Lemma 7). It says that a δ-prime ring is commutative whenever the derivation δ is nonzero and commuting on some nonzero two-sided δ-ideal. However, regarding the proof, the authors give only a one-sentence comment. Our first goal is therefore to provide an elementary and self-contained proof of the Hirano-Tominaga criterion.

A classical theorem of Herstein states, among other things, that a 2-torsion free prime ring admitting a nonzero derivation with commutative range is itself commutative (see [**1**, Theorem 2]). In [**5**] the first named author of the present note generalized this result to δ-prime rings. He also observed that a δ-prime ring (not necessarily 2-torsion free) with commutative range of δ is itself commutative whenever δ^3 is a nonzero mapping. Our second goal here is to enhance the generalized Herstein theorem and to give a new proof of it.

Some of the facts and ideas discussed in the note come from the first named author's doctoral thesis.

The remainder of the note is organized as follows. In Section 2 we prove the Hirano-Tominaga criterion. In Section 3 we present an extension of Herstein's result about the subring generated by the range of a derivation. Finally, in Section 4 we discuss some further generalizations of the Herstein commutativity criterion and provide a few additional remarks related to Section 3.

2. Hirano–Tominaga criterion

We start with two purely technical lemmas.

Lemma 2.1. *Let $\delta \in \mathrm{Der}(R)$ and $E \subseteq R$ be a δ-invariant set. Suppose that δ is commuting on E. Then δ^m is commuting on E for any $m \in \mathbb{N}$.*

Proof. We will proceed by induction on m. The assertion is obvious whenever $m \leqslant 1$. Now assume that the mappings δ^k and δ^{k-1} are commuting on E for some integer $k \geqslant 1$. Let $a \in E$. To complete the proof we need to show that $[\delta^{k+1}(a), a] = 0$. Since E is a δ-invariant set, we have $\delta(a) \in E$. Lemma 1.1 (ii) and the induction hypothesis therefore yield

$$\begin{aligned} 0 &= \delta([\delta^k(a), a]) = [\delta^{k+1}(a), a] + [\delta^k(a), \delta(a)] \\ &= [\delta^{k+1}(a), a] + [\delta^{k-1}(\delta(a)), \delta(a)] = [\delta^{k+1}(a), a]. \end{aligned}$$ □

The next lemma is a bit more complicated. To prove the lemma we will use induction again.

Lemma 2.2. *Let $\delta \in \mathrm{Der}(R)$ and S be a δ-invariant subring of the ring R. Suppose that δ is commuting on S. Then $[a, b\delta^n(a)] = 0$ for any $a, b \in S$ and any $n \in \mathbb{N} \smallsetminus \{0\}$.*

Proof. Let $x, y \in S$ and $m \in \mathbb{N}$. Since $x + y \in S$, it follows from Lemma 2.1 that

$$\begin{aligned} 0 = [\delta^m(x+y), x+y] &= [\delta^m(x), x] + [\delta^m(x), y] + [\delta^m(y), x] + [\delta^m(y), y] \\ &= [\delta^m(x), y] + [\delta^m(y), x] = [\delta^m(x), y] - [x, \delta^m(y)]. \end{aligned}$$

We have therefore shown that

$$\forall\, x, y \in S\, \forall\, m \in \mathbb{N}\ ([\delta^m(x), y] = [x, \delta^m(y)]). \tag{2.1}$$

Let $a, b \in S$. Then we have $ba \in S$. Notice also that $\delta(a)a = a\delta(a)$ (because δ is commuting on S). Using Lemma 1.1 (i) and formula (2.1), we thus obtain

$$\begin{aligned} [a, \delta(b)a] &= [a, \delta(b)]a + \delta(b)[a, a] = [\delta(a), b]a = \delta(a)ba - b\delta(a)a \\ &= \delta(a)ba - ba\delta(a) = [\delta(a), ba] = [a, \delta(ba)] = [a, \delta(b)a] + [a, b\delta(a)]. \end{aligned}$$

Consequently, $[a, b\delta(a)] = 0$. This means that the assertion of the lemma holds true if $n = 1$. Now pick some $k \in \mathbb{N} \smallsetminus \{0, 1\}$ and assume that

$$\forall\, x, y \in S\, \forall\, \ell \in \mathbb{N} \smallsetminus \{0\}\ (\ell < k \ \Rightarrow\ [x, y\delta^\ell(x)] = 0).$$

