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wt_0-DISTANCE AND BEST PROXIMITY POINTS INVOLVING b-SIMULATION FUNCTIONS

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Abstract. We define wt_0-distance which is a special type of wt-distance and obtain some best proximity point theorems involving b-simulation functions. Our results are significant, since we replace simulation function with b-simulation function, metric space with b-metric space, and w_0-distance and wt-distance with wt_0-distance. We also provide some examples to support our results.

1. Introduction and preliminaries

Fixed point theory is an important and useful tool for different branches of mathematical analysis and it has a wide range of applications in applied mathematics and sciences. Also, it may be discussed as an essential subject of nonlinear analysis. Since the first results of Banach in 1922, various authors have been studying fixed points, and, in recent years, best proximity points of mappings in metric spaces. Their discoveries are still being generalized in many directions such that there has been a number of generalizations of the usual notion of a metric space. One generalization is a w-distance introduced by Kada et al. [**8**] (also, see [**7, 16**] and references therein).

Definition 1.1. Let (X, d) be a metric space. A function $\rho : X \times X \to [0, +\infty)$ is called a w-distance on X if the following properties are satisfied:

(w_1) $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$;
(w_2) ρ is lower semi-continuous in its second variable;
(w_3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that
$\rho(z, x) \leqslant \delta$ and $\rho(z, y) \leqslant \delta$ imply $d(x, y) \leqslant \varepsilon$.

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Very recently, Kostić et al. [**11**] introduced the concept of w_0-distance, which is slightly different to the original w-distance, in regard that the lower semicontinuity with respect to both variables (when one of them is fixed) is supposed.

Definition 1.2. Let (X, d) be a metric space. Then a function $p : X \times X \to [0, \infty)$ is called a w_0-distance on X if the following are satisfied:

(p_1) $p(x, z) \leqslant p(x, y) + p(y, z)$, for any $x, y, z \in X$,
(p_2) for any $x \in X$, functions $p(x, \cdot), p(\cdot, x) : X \to [0, \infty)$ are lower semicontinuous,
(p_3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(z, x) \leqslant \delta$ and $p(z, y) \leqslant \delta$ imply $d(x, y) \leqslant \epsilon$.

Note that the notion of w_0-distance is more general than the standard notion of metric, but less general than the w-distance, as illustrated by the following examples.

Example 1.1. [**11**] Let (X, d) be a metric space. A mapping $p : X \times X \to [0, \infty)$ defined by $p(x, y) = k > 0$ for all $x, y \in X$ is a w_0-distance on X (see [**8**, Example 2]). The mapping p is not a metric, since $p(x, x) \neq 0$ for any $x \in X$.

Example 1.2. [**11**] Let $X = [0, \infty)$ be endowed with the standard metric d. Let $p : X \times X \to \mathbb{R}$ be defined as $p(x, y) = c \in (0, 1)$ for all $x, y \in X$ and let $\alpha : X \to [0, \infty)$ be defined by

$$\alpha(x) = \begin{cases} e^{-x}, & x > 0 \\ 2, & x = 0 \end{cases}$$

A function $q : X \times X \to [0, \infty)$ defined by $q(x, y) = \max\{\alpha(x), c\}$ for all $x, y \in X$ is then a w-distance on X (see Example 1 and [**8**, Lemma 3]). However, q is not a w_0-distance on X, since for any sequence $\{x_n\} \subset (0, \infty)$ such that $x_n \to 0$ we have

$$\liminf_{n\to\infty} q(x_n, y) = \liminf_{n\to\infty} \max\{e^{-x_n}, c\} = 1 < q(0, y) = \max\{\alpha(0), c\} = 2.$$

Another such generalization is a b-metric space defined by Bakhtin [**2**] and Czerwik [**4**].

Definition 1.3. Let X be a nonempty set and $b \geqslant 1$ be a real number. Suppose that the mapping $d : X \times X \to [0, \infty)$ satisfies

(d_1) $d(x, y) = 0$ if and only if $x = y$;
(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(d_3) $d(x, z) \leqslant b[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

Then d is called a b-metric and (X, d) is called a b-metric space (or metric type space).

Obviously, for $b = 1$, a b-metric space is a metric space. Also, for notions such as convergent and Cauchy sequences, completeness, continuity and etc. in b-metric spaces, we refer to [**3**, **9**].

In 2014, Hussain et al. [**6**] defined a wt-distance on b-metric spaces as an extension of w-distance on metric spaces and proved some fixed point theorems under wt-distance in a partially ordered b-metric space.

DEFINITION 1.4. Let (X, d) be a b-metric space and $b \geqslant 1$ be a given real number. A function $\rho : X \times X \to [0, +\infty)$ is called a wt-distance on X if the following properties are satisfied:

(wt_1) $\rho(x, z) \leqslant b[\rho(x, y) + \rho(y, z)]$ for all $x, y, z \in X$;
(wt_2) ρ is b-lower semi-continuous in its second variable
i.e., if $x \in X$ and $y_n \to y$ in X then $\rho(x, y) \leqslant b \liminf_n \rho(x, y_n)$;
(wt_3) for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\rho(z, x) \leqslant \delta$ and $\rho(z, y) \leqslant \delta$ imply $d(x, y) \leqslant \varepsilon$.

Let us recall that a real-valued function f defined on a b-metric space X is said to be lower b-semicontinuous at a point x_0 in X if either $\liminf_{x_n \to x_0} f(x_n) = \infty$ or $f(x_0) \leqslant \liminf_{x_n \to x_0} bf(x_n)$, whenever $x_n \in X$ and $x_n \to x_0$. Obviously, for $b = 1$, every wt-distance is a w-distance. But, a w-distance is not necessary a wt-distance. Thus, each wt-distance is a generalization of w-distance.

