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CHARACTERIZATION OF TIGHT WAVELET FRAMES WITH COMPOSITE DILATIONS IN $L^2(\mathbb{R}^n)$

Owais Ahmad

Abstract. Tight wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, the tight wavelet frames become much easier to construct than the orthonormal wavelets. Guo, Labate, Lim, Weiss, and Wilson [Electron. Res. Announc. Am. Math. Soc. **10** (2004), 78–87] introduced the theory of wavelets with composite dilations in order to provide a framework for the construction of waveforms defined not only at various scales and locations but also at various orientations. In this paper, we provide the characterization of composite wavelet system to be tight frame for $L^2(\mathbb{R}^n)$

1. Introduction

Most of the signals in nature are non-stationary and a complete representation of these signals requires frequency analysis that is local in time, resulting in the time-frequency analysis of signals. Although time-frequency analysis of signals had its origin almost 60 years ago, there has been a major development of the time-scale and time-frequency analysis approach in the last few decades and many new transforms have been introduced to analyze the non-stationary and multi-component signals in the joint time-frequency domain. In the framework of mathematical analysis and linear algebra, redundant representations are obtained by analysing vectors with respect to an overcomplete system. Then the obtained vectors are interpreted using the frame theory as introduced by Duffin and Schaeffer [**12**] and recently studied at depth, see [**9**] and the compressive list of references therein. Most commonly used coherent/structured frames are wavelet, Gabor, and wave-packet frames which are a mixture type of wavelet and Gabor frames [**9**, **11**]. Frames provide a useful model to obtain signal decompositions in cases where redundancy, robustness, over-sampling, and irregular sampling ploy a role. Today, the theory of frames has become an interesting and fruitful field of mathematics with abundant applications in signal processing, image processing, harmonic analysis, Banach

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space theory, sampling theory, wireless sensor networks, optics, filter banks, quantum computing, and medicine.

Though the importance of wavelets in signal processing applications is widely acknowledged, it is well-known that wavelets do not perform as well in higher dimensions. The situation is illustrated, for example, by the classical problem of representing a natural image using a 2-D wavelet basis. Natural images exhibit discontinuous and irregular edges along curves. Because these discontinuities are spatially distributed, they interact extensively with the elements of the wavelet basis, and the wavelet representation is not sparse, that is, "many" wavelet coefficients are needed to represent the edges accurately. This limitation has led to several new constructions [**21**], in order to handle efficiently the geometric features of multidimensional signals. These constructions include the ridgelets [**5**], the directional wavelets [**10**] and the curvelets [**6**]. The main idea, in all of these constructions, is that such representations must contain basis elements with more shapes and directions than the classical wavelet bases in order to obtain efficient representations of multidimensional signals with spatially distributed discontinuities.

One of structured frames are wavelet frames which are obtained by translating and dilating a finite number of functions. Wavelet frames are different from the orthonormal wavelets because of redundancy. By sacrificing orthonormality and allowing redundancy, wavelet frames become much easier to construct than the orthonormal wavelets. An important problem in practice is therefore to determine conditions on the wavelet function, dilation and translation parameters so that the corresponding wavelet system forms a frame. In her famous book, Daubechies [**11**] proved the first result on the necessary and sufficient conditions for wavelet frames, and then, Chui and Shi [**8**] gave an improved result. After about ten years, Casazza and Christenson [**7**] proved a stronger version of Daubechies' sufficient condition for wavelet frames in $L^2(\mathbb{R})$. The first author and his collaborators in the series of papers [**1**–**4**, **15**–**19**] studied theory of frames in various domains.

In order to study efficient representations of multidimensional functions Guo and his colleagues [**13**,**14**] introduced the concept of wavelet systems with composite dilations as a directional representation system to fit into the framework of affine systems and also allow a faithful implementation by a unified treatment of the continuum and digital realm. These systems have the following form

$$\begin{aligned}(1.1)\quad \mathcal{W}_{AB}(\psi, j, k) &= \big\{D_A D_B T_k \psi^\ell : A \in \mathcal{A},\, B \in \mathcal{B},\, k \in \mathbb{Z}^n,\, 1 \leqslant \ell \leqslant L\big\}\\ &= \big\{\psi^\ell_{j,k}(x) = q^{j/2}\psi\big(A^j B^\ell x - k\big) : j \in \mathbb{Z},\, k \in \mathbb{Z}^n,\, 1 \leqslant \ell \leqslant L\big\}\end{aligned}$$

