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MAPS PRESERVING n-TUPLE $A^*B - B^*A$ DERIVATIONS ON FACTOR VON NEUMANN ALGEBRAS

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Abstract. Let $\mathcal{A}$ be a factor von Neumann algebra and Φ preserve n-tuple new product derivations on $\mathcal{A}$, that is, for every $A_1, A_2, \dots, A_n \in \mathcal{A}$,

$$\Phi(A_1 \diamond A_2 \diamond \cdots \diamond A_n) = \Phi(A_1) \diamond A_2 \diamond \cdots \diamond A_n + A_1 \diamond \Phi(A_2) \diamond \cdots \diamond A_n + \cdots + A_1 \diamond A_2 \diamond \cdots \diamond \Phi(A_n)$$

where $A_i \diamond A_j = A_i^* A_j - A_j^* A_i$ for $i, j \in \mathbb{N}$, then Φ is additive $*$-derivation, on the condition that $\Phi\big(\alpha \frac{I}{2}\big)$ is self-adjoint operator for $\alpha \in \{1, i\}$.

1. Introduction

Let $\mathcal{R}$ be a factor von Neumann algebras. For $A, B \in \mathcal{R}$, denoted by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are $*$-Jordan product and $*$-Lie product, respectively. These products are found playing a more and more important role in some research topics, and its study has recently attracted many author's attention (for example, see [**5**, **10**, **11**, **16**]).

Recall that a map $\Phi\colon \mathcal{R} \to \mathcal{R}$ is said to be an additive derivation if

$$\Phi(A + B) = \Phi(A) + \Phi(B) \quad \text{and} \quad \Phi(AB) = \Phi(A)B + A\Phi(B)$$

for all $A, B \in \mathcal{R}$. A map Φ is additive $*$-derivation if it is an additive derivation and $\Phi(A^*) = \Phi(A)^*$. Derivations are very important maps both in theory and applications, in particular to study the topological and fundamental structures of von Neumann algebras [**4**]. This notion has been studied intensively [**3**, **12**–**14**].

Let us define λ-Jordan $*$-product by $A \bullet_\lambda B = AB + \lambda BA^*$. We say that the map Φ with the property of $\Phi(A \bullet_\lambda B) = \Phi(A) \bullet_\lambda B + A \bullet_\lambda \Phi(B)$ is a λ-Jordan $*$-derivation map. It is clear that for $\lambda = -1$ and $\lambda = 1$, the λ-Jordan $*$-derivation map is a $*$-Lie derivation and $*$-Jordan derivation, respectively [**1**].

A von Neumann algebra $\mathcal{A}$ is a self-adjoint subalgebra of some $B(H)$, the algebra of bounded linear operators acting on a complex Hilbert space, which satisfies

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the double commutant property: $\mathcal{A}'' = \mathcal{A}$ where $\mathcal{A}' = \{T \in B(H), TA = AT, \forall A \in \mathcal{A}\}$ and $\mathcal{A}'' = \{\mathcal{A}'\}'$. Denote by $\mathcal{Z}(\mathcal{A}) = \mathcal{A}' \cap \mathcal{A}$ the center of $\mathcal{A}$. A von Neumann algebra $\mathcal{A}$ is called a factor if its center is trivial, that is, $\mathcal{Z}(\mathcal{A}) = \mathbb{C}I$ [**7**, **8**].

Recently, Yu and Zhang in [**17**] proved that every non-linear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation. Also, Li, Lu and Fang in [**9**] have investigated a non-linear λ-Jordan $*$-derivation. They showed that if $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ is a von Neumann algebra without central abelian projections and λ is a non-zero scalar, then $\Phi\colon \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a non-linear λ-Jordan $*$-derivation if and only if Φ is an additive $*$-derivation.

Very recently the authors of [**2**] discussed some bijective maps preserving the new product $A^*B + B^*A$ between von Neumann algebras with no central abelian projections. In other words, Φ holds in the following condition

$$\Phi(A^*B + B^*A) = \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A).$$

They showed that such a map is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism. In [**15**], the authors have discussed such maps between unital prime *-algebras which preserve the new triple product.

In [**18**] the authors considered nonlinear $*$-Lie higher derivations on factor von Neumann algebras acting on a complex Hilbert space H with $\dim H \geqslant 2$.

Also, the authors of [**6**] obtained the following result: Let $\mathcal{A}$ and $\mathcal{B}$ be two factor von Neumann algebras with $I_{\mathcal{A}}$ and $I_{\mathcal{B}}$ the identities of them, respectively. Let $A_1, A_2, \dots, A_n \in \mathcal{A}$ and $A_1 \bullet A_2 \bullet \cdots \bullet A_n$ is the Jordan multiple $*$-product in any fixed operation order. Then a not necessarily linear bijective mapping $\Phi\colon \mathcal{A} \to \mathcal{B}$ satisfies

$$\Phi(A_1 \bullet A_2 \bullet \cdots \bullet A_n) = \Phi(A_1) \bullet \Phi(A_2) \bullet \cdots \bullet \Phi(A_n)$$

if and only if Φ is a $*$-ring isomorphism.

