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CERTAIN ALMOST KENMOTSU METRICS SATISFYING THE VACUUM STATIC EQUATION

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Abstract. We characterize the solutions of the vacuum static equation on a class of almost Kenmotsu manifolds. First, we prove that if the vacuum static equation has a non-trivial solution on $(\kappa,\mu)'$-almost Kenmotsu manifold, then it is locally isometric to some warped product spaces. Next, we prove that the vacuum static equation have only trivial solution on generalized(κ,μ)-almost Kenmotsu manifold. At last, we consider the vacuum static equation on an almost Kenmotsu manifold with conformal Reeb foliation. We also provide some important examples of almost Kenmotsu manifolds that satisfies the vacuum static equation.

1. Introduction

Let (M,g) be a Riemannian manifold of dimension n and f be a smooth function on M. For the cosmological constant Λ to maintain the mass-energy density to be non-negative, a static space-time metric $\bar{g} = -f^2dt^2 + g$ on a Lorentzian manifold $\bar{M} = R\times M$ satisfies the Einstein equation

$$\mathrm{Ric}_{\bar{g}} - \tfrac{1}{2}r_{\bar{g}}\bar{g} + \Lambda\bar{g} = -8\pi GT,$$

where $\mathrm{Ric}_{\bar{g}}$, $r_{\bar{g}}$ are the Ricci tensor and scalar curvature of $\bar{g}$ respectively and G denotes the the gravitational constant. Here $T = \mu f^2dt^2 + pg$ is the energy-momentum-stress tensor of the perfect fluid, where μ and p are non-negative, time independent mass-energy density and pressure of the perfect fluid respectively. Static space-times are important global solutions to Einstein equations in general relativity. Static space-times carrying a perfect fluid matter field (see [**11**,**15**]). On the other hand, a complete Riemannian manifold (M,g) is said to be a static space with perfect fluid if there exists a nontrivial smooth function f on M such that

$$D_gdf - f\Big(\mathrm{Ric}_g - \frac{r_g}{n-1}g\Big) = \frac{1}{n}\Big(\frac{r_g}{n-1} + \Delta_g f\Big)g, \tag{1.1}$$

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where $D_g df$ is the Hessian of f and Δ_g is the negative Laplacian of f. Vacuum static spaces are static space with particular property $\frac{r_g}{n-1} + \Delta_g f = 0$. In this case, (M, g) is said to be a vacuum static space and the Eq. (1.1) transform into

$$D_g df - f\Big(\operatorname{Ric}_g - \frac{r_g}{n-1} g\Big) = 0, \tag{1.2}$$

and this is call as the vacuum static equation. These spaces are studied by Qing and Yuan (see [**24**, **25**]). Recently, Hawan and Yun [**12**] consider vacuum static spaces with the complete divergence of the Bach tensor and Weyl tensor and also find a sufficient condition for the metric to be Bach-flat for vacuum static spaces. Note that, it was also considered by Fischer and Marsden in their study of the surjectivity of the scalar curvature function from the space of Riemannian metrics (see [**4**, **8**, **9**, **14**, **22**, **26**]). In [**8**], Fischer and Marsden conjectured that a compact Riemannian manifold (M^n, g) that admits a nontrivial solution of the vacuum static equation is necessarily an Einstein manifold, and therefore, Obata's theorem [**17**] shows that M must be a standard sphere or a Ricci flat space.