where $L = \min\{m : B^m = I,\, m \geqslant 1,\, m \in \mathbb{Z}\}$, T_k are the translations, defined by $T_k f(x) = f(x-k)$, D_A are the dilations, defined by $D_A f(x) = q^{1/2} f(Ax)$, $q = |\det A|$ and the sets $\mathcal{A}, \mathcal{B}$ which are not necessarily commuting matrix sets are countable subsets of $GL_n(\mathbb{R})$. Typically, more restraints are put on the sets $\mathcal{A}$ and $\mathcal{B}$. For instance, it is common for $\mathcal{A}$ to be a collection of invertible matrices with eigenvalues $|\lambda| > 1$ and for $\mathcal{B}$ to be a group of matrices each with determinant 1. However, in [**22**] it was shown that these constraints are not always necessary.

The composite wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ is called a *composite wavelet frame*, if there exist constants C and D, $0 < C \leqslant D < \infty$ such that

$$C\|f\|_2^2 \leqslant \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} |\langle f, \psi_{j,k}^{\ell}\rangle|^2 \leqslant D\|f\|_2^2, \tag{1.2}$$

holds for every $f \in L^2(\mathbb{R}^n)$. We call the optimal constants C and D the lower frame bound and the upper frame bound, respectively. A *tight composite wavelet frame* refers to the case when $C = D$, and a Parseval frame refers to the case when $C = D = 1$.

Motivated and Inspired by the recent work of Srivastava and Shah [**20**], we in this paper, establish the characterization of composite wavelet system to be tight frame for $L^2(\mathbb{R}^n)$.

2. Characterization of composite tight wavelet frames in $L^2(\mathbb{R}^n)$

We shall use the following conventions throughout the paper. We adopt the notation that the time domain is represented by $\mathbb{R}^n$, and its elements will be column vectors denoted by letters of the Roman alphabet, $x = (x_1, x_2, \dots, x_n)^t \in \mathbb{R}^n$. The elements of the frequency domain will be row vectors, $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$. We denote by $T^n = [-1/2, 1/2]^n$ the n-dimensional torus and hence, the subsets of $\mathbb{R}^n$ are invariant under $\mathbb{Z}^n$ translations and the subsets of T^n are often identified. We use the Fourier transform in the form

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\xi x}dx. \tag{2.1}$$

The Fourier transform of the composite wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ is given by

$$\hat{\psi}_{j,k}^{\ell}(\xi) = q^{-j/2}\hat{\psi}(A^{*-j}B^{*-\ell}\xi)e^{-2\pi iB^{-\ell}A^{-j}k\xi},$$

where A^* and B^* denotes the transpose of A and B, respectively. Before proceeding, it is useful to state a basic lemma whose proof can be found in Christensen [**9**].

Lemma 2.1. *Suppose that $\{\varphi_k\}_{k=1}^{\infty}$ is a family of elements in a Hilbert space $\mathcal{H}$ such that there exist constants $0 < C \leqslant D < \infty$ satisfying*

$$C\|\varphi\|_2^2 \leqslant \sum_{k=1}^{\infty} |\langle \varphi, \varphi_k\rangle|^2 \leqslant D\|\varphi\|_2^2, \tag{2.2}$$

for all φ belonging to a dense subset $\mathfrak{D}$ of $\mathcal{H}$. Then, the same inequalities (2.2) *are true for all $\varphi \in \mathcal{H}$, that is, $\{\varphi_k\}_{k=1}^{\infty}$ is a frame for $\mathcal{H}$.*

In view of Lemma 2.1, we will consider the following set of functions:

$$\mathcal{D} = \{\varphi \in L^2(\mathbb{R}^n) : \hat{\varphi} \in L^{\infty}(\mathbb{R}^n) \text{ and } \hat{\varphi} \text{ has compact support in } \mathbb{R}^n \setminus \{0\}\}.$$

It is clear that $\mathcal{D}$ is a dense subspace of $L^2(\mathbb{R}^n)$. Therefore, it is enough to verify that the composite wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ given by (1.1) is a frame for $L^2(\mathbb{R}^n)$ if (1.2) hold for all $\varphi \in \mathcal{D}$.

We first state a lemma whose proof can be found in [**20**] which will be used in the proofs of the main result.