LEMMA 1.1. [**6**] *Let (X, d) be a b-metric space with parameter $b \geqslant 1$ and let ρ be a wt-distance on X. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X, let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $[0, +\infty)$ converging to 0, and let x, y, $z \in X$. Then the following hold:*

(i) *If $\rho(x_n, y) \leqslant \alpha_n$ and $\rho(x_n, z) \leqslant \beta_n$ for any $n \in \mathbb{N}$, then $y = z$. In particular, if $\rho(x, y) = 0$ and $\rho(x, z) = 0$, then $y = z$;*
(ii) *if $\rho(x_n, y_n) \leqslant \alpha_n$ and $\rho(x_n, z) \leqslant \beta_n$ for any $n \in \mathbb{N}$, then y_n converges to z;*
(iii) *if $\rho(x_n, x_m) \leqslant \alpha_n$ for any $n, m \in \mathbb{N}$ with $m > n$, then $\{x_n\}$ is a Cauchy sequence;*
(iv) *if $\rho(y, x_n) \leqslant \alpha_n$ for any $n \in \mathbb{N}$, then $\{x_n\}$ is a Cauchy sequence.*

On the other hand, the notion of simulation function has been introduced and studied by Khojasteh et al. [**10**].

DEFINITION 1.5. Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a mapping. Then ζ is called a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(t, s) < s - t$ for all $t, s > 0$;
(ζ_2) if $\{t_n\}$ and $\{s_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

The set of all simulation functions will be denoted by $\mathcal{Z}$.

REMARK 1.1. Originally, simulation function was defined by Khojasteh et al. [**10**] as a mapping $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying $\zeta(0, 0) = 0$ alongside the conditions (ζ_1) and (ζ_2). In this paper, a modified definition of Argoubi et al. [**1**] is used.

Next, we give some examples of simulation functions.

Recently, Demma et al. [**5**] and Mongkolkeha et al. [**12**] introduced the b-simulation function in the framework of b-metric spaces as follows.

DEFINITION 1.6. A b-simulation function is a function $\xi : [0, +\infty)^2 \to \mathbb{R}$ satisfying the following:

(ξ_1) $\xi(bt, s) < s - bt$ for all $t, s > 0$;
(ξ_2) if $\{t_n\}, \{s_n\}$ are sequences in $(0, +\infty)$ such that

$$0 < \lim_{n\to+\infty} t_n \leqslant \varliminf_{n\to+\infty} s_n \leqslant \varlimsup_{n\to\infty} s_n \leqslant b \lim_{n\to+\infty} t_n < +\infty,$$

then $\varlimsup_{n\to\infty} \xi(bt_n, s_n) < 0$.

It is clear that if $b = 1$, then b-simulation function is in fact the simulation function in the framework of (standard) metric space.

Example 1.3. [**5**] Let $\xi : [0, +\infty)^2 \to \mathbb{R}$ be defined by

(i) $\xi(t, s) = \lambda s - t$ for all $t, s \in [0, +\infty)$, where $\lambda \in [0, 1)$.
(ii) $\xi(t, s) = \psi(s) - \varphi(t)$ for all $t, s \in [0, +\infty)$, where $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ are two continuous functions such that $\psi(t) = \varphi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leqslant \varphi(t)$ for all $t > 0$.
(iii) $\xi(t, s) = s - \frac{f(t,s)}{g(t,s)}t$ for all $t, s \in [0, +\infty)$, where $f, g : [0, +\infty)^2 \to (0, +\infty)$ are two continuous functions with respect to each variable such that $f(t, s) > g(t, s)$ for all $t, s > 0$.
(iv) $\xi(t, s) = s - \varphi(s) - t$ for all $t, s \in [0, +\infty)$, where $\varphi : [0, +\infty) \to [0, +\infty)$ is a lower semi-continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
(v) $\xi(t, s) = s\varphi(s) - t$ for all $t, s \in [0, +\infty)$, where $\varphi : [0, +\infty) \to [0, 1)$ is such that $\lim_{t\to r^+} \varphi(t) < 1$ for all $r > 0$.

Each of the functions considered in (i)–(v) is a b-simulation function.

Recently, simulation functions and b-simulation functions have been used to study the fixed point and best proximity points in metric spaces and b-metric spaces (see [**12**–**15**, **17**, **18**]).

In this paper, we introduce a special type of wt-distance, which is called the wt_0-distance. Then we extend best proximity results of Tchier et al. [**18**] and Kostić et al. [**11**] involving b-simulation functions instead of simulation function via considering wt_0-distance instead of metric space and w_0-distance.

Let (X, d) be a b-metric space, A and B two nonempty subsets of X and $T : A \to B$ a non-self mapping. In the sequel we will use the following notations

$$\begin{aligned} d(A, B) &= \inf\{d(x, y) : x \in A, y \in B\} \\ d(y, A) &= \inf\{d(x, y) : x \in A\} = d(\{y\}, A) \\ A_0 &= \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\} \\ B_0 &= \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\} \end{aligned}$$

Also, the set of all best proximity points of a non-self mapping $T : A \to B$ will be denoted by

$$B_{\text{est}}(T) = \{x \in A : d(x, Tx) = d(A, B)\}.$$

If moreover $g : A \to A$, then we have

$$B^g_{\text{est}}(T) = \{x \in A : d(gx, Tx) = d(A, B)\}.$$

2. Main results

Here we define the concept of wt_0-distance, which is slightly different from the original wt-distance of [**6**], in regard that the lower b-semicontinuity with respect to both variables (when one of them is fixed) is supposed.

DEFINITION 2.1. Let (X, d) be a b-metric space with parameter $b \geqslant 1$. Then a function $P : X \times X \to [0, \infty)$ is called a wt_0-distance on X if the following are satisfied:

(P_1) $P(x, z) \leqslant b[P(x, y) + P(y, z)]$, for any $x, y, z \in X$,
(P_2) for any $x \in X$, functions $P(x, \cdot), P(\cdot, x) : X \to [0, \infty)$ are lower b-semicontinuous,
(P_3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $P(z, x) \leqslant \delta$ and $P(z, y) \leqslant \delta$ imply $d(x, y) \leqslant \epsilon$.

Note that the notion of wt_0-distance is more general than the standard notion of b-metric, but less general than the wt-distance, as illustrated by the following examples. Also, w_0-distance is a wt_0-distance with $b = 1$; but the converse does not hold. Thus, the wt_0-distance is a generalization of w_0-distance.

