MAPS PRESERVING $n$-TUPLE $A^*B - B^*A$ DERIVATIONS ON FACTOR VON NEUMANN ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a factor von Neumann algebra and $\Phi$ preserve $n$-tuple new product derivations on $\mathcal{A}$, that is, for every $A_1, A_2, \ldots, A_n \in \mathcal{A}$,

$$\Phi(A_1 \ast A_2 \ast \cdots \ast A_n) = \Phi(A_1) \ast A_2 \ast \cdots \ast A_n + A_1 \ast \Phi(A_2) \ast \cdots \ast A_n$$

where $A_i \ast A_j = A_i^*A_j - A_j^*A_i$ for $i, j \in \mathbb{N}$, then $\Phi$ is additive $\ast$-derivation, on the condition that $\Phi(\alpha I_2)$ is self-adjoint operator for $\alpha \in \{1, i\}$.

1. Introduction

Let $\mathcal{R}$ be a factor von Neumann algebras. For $A, B \in \mathcal{R}$, denoted by $A \bullet B = AB + BA^*$ and $[A, B]_{\ast} = AB - BA^*$, which are $\ast$-Jordan product and $\ast$-Lie product, respectively. These products are found playing a more and more important role in some research topics, and its study has recently attracted many author's attention (for example, see [5,10,11,16]).

Recall that a map $\Phi: \mathcal{R} \to \mathcal{R}$ is said to be an additive derivation if

$$\Phi(A + B) = \Phi(A) + \Phi(B) \quad \text{and} \quad \Phi(AB) = \Phi(A)B + A\Phi(B)$$

for all $A, B \in \mathcal{R}$. A map $\Phi$ is additive $\ast$-derivation if it is an additive derivation and $\Phi(A^*) = \Phi(A)^*$. Derivations are very important maps both in theory and applications, in particular to study the topological and fundamental structures of von Neumann algebras [4]. This notion has been studied intensively [3,12,14].

Let us define $\lambda$-Jordan $\ast$-product by $A \ast_{\lambda} B = AB + \lambda BA^*$. We say that the map $\Phi$ with the property of $\Phi(A \ast_{\lambda} B) = \Phi(A) \ast_{\lambda} B + A \ast_{\lambda} \Phi(B)$ is a $\lambda$-Jordan $\ast$-derivation map. It is clear that for $\lambda = -1$ and $\lambda = 1$, the $\lambda$-Jordan $\ast$-derivation map is a $\ast$-Lie derivation and $\ast$-Jordan derivation, respectively [1].

A von Neumann algebra $\mathcal{A}$ is a self-adjoint subalgebra of some $B(H)$, the algebra of bounded linear operators acting on a complex Hilbert space, which satisfies

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the double commutant property: \( \mathcal{A}'' = \mathcal{A} \) where \( \mathcal{A}' = \{ T \in B(H), TA = AT, \forall A \in \mathcal{A} \} \) and \( \mathcal{A}' = (\mathcal{A}')' \). Denote by \( \mathcal{Z}(\mathcal{A}) = \mathcal{A}' \cap \mathcal{A} \) the center of \( \mathcal{A} \). A von Neumann algebra \( \mathcal{A} \) is called a factor if its center is trivial, that is, \( \mathcal{Z}(\mathcal{A}) = \mathbb{C}1 \).

Recently, Yu and Zhang in [17] proved that every non-linear \( * \)-Lie derivation from a factor von Neumann algebra into itself is an additive \( * \)-derivation. Also, Li, Lu, and Fang in [9] have investigated a non-linear \( \lambda \)-Jordan \( * \)-derivation. They showed that if \( \mathcal{A} \subseteq B(H) \) is a von Neumann algebra without central abelian projections and \( \lambda \) is a non-zero scalar, then \( \Phi: \mathcal{A} \rightarrow B(H) \) is a non-linear \( \lambda \)-Jordan \( * \)-derivation if and only if \( \Phi \) is an additive \( * \)-derivation.

