PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 112(126) (2022), 95–101 DOI: https://doi.org/10.2298/PIM2226095S

ON THE POSITION VECTOR OF SURFACE CURVES IN THE EUCLIDEAN SPACE

Absos Ali Shaikh and Pinaki Ranjan Ghosh

Dedicated to Professor Hari Mohan Srivastava on the occasion of his 80th birthday

ABSTRACT. We study curves on a surface whose position vectors lie in the plane generated by any two vectors among $\vec{T}$, $\vec{N}$ and $\vec{T} \times \vec{N}$. We find the distance function, tangential, normal and binormal components of the position vectors of such curves and obtain a condition for its invariancy under the isometry of surfaces.

1. Introduction

Now a day's in the study of space curves or surface curves, the characterization of curves by restricting its position vectors in some plane related to surface is an important area of research. In 2003, Chen [**2**] studied curves by restricting its position vectors in the rectifying plane and obtained some characterization. In 2005, Chen and Dillen [**3**] investigated the relationship among rectifying curves, centrodes and extremal curves. In 2008, Ilarslan [**6**] studied rectifying curves in Minkowski 3-space and characterized them analogously as that of Chen [**2**]. In 2018, Deshmukh et al. [**4**] also characterized rectifying curves by centrodes of an unit speed curve in Euclidean space. The authors of [**7**, **9**, **10**, **12**–**15**] have studied curves by restricting their position vectors to the rectifying, osculating and normal plane on a surface and obtained their characterization under isometry and conformal maps of surfaces.

Camci et al. [**1**] studied a surface curve by restricting its position vector in three mutually perpendicular planes on the surface and established the existence of such a curve. Shaikh and Ghosh [**11**] investigated the surface curve by restricting its position vector in the tangent plane to the surface.

Motivated from the above studies, we extend the results of [**1**] under isometry of surfaces. We consider three surface curves whose position vectors lie in three

2020 *Mathematics Subject Classification:* 53C22, 53A04, 53A05.

Key words and phrases: isometry of surfaces, normal curvature, first fundamental form, second fundamental form.

Communicated by H. M. Srivastava.

different planes generated by $\{\vec{T}, \vec{N}\}$, $\{\vec{N}, \vec{T} \times \vec{N}\}$ and $\{\vec{T}, \vec{T} \times \vec{N}\}$ respectively. We find the distance function, tangential, normal and binormal components of the position vectors of those three curves and obtain their deviation under isometry.

2. Preliminaries

Let P be a smooth surface immersed in $\mathbb{R}^3$ such that (U, ψ) be a coordinate chart at any point $p = \psi(x, y) \in P$. Then a smooth curve $\xi(s)$ of the arc length parameter s on the surface P can be written as $\xi(s) = \psi(x(s), y(s))$.

Definition 2.1. A diffeomorpfism $h \colon P \to \bar{P}$ between two smooth surfaces P and $\bar{P}$ is called an isometry if it preserves the distance between any two points.

Since P is a smooth surface hence $\psi_x = \frac{\partial \psi}{\partial x}$ and $\psi_y = \frac{\partial \psi}{\partial y}$ generates the tangent space T_pP at p. The first fundamental form of a surface P at a point p is a quadratic form on T_pP given by

$$I_p(\xi'(s)) = \langle \xi'(s), \xi'(s) \rangle = Ex'^2 + 2Fx'y' + Gy'^2,$$

where $'$ denotes the derivative with respect to arc length parameter and E $(= \langle \psi_x, \psi_x \rangle)$, F $(= \langle \psi_x, \psi_y \rangle)$ and G $(= \langle \psi_y, \psi_y \rangle)$ are coefficients of the first fundamental form of the surface P. Let $\mathbf{N}$ be the unit normal to the surface P. Then $\mathbf{N}$ is a map from P to $\mathbb{R}^3$ and the range of $\mathbf{N}$ is a subset of the 2-sphere S^2. Now $d\mathbf{N}_p$ is a map from T_pP to $T_{\mathbf{N}(p)}S^2$. Since $T_{\mathbf{N}(p)}S^2$ is parallel to T_pP, hence $d\mathbf{N}_p$ is a map from T_pP to T_pP. The second fundamental form of a surface P at a point p is a quadratic form given by

$$II_p(\xi'(s)) = -\langle d\mathbf{N}_p(\xi'(s)), \xi'(s) \rangle.$$

For a detailed study of fundamental forms of the surface we refer the reader to see [**8**] and [**5**].