It is important to note that if a complete Riemannian manifold (M, g) has a non-trivial solution f of the vacuum static equation, then the scalar curvature r of g is constant (see [**3**] and [**8**]). Further, Kobayashi [**14**] and Lafontaine [**16**] proved that if M is conformally flat and has a non trivial solution of the equation (1.2), then M is isometric to one of (a) Euclidean sphere S^n; (b) Finite quotient of $(S^1, dt^2) \times (S^{n-1}, g_0)$, where g_0 is the canonical metric; or (c) Finite quotient of a product torus $(S^1 \times S^{n-1}, dt^2 + h^2(t) g_0)$. In [**26**], Shen proved that if a 3-dimensional closed manifold (M, g) with positive scalar curvature has a non trivial solution to the vacuum static equation, then M contains a totally geodesic 2-sphere. Further, in [**5**], Corvino proved f is a nontrivial solution of (1.2) if and only if the warped product metric $g^* = g - f^2 dt^2$ is Einstein. Recently, Patra et al. consider the same special contact metrics that satisfy the Miao-Tam critical condition and the critical point equation, see [**21**, **23**]. Further, we mention that Patra–Ghosh prove the Fischer–Marsden conjecture within the framework of K-contact and (κ, μ)-contact manifolds (see [**22**]). Motivated by the above works, we characterize the solutions of the vacuum static equation on almost contact metric structures, specially, on almost Kenmotsu manifolds.

The organization of the paper is as follows. In Section 2, we provide some preliminaries and examples of almost Kenmotsu manifolds that satisfy the vacuum static equation. In Section 3, we prove that if the vacuum static equation has a non-trivial solution on $(\kappa, \mu)'$-almost Kenmotsu manifold, then it is locally isometric to some warped product spaces. Next, we prove that the vacuum static equation has only trivial solution on generalized(κ, μ)-almost Kenmotsu manifold. At last, we consider the vacuum static equation on an almost Kenmotsu manifold with conformal Reeb foliation.

2. Notes on almost contact metric manifolds

In this section, we recall some basic definitions and formulas on almost Kenmotsu manifold and some nullity distributions. A $(2n+1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if it admits a $(1,1)$ tensor field φ, a unit vector field ξ (called the Reeb vector field) and a 1-form η such that

$$\varphi^2 X = -X + \eta(X)\xi, \quad \eta(X) = g(X, \xi), \tag{2.1}$$

for any vector fields X on M. A Riemannian metric g is said to be an associated (or compatible) metric if it satisfies $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$ for all vector fields X, Y on M. Using this in (2.1) we have $\varphi(\xi) = 0$, $\eta \circ \varphi = 0$ (see [**2**]). An almost contact manifold $M(\varphi, \xi, \eta)$ together with a compatible metric g is known as almost contact metric manifold. On almost contact metric manifolds one can always define a fundamental 2-form Φ by $\Phi(X,Y) = g(X, \varphi Y)$ for all vector fields X, Y on M. An almost contact metric structure of M is said to be contact metric if $\Phi = d\eta$, and is said to be almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$. Further, an almost contact metric structure is said to be normal if

$$[\varphi, \varphi](X,Y) + 2d\eta(X,Y)\xi = 0,$$

for all vector fields X, Y on M. A normal almost Kenmotsu manifold is said to be a Kenmotsu manifold. In [**13**], Kenmotsu proved that a warped product of a line and a Kählerian manifold admits a Kenmotsu structure. In fact, a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f M^{2n}$, where I is an open interval with coordinate t, $f = ce^t$ is the warping function for some positive constant c and M^{2n} is a Kählerian manifold.

Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold. We now define two operators h and l by $h = \frac{1}{2}\pounds_\xi \varphi$ and $l = R(.,\xi)\xi$ on M, where R denotes the curvature tensor and $\pounds$ is the Lie differentiation. On an almost Kenmotsu manifold the following formulas are valid [**6, 7**]:

$$\nabla_X \xi = -\varphi^2 X - \varphi h X, \tag{2.2}$$

$$h\xi = 0, \quad l\xi = 0, \quad \operatorname{tr} h = 0, \quad \operatorname{tr}(h\varphi) = 0, \quad h\varphi = -\varphi h, \tag{2.3}$$

$$\operatorname{tr}(l) = S(\xi, \xi) = g(Q\xi, \xi) = -2n - \operatorname{tr} h^2, \tag{2.4}$$

for any vector fields X on M, where $h' = ho\varphi$, tr the trace operator, ∇ the operator of covariant differentiation of g and Q the Ricci operator associated with the $(0,2)$ Ricci tensor given by $\mathrm{Ric}_g(Y,Z) = g(QY, Z)$ for all vector fields Y, Z on M.

An almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a generalized (κ, μ)-almost Kenmotsu manifold if ξ belongs to the generalized (κ,μ)-nullity distribution, i.e.,

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)hX - \eta(X)hY\},$$

for all vector fields X, Y on M, where κ, μ are smooth functions on M. An almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ is said to be a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold if ξ belongs to the generalized $(\kappa, \mu)'$-nullity distribution, i.e.,

$$R(X,Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\} + \mu\{\eta(Y)h'X - \eta(X)h'Y\}, \tag{2.5}$$

for all vector fields X, Y on M, where κ, μ are smooth functions on M and $h' = ho\varphi$. Moreover, if both κ and μ are constants in (2.5), then M is called a $(\kappa, \mu)'$-almost

Kenmotsu manifold. Classifications of almost Kenmotsu manifolds with ξ belong to $(\kappa,\,\mu)$-nullity distribution and $(\kappa,\mu)'$-nullity distribution have been done by several authors. For more details, we refer the reader to [**6**, **7**, **27**, **28**]. For generalized (κ,μ) or $(\kappa,\mu)'$-almost Kenmotsu manifold with $h\neq 0$ the following equations were proved

$$h'^2=(\kappa+1)\varphi^2 \quad \text{or, equivalently} \quad h^2=(\kappa+1)\varphi^2, \tag{2.6}$$

$$Q\xi=2n\kappa\xi. \tag{2.7}$$

REMARK 2.1. If $D=\ker(\eta)$ is the distribution and $X\in D$ is an eigenvector of h' with eigenvalue σ, then it follows from (2.6) that $\sigma^2=-(\kappa+1)$. Hence $\kappa\leqslant -1$ and $\sigma=\pm\sqrt{-\kappa-1}$. The equality holds if and only if $h=0$ (equivalently, $h'=0$). Thus, $h'\neq 0$ if and only if $\kappa<-1$. We know that on Kenmotsu manifold, $h=0$ (equivalently, $h'=0$), and therefore, $\kappa=-1$.

Let $(V,J,\tilde g)$ be an almost Hermitian manifold and consider the warped product $M=\mathbb{R}\times_f V$ with the metric $g=g_0+f^2\tilde g$, where f is a positive function on $\mathbb{R}$ and g_0 is the standard metric on $\mathbb{R}$. define $\eta=dt$, $\xi=\frac{\partial}{\partial t}$ and the tensor field φ is defined on $\mathbb{R}\times_f V$ by $\varphi X=JX$ for any vector field X on V and $\varphi X=0$ if X is tangent on $\mathbb{R}$. Then it is easy to testify that M admits an almost contact metric structure. An interesting characterization of an almost Kenmotsu manifold through the warped product of a real line and an almost Hermitian manifold is given by the following (see [**1**]).

LEMMA 2.1. *Let V be an almost Hermitian manifold. Then the warped product $\mathbb{R}\times_f V$ is a $(0,\beta)$-trans Sasakian manifold, with $\beta=\frac{f'}{f}$ if and only if V is Kählerian.*

Now, we provide some examples of almost Kenmotsu manifolds satisfying the vacuum static equation.

EXAMPLE 2.1. Let $(V,J,\tilde g)$ be a strictly almost Kähler Einstein manifold. We set $\eta=dt$, $\xi=\frac{\partial}{\partial t}$ and the tensor field φ is defined on $\mathbb{R}\times_f N$ by $\varphi X=JX$ for vector field X on N and $\varphi X=0$ if X is tangent to $\mathbb{R}$. Consider a metric $g=g_0+f^2\tilde g$, where $f^2=ce^{2t}$, g_0 is the Euclidean metric on $\mathbb{R}$ and c is a positive constant. Then it is easy to verify (see [**6**]) that the warped product $\mathbb{R}\times_f V$, $f^2=ce^{2t}$, with the structure (φ,ξ,η,g) is an almost Kenmotsu manifold. Let $f=ce^t$ on M. Then we can easily show that f is a solution of the vacuum static equation.