Lemma 2.2. *Suppose that the composite wavelet system* $\mathcal{W}_{AB}(\psi, j, k)$ *is defined by* (1.1). *If* $\varphi \in \mathcal{D}$ *and* $\operatorname{ess\,sup}\left\{\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)|^2 : 1 \leqslant \xi \leqslant q\right\} < \infty$, *then*

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}|\langle\varphi,\psi_{j,k}^{\ell}\rangle|^2 = \int_{\mathbb{R}^n}|\hat{\varphi}(\xi)|^2\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}((A^*)^{j}(B^*)^{\ell}\xi)|^2 d\xi + S_\psi(\varphi), \tag{2.3}$$

where

$$S_\psi(\varphi) = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{Z}^n\setminus\{0\}}\int_{\mathbb{R}^n}\overline{\hat{\varphi}(\xi)}\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)\hat{\varphi}(\xi+(A^*)^{j}(B^*)^{\ell}s) \times \overline{\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi+s)}d\xi. \tag{2.4}$$

Furthermore, the iterated series in (2.4) *is absolutely convergent.*

Now we proceed to establish our main result concerning the characterization of composite wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ is defined by (1.1) to be tight frame for $L^2(\mathbb{R}^n)$.

Theorem 2.1. *The composite wavelet system* $\mathcal{W}_{AB}(\psi, j, k)$ *is defined by* (1.1) *is a tight wavelet frame for* $L^2(\mathbb{R}^n)$ *if and only if* ψ *satisfies*

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)|^2 = 1, \quad \textit{for a.e.} \quad \xi\in\mathbb{T}^n, \tag{2.5}$$

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{N}_0}\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)\overline{\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}(\xi+m))} = 0, \tag{2.6}$$

for a.e. $\xi\in\mathbb{T}^n$, $m \in q\mathbb{N}_0+\Delta$, *where* $\Delta = \{1, 2, \ldots, q-1\}$.

Proof. Let

$$t_\psi(m,\xi) = \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^n}\hat{\psi}((A^*)^{-k}(B^*)^{-\ell}\xi)\overline{\hat{\psi}((A^*)^{-k}(B^*)^{-\ell}(\xi+m))}.$$

Assume $\varphi\in\mathcal{D}$, then for each $l\in\mathbb{Z}^n$, there exists $k\in\mathbb{Z}^n$ and a unique $m\in q\mathbb{Z}^n+\Delta$ such that $l = ((A^*)^{-k}(B^*)^{-\ell}m$. Thus, by virtue of (2.1) we have that $\{l\}_{l\in\mathbb{Z}^n} = \{(A^*)^{-k}(B^*)^{-\ell}m\}_{(k,m)\in\mathbb{Z}^n\times\{q\mathbb{Z}^n+\Delta\}}$. Since the series in (2.3) is absolutely convergent, we can estimate $S_\psi(\varphi)$ as follows:

$$\begin{aligned}
S_\psi(\varphi) &= \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\overline{\hat{\varphi}(\xi)}\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)\Big\{\sum_{l\in\mathbb{Z}^n}\hat{\varphi}(\xi+(A^*)^{j}(B^*)^{\ell}l)\overline{\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi+l)}\Big\}d\xi\\
&= \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\overline{\varphi(\xi)}\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)\Big\{\sum_{k\in\mathbb{Z}^n}\sum_{m\in q\mathbb{Z}^n+\Delta}\varphi(\xi+(A^*)^{-j-k}(B^*)^{\ell}m)\\
&\qquad\times\overline{\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi+(A^*)^{-k}(B^*)^{-\ell}m)}\Big\}d\xi
\end{aligned}$$

$$= \int_{\mathbb{R}^n} \overline{\varphi(\xi)} \Big\{ \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} \sum_{m\in q\mathbb{Z}^n+\Delta} \sum_{j\in\mathbb{Z}} \varphi(\xi + (A^*)^j (B^*)^\ell m) \hat{\psi}((A^*)^{j-k}(B^*)^{-\ell}\xi) \times \overline{\hat{\psi}((A^*)^{j-k}(B^*)^{-\ell}\xi + (A^*)^{-k}(B^*)^{-\ell}m)} \Big\} d\xi$$