EXAMPLE 2.1. Let (X, d) be a b-metric space with $b = 2$, $X = \mathbb{R}$ and $d(x, y) = |x - y|^2$ for all $x, y \in \mathbb{R}$ and define wt_0-distance by $P(x, y) = |x|^2 + |y|^2$ for all $x, y \in \mathbb{R}$ (see [**6**]). The mapping P is not a metric, since $P(x, y) = 0$ only for $x = y = 0$ (and not true for all $x, y \in \mathbb{R}$).

EXAMPLE 2.2. Let (X, d) and P be as in the previous example. Let the function $\alpha : X \to [0, \infty)$ be defined as

$$\alpha(x) = \begin{cases} 1, & x \neq 0 \\ c, & x = 0 \end{cases}$$

where $c > 2$. Then it can be proved that the function $Q : X \times X \to [0, \infty)$ defined as $Q(x, y) = \max\{P(x, y), \alpha(x)\}$ for all $x, y \in X$ is also a wt-distance on X.

However, Q is not a wt_0-distance on X. Indeed, let $\{x_n\}$ be a sequence in X such that $x_n \neq 0$ for every $n \in \mathbb{N}$ and $x_n \to 0$ when $n \to \infty$. Then we have

$$Q(0, 0) = \max\{P(0, 0), \alpha(0)\} = c > 2 = 2 \liminf_{x_n \to 0} Q(x_n, 0)$$

which means that $Q(\cdot, 0)$ is not a lower 2-semicontinuous function.

Now, we introduce the notions of $\mathcal{Z}$-P-proximal contractions and extend the best proximity point results of Kostić et al. [**11**] and Tchier et al. [**18**] to b-metric spaces with a wt_0-distance.

Let (X, d) be a b-metric space, $P : X \times X \to [0, \infty)$ be a wt_0-distance on X, and let A and B be two nonempty subsets of X (which need not be equal). Also, for every $x, y \in X$, let $\nu(x, y) := \max\{P(x, y), P(y, x)\}$. It is easily checked that the function $\nu : X \times X \to [0, \infty)$ has the following properties (for all $x, y, z \in X$):

(1) $\nu(x, y) = 0 \Rightarrow x = y$; (2) $\nu(x, y) = \nu(y, x)$; (3) $\nu(x, y) \leqslant b[\nu(x, z) + \nu(z, y)]$.

Definition 2.2. A non-self-mapping $T : A \to B$ is said to be a $\mathcal{Z}$-P-proximal contraction of the first kind if there exists a b-simulation function $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ such that

$$\left.\begin{aligned} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{aligned}\right\} \Rightarrow \xi(b\nu(u,v), \nu(x,y)) \geqslant 0$$

for every $u, v, x, y \in A$.

Definition 2.3. A non-self mapping $T : A \to B$ is said to be a $\mathcal{Z}$-P-proximal contraction of the second kind if

$$\left.\begin{aligned} d(u,Tx) = d(A,B) \\ d(v,Ty) = d(A,B) \end{aligned}\right\} \Rightarrow \xi(b\nu(Tu,Tv), \nu(Tx,Ty)) \geqslant 0$$

for all $u, v, x, y \in A$, where $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is a b-simulation function.

Remark 2.1. In the case $P = d$ and $b = 1$ (i.e., when d is a standard metric), the notions of $\mathcal{Z}$-P-proximal contractions are reduced to $\mathcal{Z}$-proximal contractions of Tchier et al. [**18**]. We will apply the same terminology if $P = d$ and $b > 1$.

In Definition 2.2, if the b-simulation function ξ is given by $\xi(t,s) = \alpha s - t$ for some $\alpha \in [0,1)$, the mapping T is called a *P-proximal contraction of the first kind.* Additionally, if $P = d$ and $b = 1$, T is a proximal contraction of the first kind.

We introduce the following notation:

$$\mathcal{G}_{A,P} = \{g : (A,d) \to (A,d) \text{ is continuous} : P(x,y) \leqslant P(gx,gy),\ \forall x,y \in A\}$$
$$\mathcal{T}_{g,P} = \{T : A \to B : P(Tx,Ty) \leqslant P(Tgx,Tgy),\ \forall x,y \in A\}.$$

In the case $P = d$ and $b = 1$, $\mathcal{G}_{A,P}$ is denoted by $\mathcal{G}_A$ and $\mathcal{T}_{g,P}$ by $\mathcal{T}_g$ (see [**18**]).

Now, we state and prove our main results.

Theorem 2.1. *Let A and B be two nonempty subsets of a complete b-metric space (X,d) with a wt_0-distance P, such that A_0 is nonempty and closed. Suppose that the mappings $g : A \to A$ and $T : A \to B$ satisfy the following conditions:*

a) *T is a $\mathcal{Z}$-P-proximal contraction of the first kind;*
b) *$g \in \mathcal{G}_{A,P}$;* c) *$A_0 \subseteq g(A_0)$;* d) *$T(A_0) \subseteq B_0$.*

Then there exists a unique element $x \in A_0$ such that $d(gx,Tx) = d(A,B)$ and $P(x,x) = 0$. Moreover, for any initial $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(gx_{n+1},Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$ there exists $x_1 \in A_0$ such that $d(gx_1,Tx_0) = d(A,B)$. Similarly, for $x_1 \in A_0$ there exists $x_2 \in A_0$ such that $d(gx_2,Tx_1) = d(A,B)$. Continuing this process, for any $x_n \in A_0$ we can find $x_{n+1} \in A_0$ such that $d(gx_{n+1},Tx_n) = d(A,B)$.