Let again $a, b \in S$. To complete the proof we must show that $[a, b\delta^k(a)] = 0$. Since S is a δ-invariant set, we have $\delta^{k-\ell}(b) \in S$ for any integer ℓ satisfying $\ell \leqslant k$. Lemma 2.1 yields $\delta^k(a)a = a\delta^k(a)$. So, making use of Lemma 1.1, formula (2.1) and the induction hypothesis we get

$$\begin{aligned} [a, \delta^k(b)a] &= [a, \delta^k(b)]a + \delta^k(b)[a, a] = [\delta^k(a), b]a = \delta^k(a)ba - b\delta^k(a)a \\ &= \delta^k(a)ba - ba\delta^k(a) = [\delta^k(a), ba] = [a, \delta^k(ba)] = \left[a, \sum_{\ell=0}^{k} \binom{k}{\ell} \delta^{k-\ell}(b)\delta^\ell(a)\right] \\ &= [a, \delta^k(b)a] + [a, b\delta^k(a)] + \sum_{\ell=1}^{k-1} \binom{k}{\ell} [a, \delta^{k-\ell}(b)\delta^\ell(a)] = [a, \delta^k(b)a] + [a, b\delta^k(a)]. \end{aligned}$$

Therefore $[a, b\delta^k(a)] = 0$. □

If $\delta \in \operatorname{Der}(R)$ and S is a δ-invariant subring of the ring R, then the restriction $\delta|_S : S \to S$ is a derivation of S. This obvious remark enables us to recall a useful fact concerning δ-prime rings. Notice that the fact generalizes one of the basic properties of standard prime rings.

Proposition 2.1. *Let $\delta \in \operatorname{Der}(R)$ and I be a nonzero two-sided δ-ideal of the ring R. Suppose that R is δ-prime. Then as a ring, I is $\tilde{\delta}$-prime, where $\tilde{\delta} = \delta|_I$.*

Proof. Let $x \in I$ and $y \in I \setminus \{0\}$ be elements with the property that

$$\forall\, k \in \mathbb{N}\ (xI\delta^k(y) = \{0\}). \tag{2.2}$$

In virtue of Theorem 1.1 and the definition of $\tilde{\delta}$, we will have completed the proof, if we show that $x = 0$. Let J be the totality of sums of the form $\sum_{j=1}^{n} a_j \delta^{m_j}(y)$, where $n \in \mathbb{N} \setminus \{0\}$, $a_1, \ldots, a_n \in I$ and $m_1, \ldots, m_n \in \mathbb{N}$. Since I is a left δ-ideal of the ring R, so is J. Combining the δ-primeness of R with condition (4) in Theorem 1.1 and the fact that I is a nonzero right δ-ideal of R, we next obtain $Iz \neq \{0\}$ for any $z \in R \setminus \{0\}$. In particular $I\delta^0(y) = Iy \neq \{0\}$, and hence J is a nonzero left δ-ideal of the ring R. But from property (2.2) it follows that $x \in \operatorname{ann}_R^{\ell}(J)$. Condition (3) in Theorem 1.1 therefore yields $x = 0$. □

With the above three results in hand we are ready to recall and prove the Hirano-Tominaga criterion.

Theorem 2.1. *Let $\delta \in \operatorname{Der}(R)$ be a nonzero derivation. Suppose that the ring R is δ-prime and that δ is commuting on some nonzero two-sided δ-ideal I of R. Then the ring R is commutative.*

Proof. Let $a, b \in I$ and $n \in \mathbb{N} \setminus \{0\}$. Lemmas 2.1 and 2.2 guarantee that the elements $\delta^n(a)$ and $b\delta^n(a)$ commute with a. We therefore get

$$[a, b]\delta^n(a) = ab\delta^n(a) - ba\delta^n(a) = b\delta^n(a)a - b\delta^n(a)a = 0.$$

In other words,

$$\forall\, a, b \in I\, \forall\, n \in \mathbb{N} \setminus \{0\}\ ([a, b]\delta^n(a) = 0). \tag{2.3}$$