$$= \int_{\mathbb{R}^n} \overline{\varphi(\xi)} \Big\{ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in q\mathbb{Z}^n+\Delta} \varphi(\xi + (A^*)^j (B^*)^\ell m) \sum_{k\in\mathbb{Z}^n} \hat{\psi}((A^*)^{j-k}(B^*)^{-\ell}\xi) \times \overline{\hat{\psi}((A^*)^{-k}(B^*)^{-\ell}((A^*)^{-j}(B^*)^{-\ell}\xi + m))} \Big\} d\xi$$

$$= \int_{\mathbb{R}^n} \overline{\varphi(\xi)} \Big\{ \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in q\mathbb{Z}^n+\Delta} \varphi(\xi + (A^*)^j (B^*)^\ell m) t_\psi(m, (A^*)^{-j}(B^*)^{-\ell}\xi) \Big\} d\xi.$$

Let us collect the results we have obtained, suppose $\psi \in L^2(\mathbb{R}^n)$ and $\varphi \in \mathcal{D}$, then

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\langle \varphi, \psi^\ell_{j,k} \rangle|^2 = \int_{\mathbb{R}^n} |\varphi(\xi)|^2 \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)|^2 d\xi + \int_{\mathbb{R}^n} \overline{\varphi(\xi)} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{m\in q\mathbb{Z}^n+\Delta} \varphi(\xi + (A^*)^j(B^*)^\ell m) \times t_\psi(m, (A^*)^{-j}(B^*)^{-\ell}\xi) d\xi. \tag{2.7}$$

The last integrand is integrable and so is the first when

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)|^2$$

is locally integrable in $\mathbb{R}^n \setminus \bigcup_{j\in\mathbb{Z}} E^c_j$. Further, (2.6) implies that

$$t_\psi(m, \xi) = 0 \quad \text{for all} \quad m \in q\mathbb{Z}^n + \Delta.$$

Combining all together with (2.5) and (2.6) gives

$$\sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^n} |\langle \varphi, \psi^\ell_{j,k} \rangle|^2 = \|\varphi\|_2^2, \quad \forall\ \varphi \in \mathcal{D}$$

Since $\mathcal{D}$ is dense in $L^2(\mathbb{R}^n)$, hence the composite wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ is defined by (1.1)) is a tight frame for $L^2(\mathbb{R}^n)$.

Conversely, suppose that the composite wavelet system $\mathcal{W}_{AB}(\psi, j, k)$ is defined by (1.1) is a tight wavelet frame for $L^2(\mathbb{R}^n)$, then we need to show that the two equations (2.5) and (2.6) are satisfied.

Since $\{\psi_{j,k}(x) : j \in \mathbb{Z},\ k \in \mathbb{Z}^n\}$ is a tight wavelet frame for $L^2(\mathbb{R}^n)$, then we have

$$\sum_{j\in\mathbb{Z}} \sum_{\ell=1}^{L} \sum_{k\in\mathbb{Z}^n} |\langle \varphi, \psi^\ell_{j,k} \rangle|^2 = \|\varphi\|_2^2, \quad \forall\ \varphi \in \mathcal{D} \tag{2.8}$$

Since $\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)|^2$ is locally integrable in $\mathbb{R}^n\setminus\cup_{j\in\mathbb{Z}}E_j^c$. Therefore, for each $\xi_0\in\mathbb{R}^n\setminus\cup_{j\in\mathbb{Z}}E_j^c$, we consider $\varphi_1(\xi)=q^{M/2}\mathbf{1}_M(\xi-\xi_0)$ where $\varphi=\varphi_1$ and $\mathbf{1}_M(\xi-\xi_0)$ is the fact characteristic function of $\xi_0+\mathbb{T}^M$. Then, it follows that for $l\in\mathbb{Z}^n\setminus\{0\}$, $\overline{\varphi(\xi)}\varphi(\xi+(A^*)^j(B^*)^\ell l)\equiv 0$, since ξ and $\xi+(A^*)^j(B^*)^\ell l$ can not be in $\xi_0+\mathbb{T}^M$ simultaneously and hence, $\|\varphi_1\|_2^2=1$. Furthermore, we have

$$\begin{aligned}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n}|\langle\varphi,\psi_{j,k}^{\ell}\rangle|^2&=\|\varphi_1\|_2^2=\|\varphi_1\|_2^2=1\\&=\int_{\xi_0+\mathbb{T}^M}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}q^M|\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)|^2d\xi+S_\psi(\varphi_1).\end{aligned}$$