Very recently, the authors of [5] discussed some bijective maps preserving the new product \( A^*B + B^*A \) between von Neumann algebras with no central abelian projections. In other words, \( \Phi \) holds in the following condition

\[
\Phi(A^*B + B^*A) = \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A).
\]

They showed that such a map is a sum of a linear \( * \)-isomorphism and a conjugate linear \( * \)-isomorphism. In [15], the authors have discussed such maps between unital prime \( * \)-algebras which preserve the new triple product.

In [18] the authors considered nonlinear \( * \)-Lie higher derivations on factor von Neumann algebras acting on a complex Hilbert space \( H \) with \( \dim H \geq 2 \).

Also, the authors of [6] obtained the following result: Let \( \mathcal{A} \) and \( \mathcal{B} \) be two factor von Neumann algebras with \( I_{\mathcal{A}} \) and \( I_{\mathcal{B}} \) the identities of them, respectively. Let \( A_1, A_2, \ldots, A_n \in \mathcal{A} \) and \( A_1 \cdot A_2 \cdots A_n \) is the Jordan multiple \( * \)-product in any fixed operation order. Then a not necessarily linear bijective mapping \( \Phi: \mathcal{A} \rightarrow \mathcal{B} \) satisfies

\[
\Phi(A_1 \cdot A_2 \cdots A_n) = \Phi(A_1) \cdot \Phi(A_2) \cdots \Phi(A_n)
\]

if and only if \( \Phi \) is a \( * \)-ring isomorphism.

In this paper motivated by the above results, we consider a map (not necessarily linear) \( \Phi \) on factor von Neumann algebras \( \mathcal{A} \) which meets the following conditions:

\[
\Phi(A_1 \circ A_2 \circ \cdots \circ A_n) = \Phi(A_1) \circ A_2 \circ \cdots \circ A_n + A_1 \circ \Phi(A_2) \circ \cdots \circ A_n
\]

\[
+ \cdots + A_1 \circ A_2 \circ \cdots \circ \Phi(A_n)
\]

where \( A_i \circ A_j = A_i^*A_j - A_j^*A_i \) for \( i, j \in \mathbb{N} \). We prove that \( \Phi \) is additive \( * \)-derivation. The results of this essay will be applied to study general structures of von Neumann algebras which are related to non-linear derivatives.

When we say that \( \mathcal{A} \) is prime, that is, for \( A, B \in \mathcal{A} \) if \( AAB = \{0\} \), then \( A = 0 \) or \( B = 0 \).

2. Main result

**Theorem 2.1.** Let \( \mathcal{A} \) be a factor von Neumann algebras. Let \( \Phi: \mathcal{A} \rightarrow \mathcal{A} \) satisfies in

\[
\Phi(A_1 \circ A_2 \circ \cdots \circ A_n) = \Phi(A_1) \circ A_2 \circ \cdots \circ A_n + A_1 \circ \Phi(A_2) \circ \cdots \circ A_n
\]

\[
+ \cdots + A_1 \circ A_2 \circ \cdots \circ \Phi(A_n), \quad n \geq 2
\]

for all \( A_1, \ldots, A_n \in \mathcal{A} \) where \( A_i \circ A_j = A_i^*A_j - A_j^*A_i \) for \( i, j \in \mathbb{N} \). Then \( \Phi \) is additive \( * \)-derivation.
By taking adjoint from both side of (2.1), for odd numbers case, we obtain

\[
\Phi - \Phi^* = 0.
\]

As well as, the self-adjoint assumption clarify that \(\Phi\) is additive on each \(A_{ij}\), \(i, j = 1, 2\).

We prove the above theorem by several claims.

**Claim 2.1.** We show that \(\Phi(0) = 0\).

This claim is easy to prove.

**Claim 2.2.** We show that \(\Phi\left(\frac{i}{2}\right) = 0\), \(\Phi\left(-\frac{i}{2}\right) = 0\) and \(\Phi\left(i\frac{2}{2}\right) = 0\).