REMARK 2.2. Oguro and Sekigawa (see [**18**]) constructed a strictly almost Kähler structure on the Riemannian product $\mathbb{H}^3\times\mathbb{R}$. By virtue of this it is possible to obtain a 5-dimensional strictly almost Kenmotsu manifold on the warped product $\mathbb{R}\times_{f^2}(\mathbb{H}^3\times\mathbb{R})$, where $f^2=ce^{2t}$.

EXAMPLE 2.2. Let $(V^{2n},J,\tilde g)$ be a Kähler Einstein manifold with negative scalar curvature, i.e., $\tilde S=-2n\tilde g$. Considering the warped product $(M,g)=(\mathbb{R}\times_{f^2} V, dt^2+f^2\ \tilde g)$ with coordinate t on $\mathbb{R}$, where $f=\cosh t$. Now, we prove that the warped product $\mathbb{R}\times_{f^2} V$, with $f=\cosh t$ is an almost Kenmotsu manifold.

Defining ξ, η and φ as in the Example 2.1, we see that (M, g) admits an almost contact metric structure. Moreover, from Lemma 2.1, it is obvious that the warped product under consideration is a β-Kenmotsu manifold with $\beta = \tanh t$, which is also an almost Kenmotsu manifold. Let $f = b \sinh t$, b is a positive constant. Then it follows that f is a solution of the vacuum static equation.

3. Main Results

Let (g, f) be a non-trivial solution of the vacuum static equation on an almost Kenmotsu manifold $M^{2n+1}(\varphi, \xi, \eta, g)$. Then the vacuum static equation (1.2) can be written as

$$\nabla_X Df = f\Big\{QX - \frac{r}{2n}X\Big\}, \tag{3.1}$$

for any vector fields X on M, where D is the gradient operator of g. The covariant differentiation of (3.1) along an arbitrary vector field Y on M yields

$$\nabla_Y(\nabla_X Df) = (Yf)\Big\{QX - \frac{r}{2n}X\Big\} + f\Big\{(\nabla_Y Q)X + Q(\nabla_Y X) - \frac{r}{2n}\nabla_Y X\Big\},$$

for any vector fields X on M. Repeated Application of the above equations in the well known expression of the curvature tensor $R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ we obtain

$$\begin{aligned} R(X, Y)Df = (Xf)QY - (Yf)QX + f\{(\nabla_X Q)Y - (\nabla_Y Q)X\} \\ - \frac{r}{2n}\{(Xf)Y - (Yf)X\}, \end{aligned} \tag{3.2}$$

for all vector fields X, Y on M. Now we recall the following

LEMMA 3.1. [**28**, Lemma 3.3]. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a generalized $(\kappa, \mu)'$-almost Kenmotsu manifold with $h' \neq 0$. For $n > 1$, the Ricci operator Q of M can be expressed as*

$$Q = -2nid + 2n(\kappa + 1)\eta \otimes \xi + \mu h' - 2(n-1)h',$$

Further, if κ and μ are constants and $n \geqslant 1$, then $\mu = -2$ and hence

$$Q = -2nid + 2n(\kappa + 1)\eta \otimes \xi - 2nh', \tag{3.3}$$

In both cases, the scalar curvature of M is $2n(\kappa - 2n)$.

LEMMA 3.2. [**7**, Theorem 4.2]. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a $(\kappa, -2)'$-almost Kenmotsu manifold with $h' \neq 0$. Then M^{2n+1} is locally isometric to the warped products $\mathbb{H}^{n+1}(\kappa - 2\gamma) \times_f \mathbb{R}^n$, $B^{n+1}(\kappa + 2\gamma) \times_{f'} \mathbb{R}^n$, where $f = ce^{(1-\gamma)t}$ and $f' = c'e^{(1+\gamma)t}$, with c, c' positive constants.*

LEMMA 3.3. [**3**] and [**8**, p. 481]. *If a Riemannian metric g satisfies the vacuum static equation, then its scalar curvature is constant.*

According to Proposition 4.1 of Dileo and Pastore [**7**], on a non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifold, $\mu = -2$. Now we consider the vacuum static equation non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifold and prove our main result.