By letting $M\to\infty$, we obtain

$$1=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}|\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)|^2+\lim_{M\to\infty}S_\psi(\varphi_1).\tag{2.9}$$

Now, we estimate $S_\psi(f_1)$ as:

$$\begin{aligned}S_\psi(f_1)&=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\int_{\mathbb{R}^n}\overline{\varphi_1(\xi)}\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)\\&\qquad\times\left\{\sum_{l\in\mathbb{N}}\varphi_1(\xi+(A^*)^j(B^*)^\ell l)\overline{\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi+l)}\right\}d\xi\\&\leqslant\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^n}\int_{\mathbb{R}^n}|\varphi_1(\xi)\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi)\varphi_1(\xi+(A^*)^j(B^*)^\ell l)\\&\qquad\times\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi+l)|d\xi\\&=\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^n}q^j\int_{\mathbb{R}^n}|\varphi_1(A^*)^j(B^*)^\ell\xi)\varphi_1((A^*)^j(B^*)^\ell(\xi+l))\hat{\psi}(\xi)\hat{\psi}(\xi+l)|d\xi.\end{aligned}$$

Note that $\left|\hat{\psi}(\xi)\hat{\psi}(\xi+l)\right|\leqslant\frac{1}{2}\left(\left|\hat{\psi}(\xi)\right|^2+\left|\hat{\psi}(\xi+l)\right|^2\right)$. Therefore, we have

$$|S_\psi(\varphi_1)|\leqslant\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{l\in\mathbb{Z}^n}q^j\int_{\mathbb{R}^n}|\varphi_1((A^*)^j(B^*)^\ell\xi)\varphi_1((A^*)^j(B^*)^\ell(\xi+l)||\hat{\psi}(\xi)|^2d\xi.\tag{2.10}$$

Since $l\neq 0$, $(l\in\mathbb{Z}^n)$ and $\varphi_1\in\mathcal{D}$, there exists a constant $J>0$ such that

$$\varphi_1((A^*)^j(B^*)^\ell t)\varphi_1((A^*)^j(B^*)^\ell t+(A^*)^j(B^*)^\ell l))=0,\quad\forall\ |j|>J.$$

On the other hand, for each $|j|\leqslant J$, there exists a constant L such that

$$\varphi_1((A^*)^j(B^*)^\ell t+(A^*)^j(B^*)^\ell l))=0,\quad\forall\ l>L.$$

This means that only finite terms of the series on the R.H.S of (2.10) are non-zero. Consequently, there exits a constant C such that

$$|S_\psi(\varphi_1)|\leqslant C\|\varphi_1\|_\infty^2\|\hat{\psi}\|_2^2=Cq^m\|\hat{\psi}\|_2^2$$

which implies $\lim_{M\to\infty} |S_\psi(\varphi_1)| = 0$. Hence equation (2.9) becomes

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} |\hat{\psi}((A^*)^{-j}(B^*)^{-\ell}\xi_0)|^2 = 1.$$

Finally, we must show that if (2.8) holds for all $\varphi \in \mathcal{D}$, then equation (2.6) is true. From equalities (2.7), (2.8) and just established equality (2.5), we have

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{Z}^n+\Delta}\int_{\mathbb{R}^n}\overline{\varphi(\xi)}\varphi(\xi+(A^*)^j(B^*)^\ell m)t_\psi(m,(A^*)^{-j}(B^*)^{-\ell}\xi)d\xi = 0,\ \forall\, f\in\mathcal{D}$$

Also by polarization, we then have

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{m\in q\mathbb{Z}^n+\Delta}\int_{\mathbb{R}^n}\overline{\varphi(\xi)}\hat{g}(\xi+(A^*)^j(B^*)^\ell m)t_\psi(m,(A^*)^{-j}(B^*)^{-\ell}\xi)d\xi = 0, \quad \forall\, f,g\in\mathcal{D} \tag{2.11}$$