In this paper motivated by the above results, we consider a map (not necessarily linear) Φ on factor von Neumann algebras $\mathcal{A}$ which meets the following conditions:

$$\begin{aligned}\Phi(A_1 \diamond A_2 \diamond \ldots \diamond A_n) &= \Phi(A_1) \diamond A_2 \diamond \ldots \diamond A_n + A_1 \diamond \Phi(A_2) \diamond \ldots \diamond A_n \\ &\quad + \cdots + A_1 \diamond A_2 \diamond \ldots \diamond \Phi(A_n)\end{aligned}$$

where $A_i \diamond A_j = A_i^* A_j - A_j^* A_i$ for $i, j \in \mathbb{N}$. We prove that Φ is additive $*$-derivation. The results of this essay will be applied to study general structures of von Neumann algebras which are related to non-linear derivatives.

When we say that $\mathcal{A}$ is prime, that is, for $A, B \in \mathcal{A}$ if $A\mathcal{A}B = \{0\}$, then $A = 0$ or $B = 0$.

2. Main result

Theorem 2.1. *Let $\mathcal{A}$ be a factor von Neumann algebras. Let $\Phi : \mathcal{A} \to \mathcal{A}$ satisfies in*

$$\begin{aligned}\Phi(A_1 \diamond A_2 \diamond \ldots \diamond A_n) &= \Phi(A_1) \diamond A_2 \diamond \ldots \diamond A_n + A_1 \diamond \Phi(A_2) \diamond \ldots \diamond A_n \\ &\quad + \cdots + A_1 \diamond A_2 \diamond \ldots \diamond \Phi(A_n), \quad n \geqslant 2\end{aligned}$$

for all $A_1, \dots, A_n \in \mathcal{A}$ where $A_i \diamond A_j = A_i^* A_j - A_j^* A_i$ for $i, j \in \mathbb{N}$. Then Φ is additive $*$-derivation.

PROOF. Let P_1 be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_{\mathcal{A}} - P_1$. Denote $\mathcal{A}_{ij} = P_i\mathcal{A}P_j,\ i,j = 1,2$, then $\mathcal{A} = \sum_{i,j=1}^{2}\mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we can write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write A_{ij}, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. For showing additivity of Φ on $\mathcal{A}$, we use above partition of $\mathcal{A}$ and give some claims that prove Φ is additive on each $\mathcal{A}_{ij}$, $i,j = 1,2$.

We prove the above theorem by several claims.

CLAIM 2.1. We show that $\Phi(0) = 0$.

This claim is easy to prove.

CLAIM 2.2. We show that $\Phi\big(\frac{I}{2}\big) = 0$, $\Phi\big(-\frac{I}{2}\big) = 0$ and $\Phi\big(i\frac{I}{2}\big) = 0$.

Since

$$\Phi\Big(i\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}\Big) = \Phi\Big(i\frac{I}{2}\Big)\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2} + i\frac{I}{2}\diamond\Phi\Big(\frac{I}{2}\Big)\diamond\ldots\diamond\frac{I}{2} + \cdots + i\frac{I}{2}\diamond i\frac{I}{2}\diamond\ldots\diamond\Phi\Big(\frac{I}{2}\Big),$$

an elementary computation shows that two cases appear depending on n:

$$\Phi\Big(i\frac{I}{2}\Big) = \frac{1}{2}\Big(\Phi\Big(i\frac{I}{2}\Big) - \Phi\Big(i\frac{I}{2}\Big)^*\Big) + (n-1)\frac{i}{2}\Big(\Phi\Big(\frac{I}{2}\Big) + \Phi\Big(\frac{I}{2}\Big)^*\Big), \quad n \text{ odd},$$
$$\Phi\Big(-i\frac{I}{2}\Big) = -\frac{1}{2}\Big(\Phi\Big(i\frac{I}{2}\Big) - \Phi\Big(i\frac{I}{2}\Big)^*\Big) - (n-1)\frac{i}{2}\Big(\Phi\Big(\frac{I}{2}\Big) + \Phi\Big(\frac{I}{2}\Big)^*\Big), \quad n \text{ even}$$

which imply that

$$\begin{aligned}&\frac{1}{2}\Big(-\Phi\Big(i\frac{I}{2}\Big) - \Phi\Big(i\frac{I}{2}\Big)^*\Big) + (n-1)\frac{i}{2}\Big(\Phi\Big(\frac{I}{2}\Big) + \Phi\Big(\frac{I}{2}\Big)^*\Big) = 0, \quad n \text{ odd},\\ &\Phi\Big(-i\frac{I}{2}\Big) + \frac{1}{2}\Big(\Phi\Big(i\frac{I}{2}\Big) - \Phi\Big(i\frac{I}{2}\Big)^*\Big) + (n-1)\frac{i}{2}\Big(\Phi\Big(\frac{I}{2}\Big) + \Phi\Big(\frac{I}{2}\Big)^*\Big) = 0. \quad n \text{ even}\end{aligned} \tag{2.1}$$