Theorem 3.1. *If (g, f) be a non-trivial solution of the vacuum static equation on non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifold, then M^3 is locally isometric to the Riemannian product $\mathbb{H}^2(-4) \times \mathbb{R}$, and for $n > 1$, M^{2n+1} is locally isometric to the warped products $\mathbb{H}^{n+1}(\alpha) \times_f \mathbb{R}^n$, $B^{n+1}(\alpha') \times_{f'} \mathbb{R}^n$, where $\mathbb{H}^{n+1}(\alpha)$ is the hyperbolic space of constant curvature $\alpha = -1 - \frac{2}{n} - \frac{1}{n^2}$, $B^{n+1}(\alpha')$ is a space of constant curvature $\alpha' = -1 + \frac{2}{n} - \frac{1}{n^2}$, $f = ce^{(1-\frac{1}{n})t}$ and $f' = c'e^{(1+\frac{1}{n})t}$, with c, c' positive constants.*

Proof. Firstly, replacing X by ξ in (2.5) and then taking the scalar product of the resulting Eq. with Df and using $g(X, Df) = Xf$ and $\mu = -2$ gives

$$g(R(\xi, Y)Df, \xi) = \kappa g(Df - (\xi f)\xi, Y) - 2g(Df, h'Y), \tag{3.4}$$

for any vector fields Y on M. As the scalar curvature (from Lemma 3.1) is $2n(\kappa - 2n)$, (3.2) reduces for the manifold M^{2n+1} to

$$\begin{aligned} R(X, Y)Df = (Xf)QY - (Yf)QX + f\{(\nabla_X Q)Y - (\nabla_Y Q)X\} \\ + (2n - \kappa)\{(Xf)Y - (Yf)X\}, \end{aligned} \tag{3.5}$$

for all vector fields X, Y on M. Taking scalar product of (3.5) with ξ and using (2.7) provides

$$\begin{aligned} g(R(X, Y)Df, \xi) = \{(2n - 1)\kappa + 2n\}\{(Xf)\,\eta(Y) - (Yf)\,\eta(X)\} \\ + f\{g(Y, (\nabla_X Q)\xi) - g(X, (\nabla_Y Q)\xi)\}, \end{aligned} \tag{3.6}$$

for all vector fields X, Y on M. Taking covariant derivative of this along an arbitrary vector field X on M we have $(\nabla_X Q)\xi + Q(\nabla_X \xi) = 2n\kappa\nabla_X \xi$. Making use of (2.2) we have from the last equation that

$$(\nabla_X Q)\xi = 2n\kappa(X - \varphi hX) - Q(X - \varphi hX), \tag{3.7}$$

for any vector fields X on M. Moreover, making use of (3.7) and $\varphi h = -h\varphi$, (3.6) transforms into

$$\begin{aligned} g(R(X, Y)Df, \xi) = \{(2n - 1)\kappa + 2n\}\{(Xf)\eta(Y) - (Yf)\eta(X)\} \\ + f\{g(Q\varphi hX, Y) - g(X, Q\varphi hY)\}, \end{aligned} \tag{3.8}$$

for all vector fields X, Y on M. Now, substituting ξ by X in the Eq. (3.8) and applying $h\xi = 0$, $\varphi\xi = 0$, $Q\xi = 2n\kappa\xi$, we obtain

$$g(R(\xi, Y)Df, \xi) = \{(2n - 1)\kappa + 2n\}\{(\xi f)\eta(Y) - (Yf)\}. \tag{3.9}$$