Let us fix $m_0 \in q\mathbb{Z}^n + \Delta$ and $\xi_0 \in \mathbb{R}^n \setminus \bigcup_{j\in\mathbb{Z}} E_j^c$ such that neither $\xi_0 \neq 0$ nor $\xi_0 + m_0 \neq 0$. Setting $\varphi = \varphi_1$ and $g = g_1$ such that $\varphi_1(\xi) = q^{M/2}\mathbf{1}_M(\xi - \xi_0)$ and $\hat{g}_1(\xi) = \varphi_1(\xi - m_0)$. Then, we have $\varphi_1(\xi)\hat{g}_1(\xi+m_0) = q^M\mathbf{1}_M(\xi-\xi_0)$. Now, equality (2.11) can be written as

$$0 = q^M\int_{\xi_0+\mathbb{T}^M} t_\psi(m_0,\xi)d\xi + J_1,$$

where

$$J_1 = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{\substack{m\in q\mathbb{Z}^n+\Delta\\ (j,m)\neq(0,m_0)}}\int_{\mathbb{R}^n}\overline{\varphi_1(\xi)}\hat{g}_1(\xi+(A^*)^j(B^*)^\ell m)t_\psi(m,(A^*)^{-j}(B^*)^{-\ell}\xi)d\xi.$$

Since the first summand tends to $t_\psi(m_0,\xi_0)$ as $M\to\infty$. Therefore, we shall prove that $\lim_{M\to\infty} J_1 = 0$.

Since $m\neq 0$, $(m\in\mathbb{Z}^n)$ and $\varphi_1, g_1 \in \mathcal{D}$, there exists a constant $J_0 > 0$ such that

$$\overline{\varphi_1(\xi)}\,\hat{g}_1(\xi+(A^*)^j(B^*)^\ell m) = 0 \quad \forall\, j > J_0.$$

Therefore, we have

$$J_1 = \sum_{\ell=1}^{L}\sum_{j\leqslant J_0}\sum_{m\in q\mathbb{Z}^n+\Delta}\int_{\mathbb{R}^n}\overline{\varphi_1(\xi)}\hat{g}_1(\xi+(A^*)^j(B^*)^\ell m)t_\psi(m,(A^*)^{-j}(B^*)^{-\ell}\xi)d\xi$$

$$|J_1| \leqslant \sum_{\ell=1}^{L}\sum_{j\leqslant J_0}\sum_{m\in q\mathbb{Z}^n+\Delta} q^j\int_{\mathbb{R}^n}|\overline{\varphi_1((A^*)^j(B^*)^\ell\xi)}\hat{g}_1((A^*)^j(B^*)^\ell(\xi+m))||t_\psi(m,\xi)|d\xi.$$

Since

$$2|t_\psi(m,\xi)| \leqslant \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^n}|\hat{\psi}((A^*)^{-k}(B^*)^{-\ell}\xi)|^2 + \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^n}|\hat{\psi}((A^*)^{-k}(B^*)^{-\ell}(\xi+m))|^2,$$

hence $|J_1| \leqslant J_1^{(1)} + J_1^{(2)}$ where

$$J_1^{(1)} = \sum_{\ell=1}^{L} \sum_{j \leqslant J_0} \sum_{m \in q\mathbb{Z}^n + \Delta} q^j \int_{\mathbb{R}^n} |\varphi_1((A^*)^j (B^*)^\ell \xi)||\hat{g}_1((A^*)^j (B^*)^\ell (\xi + m))|[\sigma(\xi)]^2 d\xi,$$

with

$$\int_{\mathbb{R}^n} [\sigma(\xi)]^2 d\xi = \frac{1}{2} \sum_{\ell=1}^{L} \sum_{k \in \mathbb{Z}^n} \int_{\mathbb{R}^n} |\hat{\psi}((A^*)^{-k} (B^*)^{-\ell} \xi)|^2 d\xi = \|\hat{\psi}\|_2^2 < \infty,$$

and

$$\begin{aligned} J_1^{(2)} &= \sum_{\ell=1}^{L} \sum_{j \leqslant J_0} \sum_{m \in q\mathbb{Z}^n + \Delta} q^j \int_{\mathbb{R}^n} |\varphi_1((A^*)^j (B^*)^\ell \xi)||\hat{g}_1((A^*)^j (B^*)^\ell (\xi + m))|[\sigma(\xi + m)]^2 d\xi, \\ &= \sum_{\ell=1}^{L} \sum_{j \leqslant J_0} \sum_{m \in q\mathbb{Z}^n + \Delta} q^j \int_{\mathbb{R}^n} |\varphi_1((A^*)^j (B^*)^\ell (\eta - m))||\hat{g}_1((A^*)^j (B^*)^\ell \eta)|[\sigma(\eta)]^2 d\eta. \end{aligned}$$