By taking adjoint from both side of (2.1), for odd numbers case, we obtain

$$\frac{1}{2}\Big(-\Phi\Big(i\frac{I}{2}\Big)^* - \Phi\Big(i\frac{I}{2}\Big)\Big) - (n-1)\frac{i}{2}\Big(\Phi\Big(\frac{I}{2}\Big) + \Phi\Big(\frac{I}{2}\Big)^*\Big) = 0 \tag{2.2}$$

Hence (2.1) and (2.2), for odd case, give us $\Phi\big(\frac{iI}{2}\big) + \Phi\big(\frac{iI}{2}\big)^* = 0$. Since $\Phi\big(\frac{iI}{2}\big)$ is self-adjoint, $\Phi\big(\frac{iI}{2}\big) = 0$. By re-applying (2.2) it results that $\Phi\big(\frac{I}{2}\big) + \Phi\big(\frac{I}{2}\big)^* = 0$. So $\Phi\big(\frac{I}{2}\big) = 0$, becuse $\Phi\big(\frac{I}{2}\big)^*$ is self-adjoint. Similarly we can show that $\Phi\big(-i\frac{I}{2}\big) = 0$. Set zero valued in extension of $\Phi\big(i\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}\big)$ for even case, we have

$$\Phi\Big(-i\frac{I}{2}\Big) = 0.$$

As well as, the self-adjoint assumption clarify that $\Phi\big(-i\frac{I}{2}\big)^* = 0$.

CLAIM 2.3. For each $A \in \mathcal{A}$, we have

$$(1)\quad \Phi(-iA) = -i\Phi(A). \quad (2)\quad \Phi(iA) = i\Phi(A).$$

It is straightforward to see that

$$\Phi\Big(-iA\diamond\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}\Big) = \Phi\Big(A\diamond i\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\frac{I}{2}\Big).$$

So,

$$\Phi(-iA)\diamond\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}=\Phi(A)\diamond i\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}.$$

It follows that

$$-\Phi(-iA)^*+\Phi(-iA)=-i\Phi(A)^*-i\Phi(A). \tag{2.3}$$

On the other hand, one can check that

$$\Phi\Big(-iA\diamond i\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}\Big)=\Phi\Big(A\diamond-\frac{I}{2}\diamond\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}\Big).$$

So,

$$\Phi(-iA)\diamond i\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}=\Phi(A)\diamond-\frac{I}{2}\diamond\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}.$$

It follows that $-i\Phi(-iA)^*-i\Phi(-iA)=\Phi(A)^*-\Phi(A)$.

Equivalently, we obtain

$$-\Phi(-iA)^*+\Phi(-iA)=i\Phi(A)^*-i\Phi(A). \tag{2.4}$$

By adding equations (2.3) and (2.4), we have $\Phi(-iA)=-i\Phi(A)$.

Similarly it can be shown $\Phi(iA)=i\Phi(A)$.

CLAIM 2.4. For each $A_{11}\in\mathcal{A}_{11}$, $A_{12}\in\mathcal{A}_{12}$, we have

$$\Phi(A_{11}+A_{12})=\Phi(A_{11})+\Phi(A_{12}).$$

Let $T=\Phi(A_{11}+A_{12})-\Phi(A_{11})-\Phi(A_{12})$. We need to show that $T=0$.