Now, if there exists $n_0 \in \mathbb{N}$ such that $\nu(x_{n_0},x_{n_0-1}) = 0$, then $x_{n_0-1} = x_{n_0}$. Thus, $d(gx_{n_0-1},Tx_{n_0-1}) = d(A,B)$, i.e. x_{n_0-1} is a best proximity point of T under mapping g and the proof is finalized. Hence, we assume that $\nu(x_n,x_{n-1}) > 0$ for all $n \in \mathbb{N}$. Then $\nu(gx_n, gx_{n-1}) > 0$ for all $n \in \mathbb{N}$ because $g \in \mathcal{G}_{A,P}$. Since T is a $\mathcal{Z}$-P-proximal contraction of the first kind and $g \in \mathcal{G}_{A,P}$, we have

$$\begin{aligned} 0 &\leqslant \xi(b\nu(gx_{n+1}, gx_n), \nu(x_n, x_{n-1})) \\ &< \nu(x_n, x_{n-1}) - b\nu(gx_{n+1}, gx_n) \\ &\leqslant \nu(x_n, x_{n-1}) - b\nu(x_{n+1}, x_n). \end{aligned} \tag{2.1}$$

Therefore, $\nu(x_{n+1}, x_n) \leqslant b\nu(x_{n+1}, x_n) < \nu(x_n, x_{n-1})$ for all $n \in \mathbb{N}$, which means that the sequence $\{\nu(x_n, x_{n-1})\}$ is decreasing. Hence, there exists $r \geqslant 0$ such that $\lim_{n\to\infty} \nu(x_n, x_{n-1}) = r \geqslant 0$. Suppose that $r > 0$. Also, by (2.1), we deduce that $\nu(gx_{n+1}, gx_n) \leqslant b\nu(gx_{n+1}, gx_n) < \nu(x_n, x_{n-1})$ for every $n \in \mathbb{N}$. On the other hand, $g \in \mathcal{G}_{A,P}$ and hence $\nu(x_{n+1}, x_n) \leqslant \nu(gx_{n+1}, gx_n) \leqslant \nu(x_n, x_{n-1})$ for all $n \in \mathbb{N}$. This implies that $\lim_{n\to\infty} \nu(gx_{n+1}, gx_n) = r$. Now, using the b-simulation function property (ξ_2), we obtain

$$0 \leqslant \limsup_{n\to\infty} \xi(b\nu(gx_{n+1}, gx_n), \nu(x_n, x_{n-1})) < 0,$$

which is a contradiction. Hence, we have $r = 0$ which implies that

$$\lim_{n\to\infty} \nu(x_n, x_{n-1}) = 0. \tag{2.2}$$

Now, let us prove that

$$\lim_{m,n\to\infty} \nu(x_n, x_m) = 0. \tag{2.3}$$

If (2.3) is not true, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}, \{n_k\} \subseteq \mathbb{N} \cup \{0\}$ with $m_k > n_k \geqslant k$ such that

$$\nu(x_{n_k}, x_{m_k}) \geqslant \varepsilon \tag{2.4}$$

for all $k \in \mathbb{N}$. We can assume that m_k is a minimal index for which (2.4) holds. Then we also have

$$\nu(x_{n_k}, x_{m_k-1}) < \varepsilon \tag{2.5}$$

for any $k \in \mathbb{N}$. Using the triangle inequality for ν, by (2.4) and (2.5) we get

$$\varepsilon \leqslant \nu(x_{n_k}, x_{m_k}) \leqslant b\nu(x_{n_k}, x_{m_k-1}) + b\nu(x_{m_k-1}, x_{m_k}) < b\varepsilon + b\nu(x_{m_k-1}, x_{m_k}).$$

Passing to the limit when $k \to \infty$. By (2.2), we conclude that

$$\varepsilon \leqslant \lim_{k\to\infty} \nu(x_{n_k}, x_{m_k}) < b\varepsilon. \tag{2.6}$$

Now, we claim that

$$\lim_{k\to\infty} \nu(x_{n_k+1}, x_{m_k+1}) < \varepsilon. \tag{2.7}$$

If $\lim_{k\to\infty} \nu(x_{n_k+1}, x_{m_k+1}) \geqslant \varepsilon$, then there exist sequence $\{k_s\}$ and $\delta > 0$ such that

$$\lim_{s\to\infty} \nu(x_{n_{k_s}+1}, x_{m_{k_s}+1}) = \delta \geqslant \varepsilon. \tag{2.8}$$

Again, T is a $\mathcal{Z}$-P-proximal contraction of the first kind and

$$d(gx_{n_{k_s}+1}, Tx_{n_{k_s}}) = d(A, B) = d(gx_{m_{k_s}+1}, Tx_{m_{k_s}}).$$

Hence, by the property (ξ_1), we obtain

$$\begin{aligned} 0 &\leqslant \xi(b\nu(gx_{n_{k_s}+1}, gx_{m_{k_s}+1}), \nu(x_{n_{k_s}}, x_{m_{k_s}})) \\ &< \nu(x_{n_{k_s}}, x_{m_{k_s}}) - b\nu(gx_{n_{k_s}+1}, gx_{m_{k_s}+1}) \\ &\leqslant \nu(x_{n_{k_s}}, x_{m_{k_s}}) - b\nu(x_{n_{k_s}+1}, x_{m_{k_s}+1}) \\ &\leqslant \nu(x_{n_{k_s}}, x_{m_{k_s}}) - \nu(x_{n_{k_s}+1}, x_{m_{k_s}+1}) \end{aligned} \tag{2.9}$$

for all $k \in \mathbb{N}$. It follows from (2.6), (2.8) and (2.9) that

$$b\delta = \lim_{s\to\infty} b\nu(x_{n_{k_s}+1}, x_{m_{k_s}+1}) < \lim_{s\to\infty} \nu(x_{n_{k_s}}, x_{m_{k_s}}) < b\varepsilon,$$

which implies (2.7). Thus, the sequences $bt_{k_s} = b\nu(x_{n_{k_s}+1}, x_{m_{k_s}+1})$ and $v_{k_s} = \nu(x_{n_{k_s}}, x_{m_{k_s}})$ have the same positive limit and verify that $t_{k_s} < v_{k_s}$ (by (2.9)). By the property (ξ_2) we conclude that $0 \leqslant \limsup_{k\to\infty} \xi(bt_{k_s}, v_{k_s}) < 0$ which is a contradiction and hence (2.3) holds.

Now, using Lemma 1.1 (iii), $\{x_n\}$ is a Cauchy sequence in A_0. Since (X, d) is a complete b-metric space and A_0 is a closed subset of X, there exists $\lim_{n\to\infty} x_n = x \in A_0$. Moreover, by the continuity of g we have $\lim_{n\to\infty} gx_n = gx$. Since $gx_n \in A_0$ for all $n \in \mathbb{N}$ and A_0 is closed, we also have $gx \in A_0$. On the other hand, since $x \in A_0$ and $T(A_0) \subseteq B_0$, for x there exists $z \in A_0$ such that $d(z, Tx) = d(A, B)$.