Since

\[
\Phi\left(\frac{I}{2} \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2}\right) = \Phi\left(\frac{I}{2}\right) \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2} + \frac{I}{2} \circ \Phi\left(\frac{I}{2}\right) \circ \ldots \circ \frac{I}{2}
\]

\[
+ \ldots + \frac{I}{2} \circ \frac{I}{2} \circ \ldots \circ \Phi\left(\frac{I}{2}\right),
\]

an elementary computation shows that two cases appear depending on \(n\):

\[
\Phi\left(\frac{I}{2}\right) = \frac{1}{2} \left(\Phi\left(\frac{I}{2}\right) - \Phi\left(\frac{I}{2}\right)^*\right) + (n - 1) \frac{i}{2} \left(\Phi\left(\frac{I}{2}\right) + \Phi\left(\frac{I}{2}\right)^*\right), \quad n \text{ odd},
\]

\[
\Phi\left(-\frac{I}{2}\right) = -\frac{1}{2} \left(\Phi\left(\frac{I}{2}\right) - \Phi\left(\frac{I}{2}\right)^*\right) - (n - 1) \frac{i}{2} \left(\Phi\left(\frac{I}{2}\right) + \Phi\left(\frac{I}{2}\right)^*\right), \quad n \text{ even}
\]

which imply that

\[
\frac{1}{2} \left(\Phi\left(\frac{I}{2}\right)^* - \Phi\left(\frac{I}{2}\right)\right) = 0, \quad n \text{ odd},
\]

\[
\Phi\left(-\frac{I}{2}\right) + \frac{1}{2} \left(\Phi\left(\frac{I}{2}\right)^* - \Phi\left(\frac{I}{2}\right)\right) = 0, \quad n \text{ even}
\]

By taking adjoint from both side of (2.1), for odd numbers case, we obtain

\[
\frac{1}{2} \left(\Phi\left(\frac{I}{2}\right)^* - \Phi\left(\frac{I}{2}\right)\right) + (n - 1) \frac{i}{2} \left(\Phi\left(\frac{I}{2}\right) + \Phi\left(\frac{I}{2}\right)^*\right) = 0, \quad n \text{ odd},
\]

\[
\Phi\left(-\frac{I}{2}\right) + \frac{1}{2} \left(\Phi\left(\frac{I}{2}\right)^* - \Phi\left(\frac{I}{2}\right)\right) + (n - 1) \frac{i}{2} \left(\Phi\left(\frac{I}{2}\right) + \Phi\left(\frac{I}{2}\right)^*\right) = 0, \quad n \text{ even}
\]

Hence (2.2), for odd case, give us \(\Phi\left(\frac{i}{2}\right) + \Phi\left(\frac{I}{2}\right)^* = 0\). Since \(\Phi\left(\frac{i}{2}\right)\) is self-adjoint, \(\Phi\left(\frac{i}{2}\right) = 0\). By re-applying (2.2) it results that \(\Phi\left(\frac{I}{2}\right) + \Phi\left(\frac{I}{2}\right)^* = 0\). So \(\Phi\left(\frac{i}{2}\right) = 0\) because \(\Phi\left(\frac{i}{2}\right)^*\) is self-adjoint. Similarly we can show that \(\Phi\left(-\frac{i}{2}\right) = 0\).

Set zero valued in extension of \(\Phi\left(\frac{i}{2} \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2}\right)\) for even case, we have

\[
\Phi\left(-\frac{i}{2}\right) = 0.
\]

As well as, the self-adjoint assumption clarify that \(\Phi\left(-\frac{i}{2}\right)^* = 0\).

**Claim 2.3.** For each \(A \in \mathcal{A}\), we have

1. \(\Phi(-iA) = -i\Phi(A)\).
2. \(\Phi(iA) = i\Phi(A)\).