$$\begin{aligned}
&\Phi(A_{11}+A_{12})\diamond C_{21}\diamond I\diamond\ldots\diamond I+(A_{11}+A_{12})\diamond\Phi(C_{21})\diamond I\diamond\ldots\diamond I\\
&\quad+\cdots+(A_{11}+A_{12})\diamond C_{21}\diamond I\diamond\ldots\diamond\Phi(I)=\Phi(A_{11}+A_{12}\diamond C_{21}\diamond I\diamond\ldots\diamond I)\\
&=\Phi(A_{11}\diamond C_{21}\diamond I\diamond\ldots\diamond I)+\Phi(A_{12}\diamond C_{21}\diamond I\diamond\ldots\diamond I)\\
&=\Phi(A_{11})\diamond C_{21}\diamond I\diamond\ldots\diamond I+A_{11}\diamond\Phi(C_{21})\diamond I\diamond\ldots\diamond I\\
&\quad+\cdots+A_{11}\diamond C_{21}\diamond I\diamond\ldots\diamond\Phi(I)+\Phi(A_{12})\diamond C_{21}\diamond I\diamond\ldots\diamond I\\
&\quad+A_{12}\diamond\Phi(C_{21})\diamond I\diamond\ldots\diamond I+\cdots+A_{12}\diamond C_{21}\diamond I\diamond\ldots\diamond\Phi(I)\\
&=(\Phi(A_{11})+\Phi(A_{12}))\diamond C_{21}\diamond I\diamond\ldots\diamond I+(A_{11}+A_{12})\diamond\Phi(C_{21})\diamond I\diamond\ldots\diamond I\\
&\quad+\cdots+(A_{11}+A_{12})\diamond C_{21}\diamond I\diamond\diamond\ldots\diamond\Phi(I).
\end{aligned}$$

Then $T\diamond C_{21}\diamond I\diamond I\diamond\ldots I=0$. According to initial discussion of proof, we know $T=T_{11}+T_{12}+T_{21}+T_{22}$. So $-T_{22}^*C_{21}-T_{21}^*C_{21}+C_{21}^*T_{22}+C_{21}^*T_{21}=0$. Thus we have $T_{22}=T_{21}=0$. In a similar way, we have

$$\begin{aligned}
&\Phi(A_{11}+A_{12})\diamond C_{12}\diamond P_1\diamond I\diamond\ldots\diamond I+(A_{11}+A_{12})\diamond\Phi(C_{12})\diamond P_1\diamond\ldots\diamond I\\
&\quad+\cdots+(A_{11}+A_{12})\diamond C_{12}\diamond P_1\diamond\ldots\diamond\Phi(I)=\Phi((A_{11}+A_{12})\diamond C_{12}\diamond P_1\diamond\ldots\diamond I)\\
&=\Phi(A_{11}\diamond C_{12}\diamond P_1\diamond I\diamond\ldots\diamond I)+\Phi(A_{12}\diamond C_{12}\diamond P_1\diamond I\diamond\ldots\diamond I)\\
&=(\Phi(A_{11})+\Phi(A_{12}))\diamond C_{12}\diamond P_1\diamond I\diamond I\diamond\ldots\diamond I+(A_{11}+A_{12})\diamond\Phi(C_{12})\diamond P_1\diamond I\diamond\ldots\diamond I\\
&\quad+\cdots+(A_{11}+A_{12})\diamond C_{12}\diamond P_1\diamond I\diamond\ldots\diamond\Phi(I).
\end{aligned}$$

So $T\diamond C_{12}\diamond P_1\diamond I\diamond\ldots\diamond I=0$. An early application of the unique decomposition of $T=T_{11}+T_{12}+T_{21}+T_{22}$ shows that $-T_{11}^*C_{12}+C_{12}^*T_{11}=0$. Now, multiplying the above equality from right side by P_1 and left side by P_2, and using the primitivity assumption of $\mathcal{A}$ we show that $T_{11} = 0$. Similarly, one can show that $T_{12} = 0$ by applying C_{21} instead of C_{12} as above.

CLAIM 2.5. For each $A_{11} \in \mathcal{A}_{11}$, $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$ and $A_{22} \in \mathcal{A}_{22}$, we have

(1) $\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})$.
(2) $\Phi(A_{12} + A_{21} + A_{22}) = \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})$.

We show that $T = \Phi(A_{11} + A_{12} + A_{21}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) = 0$. From Claim 2.4, we obtain

$$\begin{aligned}
&\Phi(A_{11} + A_{12} + A_{21}) \diamond C_{21} \diamond I \diamond \ldots \diamond I + (A_{11} + A_{12} + A_{21}) \diamond \Phi(C_{21}) \diamond I \diamond \ldots \diamond I \\
&\qquad + \cdots + (A_{11} + A_{12} + A_{21}) \diamond C_{21} \diamond I \diamond \ldots \diamond \Phi(I) \\
&= \Phi(A_{11} + A_{21} + A_{12} \diamond C_{21} \diamond I \diamond \ldots \diamond I) = \Phi((A_{11} + A_{21}) \diamond C_{21} \diamond I \diamond \ldots \diamond I) \\
&\qquad + \Phi(A_{12} \diamond C_{21} \diamond I \diamond \ldots \diamond I) = \Phi(A_{11} \diamond C_{21} \diamond I \diamond \ldots \diamond I) \\
&\qquad + \Phi(A_{21} \diamond C_{21} \diamond I \diamond \ldots \diamond I) + \Phi(A_{12} \diamond C_{21} \diamond I \diamond \ldots \diamond I) \\
&= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})) \diamond C_{21} \diamond I \diamond \ldots \diamond I + (A_{11} + A_{12} + A_{21}) \\
&\qquad \diamond C_{21} \diamond I \diamond \ldots \diamond I + \cdots + (A_{11} + A_{12} + A_{21}) \diamond C_{21} \diamond I \diamond \ldots \diamond \Phi(I).
\end{aligned}$$