Let us prove that $z = gx$. If $z = gx_n$ for infinitely many $n \in \mathbb{N}$, then $z = gx$. Hence we assume that $z \neq gx$, in which case there exists $n_0 \in \mathbb{N}$ such that $z \neq gx_n$ for all $n \geqslant n_0$. If $\nu(gx_n, z) = 0$ for some $n \geqslant n_0$ then $gx_n = z$, so it must be $\nu(gx_n, z) > 0$ for all $n \geqslant n_0$. Also there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\nu(x_{n_k}, x) > 0$ for every $k \in \mathbb{N}$ (if that is not true, then there exists $n_1 \in \mathbb{N}$ such that $\nu(x_n, x) = 0$ for all $n \geqslant n_1$, and hence $\nu(x_n, x_{n-1}) = 0$ for all $n \geqslant n_1$, which is contradiction).

Since T is a $\mathcal{Z}$-P-proximal contraction of the first kind and $g \in \mathcal{G}_{A,P}$, we get

$$\begin{aligned} 0 &\leqslant \xi(b\nu(gx_{n_k+1}, z), \nu(x_{n_k}, x)) \\ &< \nu(x_{n_k}, x) - b\nu(gx_{n_k+1}, z) \\ &\leqslant \nu(gx_{n_k}, gx) - \nu(gx_{n_k+1}, z), \end{aligned}$$

which implies that

$$\nu(gx_{n_k+1}, z) < \nu(gx_{n_k}, gx) \tag{2.10}$$

for every $k \in \mathbb{N}$ such that $n_k \geqslant n_0$.

By a similar argument as before, we have $\lim_{m,n\to\infty} \nu(gx_n, gx_m) = 0$. This means that for any $\epsilon > 0$ there exists a $N_\epsilon \in \mathbb{N}$ such that $\nu(gx_n, gx_m) < \frac{\epsilon}{b}$ for all $m > n \geqslant N_\epsilon$. For a fixed $n \in \mathbb{N}$ with $n \geqslant \max\{n_0, N_\epsilon\}$ the function $P(gx_n, \cdot)$ is lower b-semicontinuous; hence, we obtain that

$$P(gx_n, gx) \leqslant \liminf_m bP(gx_n, gx_m) < \epsilon.$$

Thus,

$$\lim_{k\to\infty} P(gx_{n_k}, gx) = 0. \tag{2.11}$$

Similarly, $\lim_{k\to\infty} P(gx, gx_{n_k}) = 0$ which combined with (2.11) yields $\lim_{k\to\infty} \nu(gx_{n_k}, gx) = 0$. Now, from (2.10) we have

$$\lim_{k\to\infty} \nu(gx_{n_k+1}, z) = 0. \tag{2.12}$$

If $k \to \infty$ in

$$\nu(gx_{n_k}, z) \leqslant b\nu(gx_{n_k}, gx_{n_k+1}) + b\nu(gx_{n_k+1}, z),$$

then (2.2) and (2.12) imply that $\lim_{k\to\infty} \nu(gx_{n_k}, z) = 0$. Thus,

$$\lim_{k\to\infty} P(gx_{n_k}, z) = 0. \tag{2.13}$$

Now, using (2.11) and (2.13), Lemma 1.1 (i) implies that $z = gx$. Finally, from $d(z, Tx) = d(A, B)$ we get $d(gx, Tx) = d(A, B)$.

To prove the uniqueness, let y be in A_0 such that $d(gy, Ty) = d(A, B)$. Assume that $\nu(gx, gy) \geqslant \nu(x, y) > 0$. Since $g \in \mathcal{G}_{A,P}$ and T is a $\mathcal{Z}$-P-proximal contraction of the first kind, we obtain

$$\begin{aligned} 0 &\leqslant \xi(b\nu(gx, gy), \nu(x, y)) \\ &< \nu(x, y) - b\nu(gx, gy) \\ &\leqslant \nu(x, y) - \nu(x, y) = 0, \end{aligned}$$

which leads to a contradiction. Hence $\nu(x, y) = 0$, which implies $x = y$.

By a similar argument we prove $P(x, x) = 0$. Suppose that $\nu(x, x) = P(x, x) > 0$. Then $\nu(gx, gx) > 0$ and we have

$$\begin{aligned} 0 &\leqslant \xi(b\nu(gx, gx), \nu(x, x)) \\ &< \nu(x, x) - b\nu(gx, gx) \\ &\leqslant \nu(x, x) - \nu(x, x) = 0, \end{aligned}$$

which is a contradiction. □

The next best proximity point result for $\mathcal{Z}$-P-proximal contractions of the first kind is an immediate consequence of Theorem 2.1 by setting g as the identity mapping on A.

Corollary 2.1. *Let A and B be two nonempty subsets of a complete b-metric space (X, d) with a wt_0-distance P, such that A_0 is nonempty and closed. Suppose that a mapping $T : A \to B$ satisfies the following conditions*

a) *T is a $\mathcal{Z}$-P-proximal contraction of the first kind;* b) *$T(A_0) \subseteq B_0$.*

Then there exists a unique best proximity point $x \in A_0$ of the mapping T, such that $P(x, x) = 0$, and for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Example 2.3. Let $X = \mathbb{R}$ be endowed with the 2-metric $d(x, y) = |x - y|^2$ for all $x, y \in X$ and a wt_0-distance P defined by $P(x, y) = x^2 + y^2$ for all $x, y \in X$. Then we have $\nu(x, y) = \max\{P(x, y), P(y, x)\} = x^2 + y^2$ for all $x, y \in X$.

Let $A = [-1, 0]$ and $B = [1, 2]$, and let $T : A \to B$ be a mapping given by $Tx = 1 - x$ for all $x \in A$. Now it is easily obtained that $d(A, B) = 1$, and also $A_0 = \{0\}$ and $B_0 = \{1\}$, so that $T(A_0) = \{1\} = B_0$.