It is straightforward to see that

\[
\Phi\left(-iA \circ \frac{I}{2} \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2}\right) = \Phi\left(A \circ \frac{I}{2} \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2}\right).
\]
So,
\[ \Phi(-iA) \circ \frac{I}{2} \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2} = \Phi(A) \circ \frac{I}{2} \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2} \]

It follows that
\[ (2.3) \quad -\Phi(-iA)^* + \Phi(-iA) = -i\Phi(A)^* - i\Phi(A). \]

On the other hand, one can check that
\[ \Phi\left(-iA \circ \frac{I}{2} \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2}\right) = \Phi\left(A \circ \frac{I}{2} \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2}\right). \]

So,
\[ \Phi(-iA) \circ \frac{I}{2} \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2} = \Phi(A) \circ \frac{I}{2} \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2} \]

It follows that
\[ -i\Phi(-iA)^* + i\Phi(-iA) = \Phi(A)^* - \Phi(A). \]

Equivalently, we obtain
\[ (2.4) \quad -\Phi(-iA)^* + \Phi(-iA) = i\Phi(A)^* - i\Phi(A). \]

By adding equations (2.3) and (2.4), we have \( \Phi(-iA) = -i\Phi(A) \).

Similarly it can be shown \( \Phi(iA) = i\Phi(A) \).

**Claim 2.4.** For each \( A_{11} \in A_{11}, A_{12} \in A_{12} \), we have
\[
\Phi(A_{11} + A_{12}) = \Phi(A_{11}) + \Phi(A_{12}).
\]

Let \( T = \Phi(A_{11} + A_{12}) - \Phi(A_{11}) - \Phi(A_{12}) \). We need to show that \( T = 0 \).

\[
\Phi(A_{11} + A_{12}) \circ C_{21} \circ I \circ \ldots \circ I + (A_{11} + A_{12}) \circ \Phi(C_{21}) \circ I \circ \ldots \circ I
\]
\[ + \cdots + (A_{11} + A_{12}) \circ C_{21} \circ I \circ \ldots \circ I \circ \Phi(I) = \Phi(A_{11} + A_{12} \circ C_{21} \circ I \circ \ldots \circ I)
\]
\[ = \Phi(A_{11} \circ C_{21} \circ I \circ \ldots \circ I + A_{11} \circ \Phi(C_{21}) \circ I \circ \ldots \circ I
\]
\[ + \cdots + A_{11} \circ C_{21} \circ I \circ \ldots \circ I \circ \Phi(I) + \Phi(A_{12}) \circ C_{21} \circ I \circ \ldots \circ I
\]
\[ + A_{12} \circ \Phi(C_{21}) \circ I \circ \ldots \circ I + \cdots + A_{12} \circ C_{21} \circ I \circ \ldots \circ I \circ \Phi(I)
\]
\[ = \Phi(A_{11} + A_{12} \circ C_{21} \circ I \circ \ldots \circ I + (A_{11} + A_{12}) \circ \Phi(C_{21}) \circ I \circ \ldots \circ I
\]
\[ + \cdots + (A_{11} + A_{12}) \circ C_{21} \circ I \circ \ldots \circ I \circ \Phi(I). \]

Then \( T \circ C_{21} \circ I \circ \ldots \circ I \circ \Phi(I) = 0 \). According to initial discussion of proof, we know \( T = T_{11} + T_{12} + T_{21} + T_{22} \). So \( -T_{22}C_{21} - T_{21}C_{21} + C_{21}^2T_{22} + C_{21}T_{21} = 0 \). Thus we have \( T_{22} = T_{21} = 0 \). In a similar way, we have
\[
\Phi(A_{11} + A_{12}) \circ C_{12} \circ P_1 \circ I \circ \ldots \circ I + (A_{11} + A_{12}) \circ \Phi(C_{12}) \circ P_1 \circ \ldots \circ I
\]
\[ + \cdots + (A_{11} + A_{12}) \circ C_{12} \circ P_1 \circ \ldots \circ I \circ \Phi(I) = \Phi((A_{11} + A_{12}) \circ C_{12} \circ P_1 \circ \ldots \circ I)
\]
\[ = \Phi(A_{11} \circ C_{12} \circ P_1 \circ I \circ \ldots \circ I + A_{12} \circ C_{12} \circ P_1 \circ I \circ \ldots \circ I
\]
\[ + \cdots + A_{11} \circ C_{12} \circ P_1 \circ I \circ \ldots \circ I \circ \Phi(I) + (A_{11} + A_{12}) \circ \Phi(C_{12}) \circ P_1 \circ I \circ \ldots \circ I
\]
\[ + \cdots + (A_{11} + A_{12}) \circ C_{12} \circ P_1 \circ I \circ \ldots \circ I \circ \Phi(I). \]