It follows that $T \diamond C_{21} \diamond I \diamond \ldots \diamond I = 0$. Since $T = T_{11} + T_{12} + T_{21} + T_{22}$ we have $-T_{22}^*C_{21} - T_{21}^*C_{21} + C_{21}^*T_{22} + C_{21}^*T_{21} = 0$. Therefore, $T_{22} = T_{21} = 0$.

Claim 2.4, by an immediately computation, shows

$$\begin{aligned}
&\Phi(A_{11} + A_{12} + A_{21}) \diamond P_1 \diamond P_1 \diamond I \diamond \ldots \diamond I + (A_{11} + A_{12} \\
&\qquad + A_{21}) \diamond \Phi(P_1) \diamond P_1 \diamond I \diamond \ldots \diamond I \\
&\qquad + \cdots + (A_{11} + A_{12} + A_{21}) \diamond P_1 \diamond P_1 \diamond I \diamond \ldots \diamond \Phi(I) \\
&= \Phi((A_{11} + A_{12} + A_{21}) \diamond P_1 \diamond P_1 \diamond I \diamond \ldots \diamond I) \\
&= \Phi((A_{11} + A_{12}) \diamond P_1 \diamond P_1 \diamond I \diamond \ldots \diamond I) \\
&\qquad + \Phi(A_{21} \diamond P_1 \diamond P_1 \diamond I \diamond \ldots \diamond I) = \Phi(A_{11} \diamond P_1 \diamond P_1 \diamond I \ldots \diamond I) \\
&\qquad + \Phi(A_{12} \diamond P_1 \diamond P_1 \diamond I \diamond \ldots \diamond I) + \Phi(A_{21} \diamond P_1 \diamond P_1 \diamond I \diamond \ldots \diamond I) \\
&= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})) \diamond P_1 \diamond P_1 \diamond I \diamond \ldots \diamond I \\
&\qquad + (A_{11} + A_{12} + A_{21}) \diamond \Phi(P_1) \diamond P_1 \diamond I \diamond \ldots \diamond I \\
&\qquad + \cdots + (A_{11} + A_{12} + A_{21}) \diamond P_1 \diamond P_1 \diamond I \diamond \ldots \diamond \Phi(I).
\end{aligned}$$

Hereupon $T \diamond P_1 \diamond P_1 \diamond \ldots \diamond I = 0$. Thus $2T_{11} - 2T_{11}^* + T_{12} - T_{12}^* = 0$ which implies

$$T_{12} = 0, \quad T_{11} - T_{11}^* = 0. \tag{2.5}$$

Also from Claims 2.3 and 2.4, we have

$$\begin{aligned}
&\Phi(A_{11} + A_{12} + A_{21}) \diamond iP_1 \diamond I \diamond \ldots \diamond I + (A_{11} + A_{12} + A_{21}) \diamond \Phi(iP_1) \diamond I \diamond \ldots \diamond I \\
&\qquad + \cdots + (A_{11} + A_{12} + A_{21}) \diamond iP_1 \diamond I \diamond \ldots \diamond \Phi(I) \\
&= \Phi((A_{11} + A_{12} + A_{21}) \diamond iP_1 \diamond I \diamond \ldots \diamond I) = \Phi((A_{11} + A_{12}) \diamond iP_1 \diamond I \diamond \ldots \diamond I)
\end{aligned}$$

$$+\,\Phi(A_{21}\diamond iP_1\diamond I\diamond\ldots\diamond I)=\Phi(A_{11}\diamond iP_1\diamond I\diamond\ldots\diamond I)$$
$$+\,\Phi(A_{12}\diamond iP_1\diamond I\diamond\ldots\diamond I)+\Phi(A_{21}\diamond iP_1\diamond I\diamond\ldots\diamond I)$$
$$=((\Phi(A_{11}+\Phi(A_{12}+\Phi(A_{21}))\diamond iP_1\diamond I\diamond\ldots\diamond I+(A_{11}+A_{12}+A_{21})\diamond\Phi(iP_1)\diamond I\diamond\ldots\diamond I$$
$$+\cdots+(A_{11}+A_{12}+A_{21})\diamond iP_1\diamond I\diamond\ldots\diamond\Phi(I)$$

So $T\diamond iP_1\diamond I\diamond\ldots\diamond I=0$ which implies

$$-T_{11}-T_{11}^*=0. \tag{2.6}$$

From (2.5), (2.6), we have $T_{11}=0$.