Let the 2-simulation function $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined with $\xi(t,s) = \lambda s - t$ for all $s,t \in [0,\infty)$ (where $\lambda \in [0,1)$). Now, $d(u,Tx) = |u-Tx|^2 = d(A,B) = 1$ if and only if $|u - Tx| = |u - 1 + x| = 1$, which is only possible for $u = x = 0$, since $x,u \in [-1,0]$, and similarly, $d(v,Ty) = d(A,B)$ is equivalent with $v = y = 0$. Hence, for $u = v = x = y = 0$ we get $\xi(2\nu(u,v),\nu(x,y)) = \lambda \cdot 0 - 2 \cdot 0 = 0$ which verifies that T is a $\mathcal{Z}$-P-proximal contraction of the first kind.

We conclude that all conditions of Corollary 2.1 are satisfied, and indeed, $B_{\text{est}}(T) = \{0\}$ and $P(0,0) = 0$.

From Theorem 2.1 we can also obtain an interesting g-best proximity point result for a P-proximal contraction of the first kind.

Corollary 2.2. *Let A and B be two nonempty subsets of a complete b-metric space (X,d) with a wt_0-distance P, such that A_0 is nonempty and closed. Suppose that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions*

a) *T is a P-proximal contraction of the first kind with respect to $\alpha \in [0,1)$;*

b) *$g \in \mathcal{G}_{A,P}$;* c) *$T(A_0) \subseteq B_0$;* d) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique point $x \in A_0$ such that $d(gx,Tx) = d(A,B)$ and $P(x,x) = 0$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(gx_{n+1},Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof. Note that a P-proximal contraction of the first kind with respect to $\alpha \in [0,1)$ is a $\mathcal{Z}$-P-proximal contraction of the first kind with respect to the b-simulation function $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ defined by $\xi(t,s) = \alpha s - t$ for all $t,s \geqslant 0$. □

By taking $d = P$ in Theorem 2.1, we obtain the same result in b-metric spaces.

Corollary 2.3. *Let A and B be two nonempty subsets of a complete b-metric space (X,d), such that A_0 is nonempty and closed. Suppose that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions.*

a) *T is a $\mathcal{Z}$-proximal contraction of the first kind;*

b) *$g \in \mathcal{G}_A$;* c) *$T(A_0) \subseteq B_0$;* d) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique point $x \in A$ such that $d(gx,Tx) = d(A,B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ such that $d(gx_{n+1},Tx_n) = d(A,B)$ for all $n \in \mathbb{N} \cup \{0\}$, and $\{x_n\}$ converges to x.

Remark 2.2. In Theorem 2.1 and its corollaries, set $b = 1$. Then we obtain the main results of Kostić et al. [**11**] involving simulation functions with w_0-distance (in metric spaces) for a $\mathcal{Z}$-p-proximal contraction of the first kind.

The next result is a g-best proximity point theorem for a $\mathcal{Z}$-P-proximal contraction of the second kind.

Theorem 2.2. *Let A and B be two nonempty subsets of a complete b-metric space (X,d) with a wt_0-distance P, such that $T(A_0)$ is nonempty and closed. Suppose that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions*

a) *T is a $\mathcal{Z}$-P-proximal contraction of the second kind;*

b) *T is injective on A_0;* c) $T \in \mathcal{T}_{g,P}$; d) $T(A_0) \subseteq B_0$; e) $A_0 \subseteq g(A_0)$.

Then there exists a unique $x \in A_0$ such that $d(gx, Tx) = d(A, B)$ and $P(Tx, Tx) = 0$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

Proof. By following a similar reasoning to that in the proof of Theorem 2.1, we can construct a sequence $\{x_n\}$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$. In the constructive process of $\{x_n\}$, if we have $Tx_n = Tx_m$ for some $m > n$, then we choose $x_{m+1} = x_{n+1}$.

Since T is a $\mathcal{Z}$-P-proximal contraction of the second kind, we have

$$\xi(b\nu(Tgx_n, Tgx_{n+1}), \nu(Tx_{n-1}, Tx_n)) \geqslant 0$$

for every $n \in \mathbb{N}$. From T being injective on A_0 and $T \in \mathcal{T}_{g,P}$, using the property (ξ_1) of a b-simulation function we deduce that

$$\begin{aligned} 0 &\leqslant \xi(b\nu(Tgx_n, Tgx_{n+1}), \nu(Tx_{n-1}, Tx_n)) \\ &< \nu(Tx_{n-1}, Tx_n) - b\nu(Tgx_n, Tgx_{n+1}) \\ &\leqslant \nu(Tx_{n-1}, Tx_n) - \nu(Tx_n, Tx_{n+1}) \end{aligned} \tag{2.14}$$

for every $n \in \mathbb{N}$. Hence we have $\nu(Tx_n, Tx_{n+1}) < \nu(Tx_{n-1}, Tx_n)$ for all $n \in \mathbb{N}$, which implies that the sequence $\{\nu(Tx_{n-1}, Tx_n)\}$ is decreasing.

If there exists $n_0 \in \mathbb{N}$ such that $\nu(Tx_{n_0-1}, Tx_{n_0}) = 0$, then $Tx_{n_0-1} = Tx_{n_0}$ and by the injectivity of T on A_0 follows $x_{n_0-1} = x_{n_0}$. But then $d(gx_{n_0-1}, Tx_{n_0}) = d(gx_{n_0}, Tx_{n_0}) = d(A, B)$ and x_{n_0} is the best proximity point of T under mapping g; that is, $x_{n_0} \in B^g_{\mathrm{est}}(T)$.