So \( T \circ C_{12} \circ P_1 \circ I \circ \ldots \circ I = 0 \). An early application of the unique decomposition of \( T = T_{11} + T_{12} + T_{21} + T_{22} \) shows that \( -T_{11}^*C_{12} + C_{12}^*T_{11} = 0 \). Now, multiplying the
above equality from right side by \( P_1 \) and left side by \( P_2 \), and using the primitivity assumption of \( \mathcal{A} \) we show that \( T_{11} = 0 \). Similarly, one can show that \( T_{12} = 0 \) by applying \( C_{21} \) instead of \( C_{12} \) as above.

**Claim 2.5.** For each \( A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21} \) and \( A_{22} \in \mathcal{A}_{22} \), we have

1. \( \Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) \).
2. \( \Phi(A_{12} + A_{21} + A_{22}) = \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}) \).

We show that \( T = \Phi(A_{11} + A_{12} + A_{21}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) = 0 \). From Claim 2.4 we obtain

\[
\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) = \Phi(A_{11} + A_{12} + A_{21})
\]

It follows that \( T = 0 \). Thus \( 2T_{11} - 2T_{11}^* + T_{12} - T_{12}^* = 0 \) which implies (2.5)

\[
T_{12} = 0, \quad T_{11} - T_{11}^* = 0.
\]

Also from Claims 2.3 and 2.4 we have

\[
\Phi(A_{11} + A_{12} + A_{21}) = \Phi(\Phi) = \Phi(A_{11} + A_{12} + A_{21}) = \Phi(\Phi) = \Phi(A_{11} + A_{12} + A_{21})
\]
\[ + \Phi(A_{21} \circ iP_1 \circ I \circ \ldots \circ I) = \Phi(A_{11} \circ iP_1 \circ I \circ \ldots \circ I) \]
\[ + \Phi(A_{12} \circ iP_1 \circ I \circ \ldots \circ I) + \Phi(A_{21} \circ iP_1 \circ I \circ \ldots \circ I) \]
\[ = ((\Phi(A_{11} + \Phi(A_{12} + \Phi(A_{21}))) \circ iP_1 \circ I \circ \ldots \circ I + (A_{11} + A_{12} + A_{21}) \circ \Phi(iP_1) \circ I \circ \ldots \circ I \]
\[ + \cdots + (A_{11} + A_{12} + A_{21}) \circ iP_1 \circ I \circ \ldots \circ I) \]

So \( T \circ iP_1 \circ I \circ \ldots \circ I = 0 \) which implies

(2.6) \[ -T_{11} - T_{11}^* = 0. \]

From (2.4), (2.5), we have \( T_{11} = 0 \).

Similarly, we can show that \( \Phi(A_{12} + A_{21} + A_{22}) = \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}) \).