Combining (3.4) and (3.9), we have

$$n(\kappa + 1)(Df - (\xi f)\xi) - h'Df = 0. \tag{3.10}$$

Now, operating the last equation by h' and using $h'\xi = 0$ yields $n(\kappa + 1)hDf + h'^2 Df = 0$. Now, By virtue of (3.10) and (2.6) the preceding equation provides

$$n^2(\kappa + 1)^2(Df - (\xi f)\xi) + (\kappa + 1)\varphi^2 Df = 0.$$

Moreover, making use of (2.1) and $g(\xi, Df) = \xi f$, the last equation reduces to $(\kappa+1)\{n^2(\kappa+1)+1\}(Df-(\xi f)\xi) = 0$. Taking into account the assumption $\kappa < -1$, the foregoing equation gives

$$\{n^2(\kappa+1)+1\}(Df-(\xi f)\xi) = 0. \tag{3.11}$$

Since κ, μ are constants, we have either $n^2(\kappa+1)+1 = 0$, or $n^2(\kappa+1)+1 \neq 0$.

Case I: In this case, we have $\kappa = -1-\frac{1}{n^2}$. For $n = 1$, $\kappa = \mu = -2$ and therefore from Lemma 3.2, we conclude that M^3 is locally isometric to the Riemannian product $\mathbb{H}^2(-4)\times\mathbb{R}$ and for $n > 1$, M^{2n+1} is locally isometric to the warped products $\mathbb{H}^{n+1}(\alpha)\times_f\mathbb{R}^n$, $B^{n+1}(\alpha')\times_{f'}\mathbb{R}^n$, where $\mathbb{H}^{n+1}(\alpha)$ is the hyperbolic space of constant curvature $\alpha = -1-\frac{2}{n}-\frac{1}{n^2}$, $B^{n+1}(\alpha')$ is a space of constant curvature $\alpha' = -1+\frac{2}{n}-\frac{1}{n^2}$, $f = ce^{(1-\frac{1}{n})t}$ and $f' = c'e^{(1+\frac{1}{n})t}$, with c, c' positive constants.

Case II: In this case, it follows from (3.11) that $Df = (\xi f)\xi$. Taking covariant derivative of $Df = (\xi f)\xi$ along an arbitrary vector field X on M and using (2.1), (2.2), we have

$$\nabla_X Df = X(\xi f)\xi + (\xi f)\{X - \eta(X)\xi - \varphi hX\}.$$

By virtue of (3.1), the foregoing equation reduces to

$$f\,QX = \{(\xi f)+(\kappa-2n)f\}X + X(\xi f)\xi - (\xi f)\{\eta(X)\xi - \varphi hX\},$$

for any vector fields X on M. Comparing this with (3.3) we deduce that

$$\begin{aligned}\{\kappa f+(\xi f)\}X &- \{(\xi f)+2n(\kappa+1)f\}\eta(X)\xi \\ &+ X(\xi f)\xi + \{2nf+(\xi f)\}h'X = 0,\end{aligned} \tag{3.12}$$

for any vector fields X on M. Now, tracing (3.12) over X and noting that $\operatorname{tr} h' = 0$, we have

$$(2n+1)\{\kappa f+(\xi f)\} - \{(\xi f)+2n(\kappa+1)f\} + \xi(\xi f) = 0. \tag{3.13}$$

Next, substituting X by ξ in (3.1) and then taking its scalar product with ξ yields $\xi(\xi f) = 2n\kappa f + (2n-\kappa)f$. By virtue of this, equation (3.13) gives

$$\kappa f + (\xi f) = 0. \tag{3.14}$$

Therefore, operating (3.12) by φ and using (3.14) we get $(2nf+(\xi f))\,\varphi h'X = 0$ for any vector fields X on M. Moreover, making use of (2.1), $h\varphi = -\varphi h$ and $h\xi = 0$, the last Eq. transform into $(2nf+(\xi f))\,h'X = 0$ for any vector fields X on M. By virtue of (3.14), the equation (3.18) reduces to $(2n-\kappa)fhX = 0$ for any vector fields X on M. Taking into account the assumption $\kappa < -1$, the foregoing equation gives $f = 0$, a contraction. □