Let $\vec{T}$ be a tangent vector to the surface P at p, i.e., $\vec{T} \in T_pP$. Then $\vec{T}$ is given by $\vec{T} = a\psi_x + b\psi_y$, for some real numbers a and b not simultaneously zero. If $\vec{N}$ is the surface normal then the vector perpendicular to $\vec{T}$ and $\vec{N}$ is given by

$$\begin{aligned} \vec{B} = \vec{T} \times \vec{N} &= (a\psi_x + b\psi_y) \times (\psi_x \times \psi_y) \\ &= a(F\psi_x - E\psi_y) + b(G\psi_x - F\psi_y) \\ &= (aF + bG)\psi_x - (aE + bF)\psi_y. \end{aligned}$$

3. Curves with position vectors lie in the plane generated by $\vec{T}$ and $\vec{N}$

Let $\xi(s)$ be a curve on the surface P with position vector lies in the plane generated by $\vec{T}$ and $\vec{N}$. Then the curve $\xi(s)$ is given by

$$\begin{aligned} \xi(s) &= c\vec{T} + d\vec{N} = c(a\psi_x + b\psi_y) + d(\psi_x \times \psi_y) \\ &= \alpha\psi_x + \beta\psi_y + d(\psi_x \times \psi_y), \end{aligned} \tag{3.1}$$

where c and d are real constants not simultaneously zero and $\alpha = ac$, $\beta = bc$.

The Gauss equation for the surface patch ψ of P with normal vector $\vec{N}$ is written as

$$\begin{aligned}\psi_{xx} &= \Gamma^1_{11}\psi_x + \Gamma^2_{11}\psi_y + L\vec{N},\\ \psi_{xy} &= \Gamma^1_{12}\psi_x + \Gamma^2_{12}\psi_y + M\vec{N},\\ \psi_{yy} &= \Gamma^1_{22}\psi_x + \Gamma^2_{22}\psi_y + N\vec{N},\end{aligned}$$

where $\{L, M, N\}$ are the coefficients of the second fundamental form of P, and Γ^k_{ij} are the Christoffel symbols for $i, j, k = 1, 2$.

Let P and $\bar{P}$ be two smooth surfaces in the Euclidean space and h be an isometry between them. The isometry h takes the curve $\xi(s)$ on P to $\bar{\xi}(s)$ on $\bar{P}$, which is given by

$$\bar{\xi}(s) = h \circ \xi(s) = \alpha h_*\psi_x + \beta h_*\psi_y + dh_*\vec{N} = ch_*\vec{T} + dh_*\vec{N},$$

where c and d are constants. So $\bar{\xi}(s)$ is also a curve on $\bar{P}$ whose position vector lies in the plane generated by $h_*\vec{T}$ and $h_*\vec{N}$.

Now the tangent vector to the curve $\xi(s)$ is given by

$$\begin{aligned}\vec{t} = \xi'(s) &= \alpha'\psi_x + \beta'\psi_y + d'\vec{N} + \alpha(x'\psi_{xx} + y'\psi_{xy}) + +\beta(x'\psi_{xy} + y'\psi_{yy})\\ &\quad + d(x'\psi_{xx} + y'\psi_{xy}) \times \psi_x + d\psi_x \times (x'\psi_{xy} + y'\psi_{yy}),\\ &= A_1\psi_x + A_2\psi_y + A_3\vec{N},\end{aligned}$$

where A_1, A_2 and A_3 are obtained as

$$\begin{aligned}A_1 &= \alpha' + \alpha x'\Gamma^1_{11} + \alpha y'\Gamma^1_{12} + \beta x'\Gamma^1_{12} + \beta y'\Gamma^1_{22}\\ &\quad + dx'LG + dy'MG - dx'MF - dy'NF,\\ A_2 &= \beta' + \alpha x'\Gamma^2_{11} + \alpha y'\Gamma^2_{12} + \beta x'\Gamma^2_{12} + \beta y'\Gamma^2_{22}\\ &\quad - dx'LF - dy'MF + dx'ME + dy'NE,\\ A_3 &= d' + \alpha x'L + \alpha y'M + \beta x'M + \beta y'N\\ &\quad + dx'\Gamma^1_{11} + dy'\Gamma^1_{12} + dx'\Gamma^2_{12} + dy'\Gamma^1_{22}.\end{aligned}$$