Similarly, we can show that $\Phi(A_{12}+A_{21}+A_{22})=\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22})$.

CLAIM 2.6. For each $A_{11}\in\mathcal{A}_{11}$, $A_{12}\in\mathcal{A}_{12}$, $A_{21}\in\mathcal{A}_{21}$ and $A_{22}\in\mathcal{A}_{22}$, we have

$$\Phi(A_{11}+A_{12}+A_{21}+A_{22})=\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})+\Phi(A_{22}).$$

In order to prove this claim, we have to show that

$$T=\Phi(A_{11}+A_{12}+A_{21}+A_{22})-\Phi(A_{11})-\Phi(A_{12})-\Phi(A_{21})-\Phi(A_{22})=0.$$

From claim (2.5), we have

$$\Phi(A_{11}+A_{12}+A_{21}+A_{22})\diamond C_{12}\diamond I\diamond\ldots\diamond I$$
$$+\,(A_{11}+A_{12}+A_{21}+A_{22})\diamond\Phi(C_{12})\diamond I\diamond\ldots\diamond I$$
$$+\cdots+(A_{11}+A_{12}+A_{21}+A_{22})\diamond C_{12}\diamond I\diamond\ldots\diamond\Phi(I)$$
$$=\Phi((A_{11}+A_{12}+A_{21}+A_{22})\diamond C_{12}\diamond I\diamond\ldots\diamond I)$$
$$=(\Phi(A_{11}+A_{12}+A_{21})\diamond C_{12}\diamond I\diamond\ldots\diamond I)+\Phi(A_{22}\diamond C_{12}\diamond I\diamond\ldots\diamond I)$$
$$=\Phi(A_{11}\diamond C_{12}\diamond I\diamond\ldots\diamond I)+\Phi(A_{12}\diamond C_{12}\diamond I)+\Phi(A_{21}\diamond C_{12}\diamond I\diamond\ldots\diamond I)$$
$$+\,\Phi(A_{22}\diamond C_{12}\diamond I\diamond\ldots\diamond I)=(\Phi(A_{11})+\Phi(A_{12})+\Phi(A_{21})$$
$$+\,\Phi(A_{22}))\diamond C_{12}\diamond I\diamond\ldots\diamond I+(A_{11}+A_{12}+A_{21}+A_{22})\diamond\Phi(C_{12})\diamond I\diamond\ldots\diamond I$$
$$+\cdots+(A_{11}+A_{12}+A_{21}+A_{22})\diamond C_{12}\diamond I\diamond\ldots\diamond\Phi(I).$$

So, $T\diamond C_{12}\diamond I=0$. It follows that $C_{12}^*T_{11}+C_{12}^*T_{12}-T_{11}^*C_{12}-T_{12}^*C_{12}=0$. Thus $T_{11}=T_{12}=0$.

Similarly, by applying C_{21} instead of C_{12} in the above computation, we obtain $T_{21}=T_{22}=0$.

CLAIM 2.7. For each $A_{ij},B_{ij}\in\mathcal{A}_{ij}$ such that $i\neq j$, we have

$$\Phi(A_{ij}+B_{ij})=\Phi(A_{ij})+\Phi(B_{ij}).$$

It is easy to show that

$$(P_i+A_{ij}^*)\diamond(P_j+B_{ij})\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}=A_{ij}^*+B_{ij}^*-A_{ij}-B_{ij}.$$

So, we can write

$$\Phi(-A_{ij}-B_{ij})+\Phi(A_{ij}^*+B_{ij}^*)=\Phi\Big(\Big(P_i+A_{ij}^*\Big)\diamond(P_j+B_{ij})\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}\Big)$$

$$= \Phi(P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}$$
$$+ (P_i + A_{ij}^*) \diamond \Phi(P_j + B_{ij}) \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}$$
$$+ (P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \Phi\Big(\frac{I}{2}\Big) \diamond \ldots \diamond \frac{I}{2}$$
$$+ \cdots + (P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \frac{I}{2} \diamond \ldots \diamond \Phi\Big(\frac{I}{2}\Big)$$
$$= (\Phi(P_i) + \Phi(A_{ij}^*)) \diamond (P_j + B_{ij}) \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}$$
$$+ (P_i + A_{ij}^*) \diamond (\Phi(P_j) + \Phi(B_{ij})) \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2} + \cdots + (P_i + A_{ij}^*) \diamond (P_j + B_{ij}) \diamond \frac{I}{2} \diamond \ldots \diamond \Phi\Big(\frac{I}{2}\Big)$$
$$= \Phi\Big(P_i \diamond B_{ij} \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}\Big) + \Phi\Big(A_{ij}^* \diamond P_j \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}\Big)$$
$$= \Phi(-B_{ij}) + \Phi(B_{ij}^*) + \Phi(-A_{ij}) + \Phi(A_{ij}^*)$$