According to [**20**, Proposition 3.2], on a generalized $(\kappa,\mu)'$-almost Kenmotsu manifold $M^{2n+1}(\varphi,\xi,\eta,g)$ with $h \neq 0$ we have

$$\xi(\kappa) = -2(\kappa+1)(\mu+2). \tag{3.15}$$

It follows from Lemma 3.1 that the scalar curvature of M is $2n(\kappa-2n)$. But we know that the Riemannian metric satisfying the vacuum static equation has constant scalar curvature (see Lemma 3.3), and therefore, κ is also constant. Hence from (3.15) we get $(\kappa+1)(\mu+2)=0$. Since $h\neq 0$, $\kappa<-1$, we must have $\mu=-2$. Thus, the generalized $(\kappa,\mu)'$-almost Kenmotsu manifold reduces to $(\kappa,\mu)'$-almost Kenmotsu manifold. This shows that the last Theorem holds good for a generalized $(\kappa,\mu)'$-almost Kenmotsu manifold.

Corollary 3.1. *Theorem* 3.1 *also holds for generalized $(\kappa,\mu)'$-almost Kenmotsu manifold with $h'\neq 0$.*

Lemma 3.4. [**28**, Lemma 3.4]. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a generalized (κ,μ)-almost Kenmotsu manifold with $h'\neq 0$. For $n>1$, the Ricci operator Q of M can be expressed as*

$$Q=-2nid+2n(\kappa+1)\eta\otimes\xi-2(n-1)h'+\mu h. \tag{3.16}$$

Also, the scalar curvature of M is $2n(\kappa-2n)$.

Theorem 3.2. *The vacuum static equation have only trivial solution on generalized (κ,μ)-almost Kenmotsu manifold with $h\neq 0$.*

Proof. Since the scalar curvature is constant (from Lemma 3.3) and from Lemma 3.4, the scalar curvature of M is $2n(\kappa-2n)$. Thus, κ is constant, and equations (3.5) to (3.8) hold good here also. Making use of (3.16) and $h\varphi=-\varphi h$ we have $f\mu h^2\varphi^2X=0$ for any vector field X on M. Using (2.1), the last equation gives $(\kappa+1)f\mu h^2X=0$ for any vector field X on M. Since $h\neq 0$, $\kappa<-1$, it follows that $f\mu=0$.

Suppose $f\neq 0$ in some open set $\mathcal{O}$. Thus, we have $\mu=0$ on $\mathcal{O}$. All our next discussion will be on $\mathcal{O}$. Further, setting $X=\xi$ in (3.16) and taking scalar product of the resulting equation with Df and using $h'\xi=0$ gives $g(R(\xi,Y)Df,\xi)=\kappa g(Df-(\xi f)\xi,Y)$. By virtue of this, (3.9) reduces to $(\kappa+1)(Df-(\xi f)\xi)=0$. As $h\neq 0$, $\kappa<-1$, it follows that $Df-(\xi f)\xi=0$. Taking covariant differentiation of the last equation along an arbitrary vector field X on M together with (2.1), (2.2) and (3.1) one can find

$$fQX=(\kappa-2n)fX+X(\xi f)\xi+(\xi f)\big\{X-\eta(X)\xi-\varphi hX\big\}.$$

By virtue of (2.7) and some straightforward calculation, the last equation transform into

$$\begin{aligned}\{(\xi f)+2n(\kappa+1)f\}\eta(X)\xi-\{\kappa f+(\xi f)\}X-X(\xi f)\xi&\\ -\{2(n-1)f+(\xi f)\}h\varphi X=0.&\end{aligned} \tag{3.17}$$

Now, tracing this over X and using $\mathrm{tr}(h\varphi)=0$, we have

$$(\xi f)+2n(\kappa+1)f-(2n+1)\{\kappa f+(\xi f)\}-\xi(\xi f)=0. \tag{3.18}$$

Next, the Eq. (3.1) yields $\xi(\xi f)=(2n-1)\kappa f+2nf$. In view of this, the equation (3.18) reduces to

$$\kappa f+(\xi f)=0. \tag{3.19}$$

Moreover, operating φ in (3.17) and making use of (3.19) we obtain $\{2(n-1)f+(\xi f)\}\varphi h\varphi X=0$. Also, using (2.1), $h\xi=0$ and $h\varphi=-\varphi h$, the last equation provides $\{2(n-1)f+(\xi f)\}hX=0$, for all vector field X on M. Making use of