Now, let $\nu(Tx_{n-1}, Tx_n) > 0$ for all $n \in \mathbb{N}$. Hence, there exists

$$\lim_{n\to\infty} \nu(Tx_{n-1}, Tx_n) = r \geqslant 0.$$

Suppose that $r > 0$. From (2.14) we can also deduce that

$$\nu(Tgx_n, Tgx_{n+1}) \leqslant b\nu(Tg_n, Tgx_{n+1}) < \nu(Tx_{n-1}, Tx_n).$$

On the other hand $T \in \mathcal{T}_{g,P}$ and hence

$$\nu(Tx_n, Tx_{n+1}) \leqslant \nu(Tgx_n, Tgx_{n+1}) < \nu(Tx_{n-1}, Tx_n)$$

for all $n \in \mathbb{N}$. Passing to the limit when $n \to \infty$ we obtain that

$$\lim_{n\to\infty} \nu(Tgx_n, Tgx_{n+1}) = r.$$

Now, by the property (ξ_2) of a b-simulation function, we have

$$0 \leqslant \limsup_{n\to\infty} \xi(b\nu(Tgx_{n+1}, Tgx_n), \nu(Tx_{n-1}, Tx_n)) < 0$$

which is a contradiction, and hence $r = 0$.

We have shown that

$$\lim_{n\to\infty} \nu(Tx_{n-1}, Tx_n) = 0. \tag{2.15}$$

Next, we prove that

$$\lim_{m,n\to\infty} \nu(Tx_n, Tx_m) = 0. \tag{2.16}$$

Assume that (2.16) is not true. Then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}, \{n_k\} \subseteq \mathbb{N} \cup \{0\}$ with $m_k > n_k \geqslant k$ such that

$$\nu(Tx_{n_k}, Tx_{m_k}) \geqslant \varepsilon \tag{2.17}$$

for all $k \in \mathbb{N}$. We can assume that m_k is a minimal index for which (2.17) holds. Then we also have

$$\nu(Tx_{n_k}, Tx_{m_k-1}) < \varepsilon \tag{2.18}$$

for any $k \in \mathbb{N}$. Using the triangle inequality for ν, by (2.17) and (2.18) we have

$$\begin{aligned}\varepsilon \leqslant \nu(Tx_{n_k}, Tx_{m_k}) &\leqslant b\nu(Tx_{n_k}, Tx_{m_k-1}) + b\nu(Tx_{m_k-1}, Tx_{m_k})\\ &< b\varepsilon + b\nu(Tx_{m_k-1}, Tx_{m_k}).\end{aligned}$$

Passing to the limit when $k \to \infty$. By (2.15), we conclude that

$$\varepsilon \leqslant \limsup_{k\to\infty} \nu(Tx_{n_k}, Tx_{m_k}) < b\varepsilon. \tag{2.19}$$

Now, we claim that

$$\lim_{k\to\infty} \nu(Tx_{n_k+1}, Tx_{m_k+1}) < \varepsilon. \tag{2.20}$$

If $\lim\limits_{k\to\infty} \nu(Tx_{n_k+1}, Tx_{m_k+1}) \geqslant \varepsilon$, then there exist sequence $\{k_s\}$ and $\delta > 0$ such that

$$\lim_{s\to\infty} \nu(Tx_{n_{k_s}+1}, Tx_{m_{k_s}+1}) = \delta \geqslant \varepsilon. \tag{2.21}$$

Again, T is a $\mathcal{Z}$-P-proximal contraction of the first kind and

$$d(gx_{n_{k_s}+1}, Tx_{n_{k_s}}) = d(A, B) = d(gx_{m_{k_s}+1}, Tx_{m_{k_s}}).$$

Hence, by the property (ξ_1), we obtain

$$\begin{aligned}0 &\leqslant \xi(b\nu(Tgx_{n_{k_s}+1}, Tgx_{m_{k_s}+1}), \nu(Tx_{n_{k_s}}, Tx_{m_{k_s}}))\\ &< \nu(Tx_{n_{k_s}}, Tx_{m_{k_s}}) - b\nu(Tgx_{n_{k_s}+1}, Tgx_{m_{k_s}+1})\\ &\leqslant \nu(Tx_{n_{k_s}}, Tx_{m_{k_s}}) - b\nu(Tx_{n_{k_s}+1}, Tx_{m_{k_s}+1})\\ &\leqslant \nu(Tx_{n_{k_s}}, Tx_{m_{k_s}}) - \nu(Tx_{n_{k_s}+1}, Tx_{m_{k_s}+1})\end{aligned} \tag{2.22}$$

for all $k \in \mathbb{N}$. It follows from (2.19), (2.21) and (2.22) that

$$b\delta = \lim_{s\to\infty} b\nu(Tx_{n_{k_s}+1}, Tx_{m_{k_s}+1}) < \lim_{s\to\infty} \nu(Tx_{n_{k_s}}, Tx_{m_{k_s}}) < b\varepsilon, \tag{2.23}$$

which implies (2.20). Thus, the sequences $bt_{k_s} = b\nu(Tx_{n_{k_s}+1}, Tx_{m_{k_s}+1})$ and $v_{k_s} = \nu(Tx_{n_{k_s}}, Tx_{m_{k_s}})$ have the same positive limit and verify that $t_{k_s} < v_{k_s}$ (by (2.22)). By the property (ξ_2), we conclude that $0 \leqslant \limsup_{k\to\infty} \xi(bt_{k_s}, v_{k_s}) < 0$ which is a contradiction and hence (2.16) holds.

Now, using Lemma 1.1 (iii), $\{Tx_n\}$ is a Cauchy sequence. Since (X, d) is a complete b-metric space and $T(A_0)$ is a closed subset of X, there exists $\lim_{n\to\infty} Tx_n = Tu \in T(A_0) \subseteq B_0$. Moreover, there exists $z \in A_0$ such that $d(z, Tu) = d(A, B)$. Since $A_0 \subseteq g(A_0)$, we obtain that $z = gx$ for some $x \in A_0$, and hence

$$d(gx, Tu) = d(A, B). \tag{2.24}$$

If $x_n = x$ holds for infinite values of $n \in \mathbb{N}$, then $Tx = Tu$. Therefore, we can assume that there exists $n_0 \in \mathbb{N}$ such that $x_n \neq x$ for all $n \geqslant n_0$. Also, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\nu(Tx_{n_k}, Tu) > 0$ for all $k \in \mathbb{N}$. Again, since T is a $\mathcal{Z}$-P-proximal contraction of the second kind, we get

$$0 \leqslant \xi(b\nu(Tgx_{n_k+1}, Tgx), \nu(Tx_{n_k}, Tu)) < \nu(Tx_{n_k}, Tu) - b\nu(Tgx_{n_k+1}, Tgx)$$

and hence

$$\nu(Tx_{n_k+1}, Tx) \leqslant b\nu(Tgx_{n_k+1}, Tgx) < \nu(Tx_{n_k}, Tu) \tag{2.25}$$

for all $k \in \mathbb{N}$ such that $n_k \geqslant n_0$, since $T \in \mathcal{T}_{g,P}$.