**Claim 2.6.** For each \( A_{11} \in \mathcal{A}_{11}, A_{12} \in \mathcal{A}_{12}, A_{21} \in \mathcal{A}_{21} \) and \( A_{22} \in \mathcal{A}_{22} \), we have

\[ \Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}). \]

In order to prove this claim, we have to show that

\[ T = \Phi(A_{11} + A_{12} + A_{21} + A_{22}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) - \Phi(A_{22}) = 0. \]

From claim (2.5), we have

\[ \Phi(A_{11} + A_{12} + A_{21} + A_{22}) \circ C_{12} \circ I \circ \ldots \circ I \]
\[ = (A_{11} + A_{12} + A_{21} + A_{22}) \circ \Phi(C_{12}) \circ I \circ \ldots \circ I \]
\[ + \cdots + (A_{11} + A_{12} + A_{21} + A_{22}) \circ C_{12} \circ I \circ \ldots \circ I \]
\[ = \Phi((A_{11} + A_{12} + A_{21} + A_{22}) \circ C_{12} \circ I \circ \ldots \circ I) \]
\[ = \Phi(A_{11} + A_{12} + A_{21}) \circ C_{12} \circ I \circ \ldots \circ I \]
\[ + \Phi(A_{22} \circ C_{12}) \circ I \circ \ldots \circ C_{12} \circ I \]
\[ + \cdots + (A_{11} + A_{12} + A_{21} + A_{22}) \circ C_{12} \circ I \circ \ldots \circ I \]
\[ = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) \]
\[ + \Phi(A_{22} \circ C_{12}) \circ I \circ \ldots \circ I \]
\[ + \cdots + (A_{11} + A_{12} + A_{21} + A_{22}) \circ C_{12} \circ I \circ \ldots \circ I \]

So, \( T \circ C_{12} \circ I = 0 \). It follows that \( C_{12}^\dagger T_{11} + C_{12}^\dagger T_{12} - T_{11}^* C_{12} - T_{12}^* C_{12} = 0 \). Thus \( T_{11} = T_{12} = 0 \).

Similarly, by applying \( C_{21} \) instead of \( C_{12} \) in the above computation, we obtain \( T_{21} = T_{22} = 0 \).

**Claim 2.7.** For each \( A_{ij}, B_{ij} \in \mathcal{A}_{ij} \) such that \( i \neq j \), we have

\[ \Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}). \]

It is easy to show that

\[ (P_i + A_{ij}^*) \circ (P_j + B_{ij}) \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2} = A_{ij}^* + B_{ij}^* - A_{ij} - B_{ij}. \]

So, we can write

\[ \Phi(-A_{ij} - B_{ij}) + \Phi(A_{ij}^* + B_{ij}^*) = \Phi\left(\left(P_i + A_{ij}^*\right) \circ (P_j + B_{ij}) \circ \frac{I}{2} \circ \ldots \circ \frac{I}{2}\right) \]
Thus we showed that

\[
\Phi(P_i + A_i^*) \circ (P_j + B_{ij}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}
\]

By an easy computation, we can write

\[
(2.7) \quad \Phi(iA_{ij} + iB_{ij}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2} = iA_{ij} + iB_{ij} + iA_{ij}^* + iB_{ij}^*.
\]

Therefore, we show that

\[
\Phi(-A_{ij} - B_{ij}) + \Phi(A_{ij}^* + B_{ij}^*) = \Phi(-A_{ij}) + \Phi(-B_{ij}) + \Phi(A_{ij}^*) + \Phi(B_{ij}^*).
\]

By an easy computation, we can write

\[
(P_i + A_i^*) \circ (iP_j + iB_{ij}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2} = iA_{ij} + iB_{ij} + iA_{ij}^* + iB_{ij}^*.
\]