(3.19), the preceding equation becomes $\{\kappa - 2(n-1)\} fhX = 0$. Since $h \neq 0$, the foregoing equation gives $\{\kappa - 2(n-1)\} f = 0$. As $\kappa < -1$, this shows that f is trivial. $\square$

In [**19**], Pastore and Saltarelli proved that the Reeb foliation on an almost Kenmotsu manifold is conformal if and only if $h = 0$.

THEOREM 3.3. *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be an almost Kenmotsu manifold with conformal Reeb foliation. If (g, f) is a non-trivial solution of the vacuum static equation, then M is a manifold is of constant scalar curvature $r = -2n(2n+1)$.*

PROOF. Since the Reeb foliation is conformal, using $h = 0$ in equation (3.10) yields

$$R(X, Y)\xi = \eta(X)(Y) - \eta(Y)(X), \tag{3.20}$$

for all vector fields X, Y on M. By virtue of this, from (2.3) we have $Q\xi = -2n\xi$, Substituting ξ by X in (3.2) and taking the scalar product of this equation with ξ and using the above equation we get

$$g(R(\xi, Y)Df, \xi) = \Big(2n + \frac{r}{2n}\Big)\{(Yf) - (\xi f)\eta(Y)\},$$

for all vector fields Y on M. Also, equation (3.20) gives

$$g(R(\xi, Y)Df, \xi) = (\xi f)\,\eta(Y) - (Yf).$$

for all vector fields Y on M. Combining above two equations yields

$$\Big(2n + 1 + \frac{r}{2n}\Big)\{Df - (\xi f)\xi\} = 0. \tag{3.21}$$

Suppose that $r \neq -2n(2n+1)$ and then from (3.21) we have $Df = (\xi f)\xi$. Taking covariant derivative of this along an arbitrary vector field X on M and using (2.1), (2.2), we acquire $\nabla_X Df = X(\xi f)\xi + (\xi f)(X - \eta(X)\xi - \varphi hX)$. Since the Reeb foliation is conformal, using $h = 0$ and (3.1) in the foregoing equation we get

$$fQX = \{X(\xi f) - (\xi f)\eta(X)\}\xi + \Big\{(\xi f) + \frac{rf}{2n}\Big\}X,$$

for all vector fields X on M. Making use of $g(\nabla_X Df, Y) = g(\nabla_Y Df, X)$ for all vector fields X, Y on M and $Df = (\xi f)\xi$ in the last equation, we obtain

$$fQX = \Big\{(\xi f) + \frac{rf}{2n}\Big\}X + \{\xi(\xi f) - (\xi f)\}\eta(X)\xi, \tag{3.22}$$

for any vector fields X on M. From (3.1) and (2.4), we have

$$\xi(\xi f) = -\frac{rf}{2n} - 2nf. \tag{3.23}$$

Next, taking trace of (3.22), we obtain $\frac{rf}{2n} + 2n(\xi f) + \xi(\xi f) = 0$. Making use of (3.23) in the last equation yields $\xi f = f$. By virtue of this, equation (3.22) gives $\{r + 2n(2n+1)\} f = 0$. Since f is a non-trivial solution of the vacuum static equation, it follows from the above equation that $r = -2n(2n+1)$, a contradiction. $\square$

Remark 3.1. On any Kenmotsu manifold we obtain a result (see [**10**])

$$(\nabla_\xi Q)X = -2QX - 4nX,$$

for any vector field X on M. The g-trace of this gives $\xi r = -2(r + 2n(2n+1))$. Thus, Lemma 3.3 shows that if a Kenmotsu metric g satisfies the vacuum static equation, then the scalar curvature r of g is $-2n(2n+1)$.

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