Therefore, we show that

$$\Phi(-A_{ij} - B_{ij}) + \Phi(A_{ij}^* + B_{ij}^*) = \Phi(-A_{ij}) + \Phi(-B_{ij}) + \Phi(A_{ij}^*) + \Phi(B_{ij}^*). \tag{2.7}$$

By an easy computation, we can write

$$(P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2} \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2} = iA_{ij} + iB_{ij} + iA_{ij}^* + iB_{ij}^*.$$

Then, we have

$$\Phi(iA_{ij} + iB_{ij}) + \Phi(iA_{ij}^* + iB_{ij}^*) = \Phi\Big((P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}\Big)$$
$$= \Phi(P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2} \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}$$
$$+ (P_i + A_{ij}^*) \diamond \Phi(iP_j + iB_{ij}) \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}$$
$$+ (P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \Phi\Big(\frac{I}{2}\Big) \diamond \ldots \diamond \frac{I}{2}$$
$$+ \cdots + (P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2} \diamond \ldots \diamond \Phi\Big(\frac{I}{2}\Big)$$
$$= (\Phi(P_i) + \Phi(A_{ij}^*)) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}$$
$$+ (P_i + A_{ij}^*) \diamond (\Phi(iP_j) + \Phi(iB_{ij})) \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2} + \cdots +$$
$$+ (P_i + A_{ij}^*) \diamond (iP_j + iB_{ij}) \diamond \frac{I}{2} \diamond \ldots \diamond \Phi\Big(\frac{I}{2}\Big)$$
$$= \Phi\Big(P_i \diamond iB_{ij} \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}\Big) + \Phi\Big(A_{ij}^* \diamond iP_j \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2}\Big)$$
$$= \Phi(iB_{ij}) + \Phi(iB_{ij}^*) + \Phi(iA_{ij}) + \Phi(iA_{ij}^*).$$

Thus we showed that

$$\Phi(iA_{ij} + iB_{ij}) + \Phi(iA_{ij}^* + iB_{ij}^*) = \Phi(iA_{ij}) + \Phi(iB_{ij}) + \Phi(iA_{ij}^*) + \Phi(iB_{ij}^*).$$

From claim 2.3 and the above equation, we have

$$-\Phi(A_{ij}+B_{ij})-\Phi(A_{ij}^*+B_{ij}^*)=-\Phi(B_{ij})-\Phi(B_{ij}^*)-\Phi(A_{ij})-\Phi(-A_{ij}^*). \tag{2.8}$$

By adding equations (2.7) and (2.8), we obtain $\Phi(A_{ij}+B_{ij})=\Phi(A_{ij})+\Phi(B_{ij})$.

CLAIM 2.8. For each $A_{ii}, B_{ii}\in\mathcal{A}_{ii}$ such that $1\leqslant i\leqslant 2$, we have

$$\Phi(A_{ii}+B_{ii})=\Phi(A_{ii})+\Phi(B_{ii}).$$

We show that $T=\Phi(A_{ii}+B_{ii})-\Phi(A_{ii})-\Phi(B_{ii})=0$. We can write

$$\begin{aligned}\Phi(A_{ii}+B_{ii})\diamond P_j\diamond I\diamond I\diamond\ldots\diamond I+(A_{ii}+B_{ii})\diamond\Phi(P_j)\diamond I\diamond\ldots\diamond I&\\ +\cdots+(A_{ii}+B_{ii})\diamond P_j\diamond I\diamond\ldots\diamond\Phi(I)&\\ =\Phi((A_{ii}+B_{ii})\diamond P_j\diamond I\diamond\ldots\diamond I)=\Phi(A_{ii}\diamond P_j\diamond I\diamond\ldots\diamond I)&\\ +\Phi(B_{ii}\diamond P_j\diamond I\diamond\ldots\diamond I)=\Phi(A_{ii})\diamond P_j\diamond I\diamond\ldots\diamond I+A_{ii}\diamond\Phi(P_j)\diamond I\diamond\ldots\diamond I&\\ +\cdots+A_{ii}\diamond P_j\diamond I\diamond\ldots\diamond\Phi(I)+\Phi(B_{ii})\diamond P_j\diamond I\diamond\ldots\diamond I&\\ +B_{ii}\diamond\Phi(P_j)\diamond I\diamond\ldots\diamond I+\cdots+B_{ii}\diamond P_j\diamond I\diamond\ldots\diamond\Phi(I)&\\ =(\Phi(A_{ii})+\Phi(B_{ii}))\diamond P_j\diamond I\diamond I\diamond\ldots\diamond I+(A_{ii}+B_{ii})\diamond\Phi(P_j)\diamond I\diamond\ldots\diamond I&\\ +\cdots+(A_{ii}+B_{ii})\diamond P_j\diamond I\diamond\ldots\diamond\Phi(I)&\end{aligned}$$

So, we have $T\diamond P_j\diamond I\diamond\ldots\diamond I=0$. Therefore, we obtain $T_{ij}=T_{ji}=T_{jj}=0$.