Hence the normal $\vec{n}$ to the curve $\xi(s)$ is given by

$$\begin{aligned}\vec{n} = \frac{1}{\kappa}\vec{t}' &= \frac{1}{\kappa}\big\{(A_1'\psi_x + A_2'\psi_y + A_3'\vec{N}) + A_1(x'\psi_{xx} + y'\psi_{xy})\\ &\quad + A_2(x'\psi_{xy} + y'\psi_{yy}) + A_3(x'\psi_{xx} + y'\psi_{xy}) \times \psi_y\\ &\qquad + A_3\psi_x \times (x'\psi_{xy} + y'\psi_{yy})\big\}\\ &= B_1\psi_x + B_2\psi_y + B_3\vec{N},\end{aligned}$$

where B_1, B_2 and B_3 are expressed as

$$\begin{aligned}
B_1 &= A_1' + A_1x'\Gamma_{11}^1 + A_1y'\Gamma_{12}^1 + A_2x'\Gamma_{12}^1 + A_2y'\Gamma_{22}^1 \\
&\qquad + A_3x'GL + A_3y'GM - A_3x'FM - A_3y'FN, \\
B_2 &= A_2' + A_1x'\Gamma_{11}^2 + A_1y'\Gamma_{12}^2 + A_2x'\Gamma_{12}^2 + A_2y'\Gamma_{22}^2 \\
&\qquad - A_3x'FL - A_3y'FM + A_3x'EM + A_3y'EN, \\
B_3 &= A_3' + A_1x'L + A_1y'M + A_2x'M + A_2y'N \\
&\qquad + A_3x'\Gamma_{11}^1 + A_3y'\Gamma_{12}^1 + A_3x'\Gamma_{12}^2 + A_3y'\Gamma_{22}^2.
\end{aligned}$$

Again the binormal vector $\vec{b}$ of $\xi(s)$ is written as

$$\begin{aligned}
\vec{b} &= (A_1\psi_x + A_2\psi_y + A_3\vec{N}) \times (B_1\psi_x + B_2\psi_y + B_3\vec{N}) \\
&= C_1\psi_x + C_2\psi_y + C_3\vec{N},
\end{aligned}$$

where C_1, C_2 and C_3 are obtained as

$$\begin{aligned}
C_1 &= F(A_3B_1 - A_1B_3) + G(A_3B_2 - A_2B_3), \\
C_2 &= E(A_1B_3 - A_3B_1) + F(A_2B_3 - A_3B_2), \\
C_3 &= A_1B_2 - A_2B_1.
\end{aligned}$$

THEOREM 3.1. *Let $\xi(s)$ be a unit speed parametrized curve on P whose position vector lies in the plane generated by $\vec{T}$ and $\vec{N}$. Then the following statements hold:*

(a) *The distance function $\rho = \|\xi\|$ is given by $\rho^2 = \alpha^2E + 2\alpha\beta F + \beta^2G + d^2$.*

(b) *The component of the position vector of the curve $\xi(s)$ along the tangent vector to the curve is given by*

$$\langle \vec{t}, \xi(s)\rangle = (\alpha E + \beta F)A_1 + (\alpha F + \beta G)A_2 + dA_3.$$

(c) *The component of the position vector of the curve $\xi(s)$ along the normal vector to the curve is given by*

$$\langle \vec{n}, \xi(s)\rangle = (\alpha E + \beta F)B_1 + (\alpha F + \beta G)B_2 + dB_3.$$

(d) *The component of the position vector of the curve $\xi(s)$ along the binormal vector to the curve is given by*

$$\langle \vec{b}, \xi(s)\rangle = (\alpha E + \beta F)C_1 + (\alpha F + \beta G)C_2 + dC_3.$$

PROOF. Let $\xi(s)$ be a unit speed parametrized curve on P with curvature $\kappa > 0$ whose position vector lies in the plane generated by $\vec{T}$ and $\vec{N}$. Then

$$\xi(s) = \alpha\psi_x + \beta\psi_y + d\vec{N}.$$

Therefore

$$\begin{aligned}
\rho = \langle \xi(s), \xi(s)\rangle &= \langle \alpha\psi_x + \beta\psi_y + d\vec{N}, \alpha\psi_x + \beta\psi_y + d\vec{N}\rangle \\
&= \alpha^2E + 2\alpha\beta F + \beta^2G + d^2.
\end{aligned}$$

Hence (a) is proved.