From (2.16) we obtain that for any $\epsilon > 0$ there exists a $N_\epsilon \in \mathbb{N}$ such that $\nu(Tx_n, Tx_m) < \frac{\epsilon}{b}$ for every $m > n \geqslant N_\epsilon$. Then, using the property (P_2) of a wt_0-distance we have

$$P(Tx_n, Tu) \leqslant \liminf_{m\to\infty} bP(Tx_n, Tx_m) < \epsilon$$

for any fixed $n \geqslant \max\{n_0, N_\epsilon\}$, which implies that

$$\lim_{k\to\infty} P(Tx_{n_k}, Tu) = 0 \tag{2.26}$$

and similarly $\lim_{k\to\infty} P(Tu, Tx_{n_k}) = 0$, hence $\lim_{k\to\infty} \nu(Tx_{n_k}, Tu) = 0$. Combine this and (2.25) to get $\lim_{k\to\infty} \nu(Tx_{n_k+1}, Tx) = 0$. Let $k \to \infty$ in

$$\nu(Tx_{n_k}, Tx) \leqslant b\nu(Tx_{n_k}, Tx_{n_k+1}) + b\nu(Tx_{n_k+1}, Tx)$$

. From (2.15) we obtain $\lim_{k\to\infty} \nu(Tx_{n_k}, Tx) = 0$. Hence, we have

$$\lim_{k\to\infty} P(Tx_{n_k}, Tx) = 0. \tag{2.27}$$

Thus, by (2.26) and (2.27) and Lemma 1.1 (i), we conclude that $Tx = Tu$. Now, by substituting $Tx = Tu$ in (2.24), we get $d(gx, Tx) = d(A, B)$.

To show the uniqueness, let y be in A_0 such that $d(gy, Ty) = d(A, B)$, i.e., $y \in B^g_{\mathrm{est}}(T)$. Assume that $\nu(Tgx, Tgy) \geqslant \nu(Tx, Ty) > 0$. Since $T \in \mathcal{T}_{g,P}$ is a $\mathcal{Z}$-P-proximal contraction of the second kind, we have

$$\begin{aligned} 0 &\leqslant \xi(b\nu(Tgx, Tgy), \nu(Tx, Ty)) \\ &< \nu(Tx, Ty) - b\nu(Tgx, Tgy) \\ &\leqslant \nu(Tx, Ty) - \nu(Tx, Ty) = 0, \end{aligned}$$

which is a contradiction. Hence, $\nu(Tx, Ty) = 0$, which means that $Tx = Ty$. Injectivity of T on A_0 then yields $x = y$.

Finally, suppose that $\nu(Tx, Tx) = P(Tx, Tx) > 0$. Then $\nu(Tgx, Tgx) > 0$. Using a similar argument as above, we have

$$\begin{aligned} 0 &\leqslant \xi(b\nu(Tgx, Tgx), \nu(Tx, Tx)) \\ &< \nu(Tx, Tx) - b\nu(Tgx, Tgx) \\ &\leqslant \nu(Tx, Tx) - \nu(Tx, Tx) = 0, \end{aligned}$$

which is a contradiction. Therefore, $P(Tx, Tx) = 0$. $\square$

The following best proximity point result is a special case of Theorem 2.2 when g is an identity map on A.

Corollary 2.4. *Let A and B be two nonempty subsets of a complete b-metric space (X, d) with a wt_0-distance P, such that $T(A_0)$ is nonempty and close. Suppose that the mapping $T : A \to B$ satisfies the following conditions*

a) *T is a $\mathcal{Z}$-P-proximal contraction of the second kind;*

b) *T is injective on A_0;* c) *$T(A_0) \subseteq B_0$.*

Then there exists a unique best proximity point $x \in A_0$ of T with $P(Tx, Tx) = 0$, and for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A_0$ converging to x, such that $d(x_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$.

By taking $d = P$ in Theorem 2.2, we obtain the same result in b-metric spaces.

Corollary 2.5. *Let A and B be two nonempty subsets of a complete b-metric space (X, d), such that $T(A_0)$ is nonempty and closed. Suppose that the mappings $T : A \to B$ and $g : A \to A$ satisfy the following conditions*

a) *T is $\mathcal{Z}$-proximal contraction of the second kind;*

b) *T is injective on A_0;* c) *$T \in \mathcal{T}_g$;* d) *$T(A_0) \subseteq B_0$;* e) *$A_0 \subseteq g(A_0)$.*

Then there exists a unique point $x \in A$ such that $d(gx, Tx) = d(A, B)$. Moreover, for every $x_0 \in A_0$ there exists a sequence $\{x_n\} \subseteq A$ such that $d(gx_{n+1}, Tx_n) = d(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$ and $\lim_{n\to\infty} x_n = x$.

Remark 2.3. In Theorem 2.2 and its corollaries, set $b = 1$. Then we obtain the main results of Kostić et al. [**11**] involving simulation functions with w_0-distance (in metric spaces) for a $\mathcal{Z}$-p-proximal contraction of the second kind.

Remark 2.4. In Corollaries 2.3 and 2.5, set $b = 1$. Then we obtain the main results of Tchier et al. [**18**].

3. Conclusion and suggestions

We considered a special type of wt-distance and obtained some interesting results about best proximity points, under which can be generalized, improved, enriched and unified a number of recently announced results in the existing literature such as Kostić et al. [**11**], Tchier et al. [**18**] and others. Also, we consider some various examples about our definitions and results to illuminate our work. Since wt_0-distance is a notion between b-metric spaces and wt-distances (similarly, w_0-distance is a notion between metric spaces and w-distances), we suggest to readers and researchers to work on these distances (both w_0 and wt_0) in fixed point theory and best proximity results as a new and different work.

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