Thus $J_1^{(2)}$ has the same form as $J_1^{(1)}$ with the roles of φ_1 and $\hat{g}_1$ interchanged. As $\varphi_1(\xi) = q^{M/2} \mathbf{1}_M(\xi - \xi_0)$, therefore, we deduce that

$$J_1^{(1)} = \sum_{\ell=1}^{L} \sum_{j \leqslant J_0} \sum_{m \in q\mathbb{Z}^n + \Delta} q^j q^{M/2} \int_{(A^*)^j (B^*)^\ell \xi_0 + \mathbb{T}^{-j+M}} |\hat{g}_1((A^*)^j (B^*)^\ell (\xi + m))|[\sigma(\xi)]^2 d\xi.$$

Now, if $\hat{g}_1((A^*)^j (B^*)^\ell (\xi + m)) \neq 0$, then we must have

$$(A^*)^j (B^*)^\ell \xi + (A^*)^j (B^*)^\ell m \in \xi_0 + \mathbb{T}^M + m_0$$

and $|(A^*)^j (B^*)^\ell m| \leqslant q^{-M}$, hence $|m| \leqslant q^{-M-j}$. Thus,

$$\begin{aligned} J_1^{(1)} &= \sum_{\ell=1}^{L} \sum_{j \leqslant J_0} q^j q^{M/2} \int_{(A^*)^j (B^*)^\ell \xi_0 + \mathbb{T}^{-j+M}} [\sigma(\xi)]^2 \sum_{m \in q\mathbb{N}_0 + \Delta} |\hat{g}_1((A^*)^j (B^*)^\ell (\xi + m))| d\xi \\ &\leqslant \sum_{\ell=1}^{L} \sum_{j \leqslant J_0} q^j q^{M/2} \int_{(A^*)^j (B^*)^\ell \xi_0 + \mathbb{T}^{-j+M}} [\sigma(\xi)]^2 q^{-M-j} q^{M/2} d\xi \\ &= \sum_{\ell=1}^{L} \sum_{j \leqslant J_0} \int_{(A^*)^j (B^*)^\ell \xi_0 + \mathbb{T}^{-j+M}} [\sigma(\xi)]^2 d\xi. \end{aligned}$$

For given $\xi_0 \neq 0$, we choose $q^{J_0} < |\xi_0| = q^{-M}$. Then, we have

$$(A^*)^j (B^*)^\ell \xi_0 + \mathbb{T}^{-j+M} \subset \mathbb{T}^{-J_0+M} \quad \forall\ j \leqslant J_0, \tag{2.12}$$

as $|\mathtt{p}^{-j} \xi_0| = q^j q^{-M} \leqslant q^{J_0} q^{-M}$ and $\mathbb{T}^{-j+M} \subset \mathbb{T}^{-J_0+M}$.

On the other hand, for any $j_1 < j_2 \leqslant J_0$, we claim that

$$\left\{(A^*)^{j_1} (B^*)^\ell \xi_0 + \mathbb{T}^{-j_1+M}\right\} \cap \left\{(A^*)^{j_2} (B^*)^\ell \xi_0 + \mathbb{T}^{-j_2+M}\right\} = \emptyset. \tag{2.13}$$

In fact, for any $x \in (A^*)^{j_1} (B^*)^\ell \xi_0 + \mathbb{T}^{-j_1+M}$ and $y \in (A^*)^{j_2} (B^*)^\ell \xi_0 + \mathbb{T}^{-j_2+M}$, write $x = (A^*)^{j_1} (B^*)^\ell \xi_0 + x_1$ and $y = (A^*)^{j_2} (B^*)^\ell \xi_0 + y_1$; then

$$|x - y| = \max\{(A^*)^{j_1} (B^*)^\ell \xi_0 - (A^*)^{j_2} (B^*)^\ell \xi_0|,\ |x_1 - y_1|\} = q^{j_2 - M} \neq 0.$$

This shows that (2.13) holds. Applying (2.12) and (2.13) to the last inequality for $J_1^{(1)}$, we obtain

$$J_1^{(1)} \leqslant \int_{\mathbb{T}^{-J_0+M}} [\sigma(\xi)]^2 d\xi \to 0 \quad \text{as} \quad M \to \infty. \qquad \square$$

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Department of Mathematics
National Institute of Technology
Srinagar
Jammu and Kashmir
India
siawoahmad@gmail.com

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