Then, we have

\[
\Phi(iA_{ij} + iB_{ij}) + \Phi(iA_{ij}^* + iB_{ij}^*) = \Phi((P_i + A_i^*) \circ (iP_j + iB_{ij}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2})
\]

\[
= \Phi(P_i + A_i^*) \circ (iP_j + iB_{ij}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}
\]

\[
+ (P_i + A_i^*) \circ \Phi(iP_j + iB_{ij}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}
\]

\[
+ (P_i + A_i^*) \circ (iP_j + iB_{ij}) \circ \Phi\left(\frac{I}{2}\right) \circ \cdots \circ \frac{I}{2}
\]

\[
+ \cdots + (P_i + A_i^*) \circ (iP_j + iB_{ij}) \circ \frac{I}{2} \circ \cdots \circ \Phi\left(\frac{I}{2}\right)
\]

\[
= (\Phi(P_i) + \Phi(A_i^*)) \circ (iP_j + iB_{ij}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}
\]

\[
+ (P_i + A_i^*) \circ \Phi(iP_j + iB_{ij}) \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2} + \cdots +
\]

\[
+ (P_i + A_i^*) \circ (iP_j + iB_{ij}) \circ \Phi\left(\frac{I}{2}\right) \circ \cdots \circ \frac{I}{2}
\]

\[
= \Phi\left(P_i \circ iB_{ij} \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}\right) + \Phi\left(A_i^* \circ iP_j \circ \frac{I}{2} \circ \cdots \circ \frac{I}{2}\right)
\]

\[
= \Phi(iB_{ij}) + \Phi(iB_{ij}^*) + \Phi(iA_{ij}) + \Phi(iA_{ij}^*).
\]

Thus we showed that

\[
\Phi(iA_{ij} + iB_{ij}) + \Phi(iA_{ij}^* + iB_{ij}^*) = \Phi(iA_{ij}) + \Phi(iB_{ij}) + \Phi(iA_{ij}^*) + \Phi(iB_{ij}^*).
\]
From claim \(2.23\) and the above equation, we have
\[
(2.8) \quad -\Phi(A_{ij} + B_{ij}) - \Phi(A_{ij}^* + B_{ij}^*) = -\Phi(B_{ij}) - \Phi(B_{ij}^*) - \Phi(A_{ij}) - \Phi(-A_{ij}).
\]
By adding equations \((2.7)\) and \((2.8)\), we obtain \(\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij})\).

**Claim 2.8.** For each \(A_{ii}, B_{ii} \in A_{ii}\) such that \(1 \leq i \leq 2\), we have
\[
\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).
\]
We show that \(T = \Phi(A_{ii} + B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii}) = 0\). We can write
\[
\Phi(A_{ii} + B_{ii}) \odot P_j \odot I \odot \cdots \odot I + (A_{ii} + B_{ii}) \odot \Phi(P_j) \odot I \odot \cdots \odot I
+ \cdots + (A_{ii} + B_{ii}) \odot P_j \odot I \odot \cdots \odot I + \Phi(A_{ii}) \odot P_j \odot I \odot \cdots \odot I
+ \cdots + A_{ii} \odot P_j \odot I \odot \cdots \odot I + A_{ii} \odot \Phi(P_j) \odot I \odot \cdots \odot I
+ B_{ii} \odot \Phi(P_j) \odot I \odot \cdots \odot I + \cdots + B_{ii} \odot P_j \odot I \odot \cdots \odot I
= (\Phi(A_{ii}) + \Phi(B_{ii})) \odot P_j \odot I \odot \cdots \odot I + (A_{ii} + B_{ii}) \odot \Phi(P_j) \odot I \odot \cdots \odot I
+ \cdots + (A_{ii} + B_{ii}) \odot P_j \odot I \odot \cdots \odot I
\]
So, we have \(T \odot P_j \odot I \odot \cdots \odot I = 0\). Therefore, we obtain \(T_{ij} = T_{ji} = T_{ji} = 0\).