Let $x, y, z \in I$ and $n \in \mathbb{N} \setminus \{0\}$. Since $yz \in I$, formula (2.3) and Lemma 1.1 (i) yield

$$0 = [x, yz]\delta^n(x) = [x, y]z\delta^n(x) + y[x, z]\delta^n(x) = [x, y]z\delta^n(x).$$

So we have proved that

$$\forall\, x, y \in I\, \forall\, k \in \mathbb{N}\ ([x, y]I\delta^k(\delta(x)) = \{0\}). \tag{2.4}$$

Consider now the set $I_0 = \{x \in I : \delta(x) = 0\}$. The facts that the ring R is δ-prime, δ is a nonzero derivation and I is a nonzero δ-ideal, together with Lemma 1.2 (i), yield $I_0 \neq I$. Let $x \in I \setminus I_0$. It follows from Proposition 2.1 that I (regarded as a ring) is $\tilde{\delta}$-prime, where $\tilde{\delta} = \delta|_I$. Combining the $\tilde{\delta}$-primeness of I and formula (2.4) with Theorem 1.1, we get $[x, y] = 0$ for any $y \in I$. So in other words $x \in \mathrm{Z}(I)$. We have therefore shown that $I = I_0 \cup \mathrm{Z}(I)$. But I_0 and $\mathrm{Z}(I)$ are additive subgroups of I. Hence the equality $I = I_0 \cup \mathrm{Z}(I)$ and the fact that $I_0 \neq I$ together imply $I = \mathrm{Z}(I)$. This means that I is a nonzero commutative δ-ideal of the ring R. The commutativity of R now follows from Lemma 1.2 (ii). □

3. A structure of the subring generated by the range of a derivation

The following result extends Theorem 1 of Herstein's paper [**1**]. It should be mentioned that [**5**] offers a slight modification of the theorem.

THEOREM 3.1. *Let $\delta \in \mathrm{Der}(R)$ and S denote the subring of R generated by $\delta(R)$. Then*

(i) *S contains a nonzero left δ-ideal and a nonzero right δ-ideal of the ring R whenever δ^2 is a nonzero mapping,*

(ii) *S contains a nonzero two-sided δ-ideal of the ring R whenever δ^3 is a nonzero mapping.*

PROOF. Suppose that δ^2 is a nonzero mapping. Let I and J be, respectively, the left and right δ-ideals of the ring R generated by $\delta^2(R)$. Then $I \neq \{0\}$ and $J \neq \{0\}$. Observe that I is the same thing as the totality of finite sums whose terms belong to $\delta^2(R) \cup \{a\delta^2(x) : a, x \in R\}$. Analogously, J is the same thing as the totality of finite sums whose terms belong to $\delta^2(R) \cup \{\delta^2(x)a : a, x \in R\}$. Since $\delta^2(R) = \delta(\delta(R)) \subseteq S$, to complete the proof of assertion (i), we must show that for any $a, x \in R$ the products $a\delta^2(x)$ and $\delta^2(x)a$ lie in S. But if $a, x \in R$, then

$$\delta(a\delta(x)) = \delta(a)\delta(x) + a\delta^2(x),$$
$$\delta(a\delta(x)) \in S, \quad \delta(a)\delta(x) \in S$$

which immediately yields $a\delta^2(x) \in S$. An analogous argument shows that $\delta^2(x)a \in S$ for any $a, x \in R$.

Assertion (ii) is proved in [**5**] (see Proposition 1 therein). □

As an immediate corollary to the above theorem we obtain a quite interesting remark on derivations of the division rings and, more generally, simple rings. Before stating the remark, notice that a division ring has no nontrivial one-sided ideals.

COROLLARY 3.1. *Let $\delta \in \mathrm{Der}(R)$. Suppose that one of the following conditions is satisfied:*

(a) *R is a division ring and δ^2 is a nonzero mapping,*

(b) *R is a simple ring and δ^3 is a nonzero mapping.*

Then R is generated (as a ring) by $\delta(R)$.

Let us now introduce some additional notation. Given an integer $n \geqslant 1$ and a ring R, we will denote the full ring of $n \times n$ matrices over R and the ring of all strictly upper triangular $n \times n$ matrices over R by $\mathbb{M}_n(R)$ and $\mathbb{T}_n^0(R)$, respectively. The zero matrix belonging to $\mathbb{M}_n(R)$ will be denoted by $O_{n\times n}$.