On the other hand, for every $C_{ij}\in\mathcal{A}_{ij}$, we have

$$\begin{aligned}\Phi(A_{ii}+B_{ii})\diamond C_{ij}\diamond I\diamond\ldots\diamond I+(A_{ii}+B_{ii})\diamond\Phi(C_{ij})\diamond I\diamond\ldots\diamond I&\\ +\cdots+(A_{ii}+B_{ii})\diamond C_{ij}\diamond I\diamond\ldots\diamond\Phi(I)&\\ =\Phi((A_{ii}+B_{ii})\diamond C_{ij}\diamond I\diamond\ldots\diamond I)=\Phi(A_{ii}\diamond C_{ij}\diamond I\diamond\ldots\diamond I)&\\ +\Phi(B_{ii}\diamond C_{ij}\diamond I\diamond\ldots\diamond I)=(\Phi(A_{ii})+\Phi(B_{ii}))\diamond C_{ij}\diamond I\diamond\ldots\diamond I&\\ +(A_{ii}+B_{ii})\diamond\Phi(C_{ij})\diamond I\diamond\ldots\diamond I+(A_{ii}+B_{ii})\diamond C_{ij}\diamond\Phi(I)\diamond\ldots\diamond I&\\ +\cdots+(A_{ii}+B_{ii})\diamond C_{ij}\diamond I\diamond\ldots\diamond\Phi(I)&\end{aligned}$$

So $T\diamond C_{ij}\diamond I\diamond\ldots\diamond I=0$ which shows that $T_{ii}\diamond C_{ij}\diamond I\diamond\ldots\diamond I=0$. Thus we see that $-T_{ii}^*C_{ij}+C_{ij}^*T_{ii}=0$. Again multiplying the above equality from right by P_i and from left by P_j and using the well-known fact that $\mathcal{A}$ is prime, we have $T_{ii}=0$.

Hence, the additivity of Φ comes from the above claims.

In the rest of this paper, we show that Φ is $*$-derivation.

CLAIM 2.9. Φ preserves star.

Since $\Phi\big(\frac{I}{2}\big)=0$ then we can write $\Phi\big(A\diamond\frac{I}{2}\diamond\frac{I}{2}\diamond\diamond\ldots\diamond\frac{I}{2}\big)=\Phi(A)\diamond\frac{I}{2}\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}$. Then $\Phi(A-A^*)=\Phi(A)-\Phi(A)^*$. So, we showed that Φ preserves star.

CLAIM 2.10. We prove that Φ is derivation.

For every $A,B\in\mathcal{A}$, a simple calculation shows that

$$\begin{aligned}\Phi(-2AB+2B^*A^*)&=\Phi\Big(A^*\diamond B\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}\diamond I\Big)\\ &=\Phi(A^*)\diamond B\diamond\frac{I}{2}\diamond\ldots\diamond\frac{I}{2}\diamond I\end{aligned}$$

$$+\cdots + A^* \diamond B \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2} \diamond \Phi(I)$$
$$= 2B^*\Phi(A)^* - 2\Phi(A)B - 2A\Phi(B) + \Phi(B)^*A^*.$$

So we have

$$\Phi(B^*A^* - AB) = \Phi(B)^*A^* + B^*\Phi(A)^* - \Phi(A)B - A\Phi(B). \tag{2.9}$$

From Claims 2.3 and 2.9, we have

$$\begin{aligned}\Phi(-2B^*A^* - 2AB) &= \Phi\Big(iA^* \diamond B \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2} \diamond I\Big)\\ &= \Phi(iA^*) \diamond B \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2} \diamond I\\ &\quad + \cdots + (iA^*) \diamond B \diamond \frac{I}{2} \diamond \ldots \diamond \frac{I}{2} \diamond \Phi(I).\end{aligned}$$

Then

$$i\Phi(-B^*A^* - AB) = i(-\Phi(B)^*A^* - B^*\Phi(A)^* - \Phi(A)B - A\Phi(B)).$$

So we can show that

$$\Phi(-B^*A^* - AB) = -\Phi(B)^*A^* - B^*\Phi(A)^* - \Phi(A)B - A\Phi(B). \tag{2.10}$$

Eventually, equations (2.9) and (2.10) shows that $\Phi(AB) = \Phi(A)B + A\Phi(B)$. □

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