On the other hand, for every \(C_{ij} \in A_{ij}\), we have
\[
\Phi(A_{ii} + B_{ii}) \odot C_{ij} \odot I \odot \cdots \odot I + (A_{ii} + B_{ii}) \odot \Phi(C_{ij}) \odot I \odot \cdots \odot I
+ \cdots + (A_{ii} + B_{ii}) \odot C_{ij} \odot I \odot \cdots \odot I
= \Phi((A_{ii} + B_{ii}) \odot C_{ij} \odot I \odot \cdots \odot I) = \Phi(A_{ii} \odot C_{ij} \odot I \odot \cdots \odot I)
+ \Phi(B_{ii} \odot C_{ij} \odot I \odot \cdots \odot I) = (\Phi(A_{ii}) + \Phi(B_{ii})) \odot C_{ij} \odot I \odot \cdots \odot I
+ (A_{ii} + B_{ii}) \odot \Phi(C_{ij}) \odot I \odot \cdots \odot I + (A_{ii} + B_{ii}) \odot C_{ij} \odot \Phi(I) \odot \cdots \odot I
+ \cdots + (A_{ii} + B_{ii}) \odot C_{ij} \odot I \odot \cdots \odot \Phi(I)
\]
So \(T \odot C_{ij} \odot I \odot \cdots \odot I = 0\) which shows that \(T_{ij} \odot C_{ij} \odot I \odot \cdots \odot I = 0\). Thus we see that \(-T_{ii}C_{ij} + C_{ij}^*T_{ii} = 0\). Again multiplying the above equality from right by \(P_i\) and from left by \(P_j\) and using the well-known fact that \(A\) is prime, we have \(T_{ii} = 0\).

Hence, the additivity of \(\Phi\) comes from the above claims.

In the rest of this paper, we show that \(\Phi\) is \(*\)-derivation.

**Claim 2.9.** \(\Phi\) preserves star.

Since \(\Phi\left(\frac{1}{2}\right) = 0\) then we can write \(\Phi\left(A \odot \frac{1}{2} \odot \frac{1}{2} \cdots \odot \frac{1}{2}\right) = \Phi(A) \odot \frac{1}{2} \odot \frac{1}{2} \cdots \odot \frac{1}{2}\). Then \(\Phi(A - A^*) = \Phi(A) - \Phi(A^*)\). So, we showed that \(\Phi\) preserves star.

**Claim 2.10.** We prove that \(\Phi\) is derivation.

For every \(A, B \in A\), a simple calculation shows that
\[
\Phi(-2AB + 2B^*A^*) = \Phi\left(A^* \odot B \odot \frac{I}{2} \odot \cdots \odot \frac{I}{2} \odot I\right)
= \Phi(A^*) \odot B \odot \frac{I}{2} \odot \cdots \odot \frac{I}{2} \odot I
\]
\[
+ \cdots + A^* \circ B \circ I_2 \circ \cdots \circ I_2 \circ \Phi(I) = 2B^* \Phi(A)^* - 2\Phi(A)B - 2A\Phi(B) + \Phi(B)^* A^*.
\]
So we have
\[
(2.9) \quad \Phi(B^* A^* - AB) = \Phi(B)^* A^* + B^* \Phi(A)^* - \Phi(A)B - A\Phi(B).
\]
From Claims 2.3 and 2.9, we have
\[
\Phi(-2B^* A^* - 2AB) = \Phi \left( iA^* \circ B \circ I_2 \circ \cdots \circ I_2 \circ I \right)
= \Phi \left( iA^* \right) \circ B \circ I_2 \circ \cdots \circ I_2 \circ I \\
+ \cdots + (iA^*) \circ B \circ I_2 \circ \cdots \circ I_2 \circ \Phi(I).
\]
Then
\[
i\Phi(-B^* A^* - AB) = i \left( (-\Phi(B)^* A^* - B^* \Phi(A)^* - \Phi(A)B - A\Phi(B)) \right).
\]
So we can show that
\[
(2.10) \quad \Phi(-B^* A^* - AB) = -\Phi(B)^* A^* - B^* \Phi(A)^* - \Phi(A)B - A\Phi(B).
\]
Eventually, equations (2.9) and (2.10) shows that \( \Phi(AB) = \Phi(A)B + A\Phi(B) \). \( \square \)

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References

11. L. Molnár, A condition for a subspace of \( B(H) \) to be an ideal, Linear Algebra Appl. 235 (1996), 229–234.