It is worth pointing out that Theorem 3.1 provides only sufficient conditions for the existence of nonzero δ-ideals contained in the subring generated by the range of a derivation δ.

EXAMPLE 3.1. Let R be a nonzero ring with identity. Recall that

$$\mathrm{Z}(\mathbb{T}_3^0(R)) = \left\{ \begin{bmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : x \in R \right\},$$

and consider an arbitrary noncentral element P of the ring $\mathbb{T}_3^0(R)$. Since $AB \in \mathrm{Z}(\mathbb{T}_3^0(R))$ for all $A, B \in \mathbb{T}_3^0(R)$, the range of the inner derivation $\partial_P : \mathbb{T}_3^0(R) \to \mathbb{T}_3^0(R)$ is a nonzero additive subgroup of $\mathrm{Z}(\mathbb{T}_3^0(R))$. But if $A \in \mathbb{T}_3^0(R)$ and $C \in \mathrm{Z}(\mathbb{T}_3^0(R))$, then $AC = O_{3\times 3}$. It follows that each additive subgroup of $\mathrm{Z}(\mathbb{T}_3^0(R))$ is a two-sided ideal of $\mathbb{T}_3^0(R)$. Consequently, the range of ∂_P is a nonzero two-sided ∂_P-ideal of the ring $\mathbb{T}_3^0(R)$ though ∂_P^2 is the zero mapping.

To conclude the section, we need an example showing that if the assumptions of Theorem 3.1 are not satisfied, the subring generated by the range of δ may not contain nonzero δ-ideals of the ring R. The example below has been developed on the basis of an example given in [**1**].

Example 3.2. Once again, let R be a nonzero ring with identity. Consider the derivation $\delta \in \mathrm{Der}(\mathbb{M}_2(R))$ defined by

$$\delta \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} r & s-p \\ 0 & -r \end{bmatrix}.$$

One can easily check that δ^3 is the zero mapping. Since

$$\forall\, p,q,r,s \in R \ \left(\delta^2 \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} 0 & -2r \\ 0 & 0 \end{bmatrix} \right),$$

the mapping δ^2 is in turn nonzero if and only if $\mathrm{char}(R)$, the characteristic of the ring R, is different from 2.

Let $\mathfrak{S}$ denote the subring of $\mathbb{M}_2(R)$ generated by $\delta(\mathbb{M}_2(R))$. It is evident that $\mathfrak{S}$ consists of upper triangular matrices. Observe also that

$$\mathfrak{S} = \delta(\mathbb{M}_2(R)) = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} : \, x, y \in R \right\}$$

whenever $\mathrm{char}(R) = 2$.

Finally, since for any $x, y, z \in R$ we have

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ x & y \end{bmatrix}, \quad \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} y & 0 \\ z & 0 \end{bmatrix}, \tag{3.1}$$

the set $\{A \in \mathbb{M}_2(R) : \, A \text{ is upper triangular}\}$, and hence the subring $\mathfrak{S}$, contains no nonzero two-sided ideal of $\mathbb{M}_2(R)$. Equalities (3.1) also yield that the set

$$\left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} : \, x, y \in R \right\}$$

contains no nonzero one-sided ideal of $\mathbb{M}_2(R)$. Therefore if $\mathrm{char}(R) = 2$, no nonzero one-sided ideal of the ring $\mathbb{M}_2(R)$ is contained in $\mathfrak{S}$.

It seems to be worth noticing that δ is in fact the inner derivation induced by the nilpotent matrix

$$Q = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

4. Further generalizations of the Herstein commutativity criterion

We are now ready to state and prove the announced enhancement of the generalized Herstein commutativity criterion.

THEOREM 4.1. *Let $\delta \in \mathrm{Der}(R)$. Suppose that the ring R is δ-prime and admits a two-sided δ-ideal I with the following properties:*

(a) $\delta^2(z) \neq 0$ *for some* $z \in I$,

(b) $\delta(x)\delta(y) = \delta(y)\delta(x)$ *for any* $x, y \in I$.

Then R is commutative.

PROOF. Recall first that $\tilde{\delta} = \delta|_I$ is a derivation of I. Let S be the subring of I generated by $\tilde{\delta}(I)$. Property (a) means that $\tilde{\delta}^2$ is a nonzero mapping. Hence in virtue of Theorem 3.1, the subring S contains a nonzero one-sided $\tilde{\delta}$-ideal of I. Let us denote this $\tilde{\delta}$-ideal by J. It follows from property (b) that the subring S is commutative. Therefore so is J. But Proposition 2.1 guarantees the $\tilde{\delta}$-primeness of I (as a ring). Since J is a nonzero commutative one-sided $\tilde{\delta}$-ideal of I, Lemma 1.2 (ii) yields that I is itself commutative. So, applying the lemma to I and the whole ring R completes the proof. □

In the rest of the note we will use a well known fact about rings of matrices (see, for instance, [**6**, Theorem 4.28]; the proof given there can be easily adapted to nonunital rings and to the semiprime case).

PROPOSITION 4.1. *Let R be an arbitrary ring and $n \in \mathbb{N} \setminus \{0\}$. Then the following conditions are equivalent:*

(1) *R is prime,*

(2) *the full matrix ring $\mathbb{M}_n(R)$ is prime.*

Moreover, the equivalence remains true if both occurrences of the word "prime" are replaced by "semiprime".

This fact enables us to give an example showing that property (a) in Theorem 4.1 generally cannot be weakened to "$I \neq \{0\}$ and δ is a nonzero derivation" (cf. Theorem 2.1 and Lemma 1.2 (i)).

EXAMPLE 4.1. Let R be an integral domain of characteristic 2. Then by Proposition 4.1 the ring $\mathbb{M}_2(R)$ is prime, and hence d-prime for any $d \in \mathrm{Der}(\mathbb{M}_2(R))$. Consider now the derivation δ defined in Example 3.2. It is easy to see that the elements of the set

$$\delta(\mathbb{M}_2(R)) = \left\{ \begin{bmatrix} x & y \\ 0 & x \end{bmatrix} : x, y \in R \right\}$$

pairwise commute. In other words, $\delta(A)\delta(B) = \delta(B)\delta(A)$ for any $A, B \in \mathbb{M}_2(R)$. But the ring $\mathbb{M}_2(R)$ is noncommutative.

It turns out that property (a) can be weakened in the case where the ring is 2-torsion free. This is a consequence of the following result due to Hirano and Tominaga (see [**3**, Lemma 5]).

LEMMA 4.1. *Let $\delta \in \operatorname{Der}(R)$ be a nonzero derivation and let I be a nonzero two-sided δ-ideal of the ring R. Suppose that R is δ-prime and 2-torsion free. Then $\delta^2(z) \neq 0$ for some $z \in I$.*

We therefore obtain a bit different enhancement of the generalized Herstein criterion.

COROLLARY 4.1. *Let $\delta \in \operatorname{Der}(R)$ be a nonzero derivation. Suppose that the ring R is δ-prime and 2-torsion free. Moreover, suppose that it admits a nonzero two-sided δ-ideal I satisfying the condition*

$$\forall\, x, y \in I\ (\delta(x)\delta(y) = \delta(y)\delta(x)).$$

Then R is commutative.

Now we will go back to subrings generated by ranges of derivations. Before stating some additional remarks, let us discuss a property of δ-semiprime rings. The lemma below, as well as its proof, was sketched by the anonymous referee of the earlier version of the note.

LEMMA 4.2. *Let $\delta \in \operatorname{Der}(R)$. Suppose that the ring R is δ-semiprime and 2-torsion free. Then δ^2 is a nonzero mapping whenever so is δ.*

PROOF. Assume that δ^2 is the zero mapping. Hence for all $a, b \in R$ we have

$$0 = \delta^2(ab) = \delta^2(a)b + 2\delta(a)\delta(b) + a\delta^2(b) = 2\delta(a)\delta(b).$$

Combining the above equalities with the fact that R is 2-torsion free gives

$$\forall\, a, b \in R\ (\delta(a)\delta(b) = 0). \tag{4.1}$$

Let J be the right δ-ideal of the ring R generated by $\delta(R)$. Evidently,

$$J = \left\{\delta(a) + \sum_{i=1}^{n} \delta(b_i)c_i : n \in \mathbb{N} \setminus \{0\} \text{ and } a, b_1, \dots, b_n, c_1, \dots, c_n \in R\right\}.$$

Consider next any $n \in \mathbb{N} \setminus \{0\}$ and any $a, b_1, \dots, b_n, c_1, \dots, c_n \in R$. It follows from the Leibniz rule that

$$\delta(a) + \sum_{i=1}^{n} \delta(b_i)c_i = \delta(a) + \sum_{i=1}^{n} (\delta(b_ic_i) - b_i\delta(c_i)) = \delta(p) - \sum_{i=1}^{n} b_i\delta(c_i),$$

where $p = a + \sum_{i=1}^{n} b_ic_i$. So if $m \in \mathbb{N} \setminus \{0\}$ and $x, y_1, \dots, y_m, z_1, \dots, z_m \in R$, then using property (4.1) we get

$$\begin{aligned}\left(\delta(a) + \sum_{i=1}^{n} \delta(b_i)c_i\right)\left(\delta(x) + \sum_{j=1}^{m} \delta(y_j)z_j\right) &= \left(\delta(p) - \sum_{i=1}^{n} b_i\delta(c_i)\right)\left(\delta(x) + \sum_{j=1}^{m} \delta(y_j)z_j\right)\\ &= \delta(p)\delta(x) + \sum_{j=1}^{m} \delta(p)\delta(y_j)z_j - \sum_{i=1}^{n} b_i\delta(c_i)\delta(x) - \sum_{i=1}^{n}\sum_{j=1}^{m} b_i\delta(c_i)\delta(y_j)z_j = 0.\end{aligned}$$

But this means that $J^2 = \{0\}$. Consequently, the δ-semiprimeness of the ring R yields $J = \{0\}$. We therefore have $\delta(R) = \{0\}$. In other words, δ is the zero derivation. □

Lemma 4.2 and Theorem 3.1 together imply the following useful fact.

COROLLARY 4.2. *Let $\delta \in \mathrm{Der}(R)$ be a nonzero derivation and let S denote the subring of R generated by $\delta(R)$. Suppose that the ring R is δ-semiprime and 2-torsion free. Then S contains a nonzero left δ-ideal and a nonzero right δ-ideal of R.*

It is worth noticing here that Lemma 1.2 (i) does not work in δ-semiprime rings (even if R is 2-torsion free and I is a two-sided δ-ideal). Let us recall a standard example.

EXAMPLE 4.2. Consider $\mathbb{Z}_{15}$, the ring of integers modulo 15, and some $n \in \mathbb{N} \setminus \{0, 1\}$. Since $\mathbb{Z}_{15}$ is semiprime, Proposition 4.1 yields that so is the matrix ring $\mathbb{M}_n(\mathbb{Z}_{15})$. Consequently, $\mathbb{M}_n(\mathbb{Z}_{15})$ is δ-semiprime for any derivation $\delta \in \mathrm{Der}(\mathbb{M}_n(\mathbb{Z}_{15}))$. Notice also that the ring $\mathbb{M}_n(\mathbb{Z}_{15})$ is 2-torsion free. Let $\mathfrak{I}$ be the set of all $n \times n$ matrices whose elements belong to $\{0, 5, 10\}$. One can easily check that $\mathfrak{I}$ is a two-sided ideal of $\mathbb{M}_n(\mathbb{Z}_{15})$. However, if all elements of a nonscalar matrix $T \in \mathbb{M}_n(\mathbb{Z}_{15})$ are divisible by 3, then $\partial_T : \mathbb{M}_n(\mathbb{Z}_{15}) \to \mathbb{M}_n(\mathbb{Z}_{15})$ is a nonzero derivation vanishing on $\mathfrak{I}$.

We conclude the section with a remark about division rings. Notice that every division ring is prime and that a prime ring R with identity is 2-torsion free if and only if $\mathrm{char}(R) \neq 2$. Recall also that an element of a ring is noncentral if and only if the inner derivation induced by this element is nonzero. Hence combining Lemma 4.1 or Lemma 4.2 with Corollary 3.1 gives the fact stated below.

COROLLARY 4.3. *Let R be a division ring of characteristic different from 2. Moreover, let $\delta \in \mathrm{Der}(R)$ and $a \in R$. Then*

(i) *R is generated (as a ring) by $\delta(R)$ if and only if δ is a nonzero derivation,*
(ii) *R is generated (as a ring) by $\{[a, x] : x \in R\}$ if and only if a is a noncentral element.*

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(Received 05 12 2020)
(Revised 11 10 2022)