The component of the position vector of $\xi(s)$ along $\vec{t}$ is given by

$$\begin{aligned}\langle \vec{t}, \xi(s)\rangle &= \langle A_1\psi_x + A_2\psi_y + A_3\vec{N}, \alpha\psi_x + \beta\psi_y + d\vec{N}\rangle\\ &= \alpha A_1 E + \alpha A_2 F + \beta A_1 F + \beta A_2 G + dA_3\\ &= (\alpha E + \beta F)A_1 + (\alpha F + \beta G)A_2 + dA_3.\end{aligned}$$

This proves (b).

The component of the position vector of the curve $\xi(s)$ along the normal vector to the curve is given by

$$\begin{aligned}\langle \vec{n}, \xi(s)\rangle &= \langle B_1\psi_x + B_2\psi_y + B_3\vec{N}, \alpha\psi_x + \beta\psi_y + d\vec{N}\rangle\\ &= \alpha B_1 E + \alpha B_2 F + \beta B_1 F + \beta B_2 G + dB_3\\ &= (\alpha E + \beta F)B_1 + (\alpha F + \beta G)B_2 + dB_3.\end{aligned}$$

This proves (c).

The component of the position vector of the curve $\xi(s)$ along the binormal vector to the curve is given by

$$\begin{aligned}\langle \vec{b}, \xi(s)\rangle &= \langle C_1\psi_x + C_2\psi_y + C_3\vec{N}, \alpha\psi_x + \beta\psi_y + d\vec{N}\rangle,\\ &= \alpha C_1 E + \alpha C_2 F + \beta C_1 F + \beta C_2 G + dC_3,\\ &= (\alpha E + \beta F)C_1 + (\alpha F + \beta G)C_2 + dC_3.\end{aligned}$$

This proves (d). □

Theorem 3.2. *Let $h\colon P \to \bar{P}$ be an isometry and $\xi(s)$ be a unit speed parametrized curve on P whose position vector lies in the plane generated by $\vec{T}$ and $\vec{N}$. Then the following statements hold:*

(a) *The distance function is invariant under isometry.*
(b) *Tangential component of the position vector is given by*

$$\begin{aligned}\langle \bar{\vec{t}}, \bar{\xi}\rangle - \langle \vec{t}, \xi\rangle = (\alpha E + \beta F)(\alpha + dG - dF)(\bar{L} - L)x' + (\alpha x' + \beta y' dy' G - dx' F\\ - dy' F + dx' E)(\bar{M} - M) + (\beta y' - dy' F + dy' E)(\bar{N} - N).\end{aligned}$$

Proof. The distance function of the curve $\bar{\xi}$ on $\bar{P}$ is given by

$$\begin{aligned}\bar{\rho}^2 &= \alpha^2\bar{E} + 2\alpha\beta\bar{F} + \beta^2\bar{G} + d^2\\ &= \alpha^2 E + 2\alpha\beta F + \beta^2 G + d^2 = \rho^2.\end{aligned}$$

Hence (a) is proved.

Now

$$\langle \bar{\vec{t}}, \bar{\xi}\rangle - \langle \vec{t}, \xi\rangle = (\alpha E + \beta F)(\bar{A}_1 - A_1) + (\alpha F + \beta G)(\bar{A}_2 - A_2) + d(\bar{A}_3 - A_3)$$

$$\begin{aligned}\bar{A}_1 - A_1 &= dx' G(\bar{L} - L) + d(y'G - x'F)(\bar{M} - M) - dy' F(\bar{N} - N),\\ \bar{A}_2 - A_2 &= -dx' F(\bar{L} - L) + d(y'F - x'E)(\bar{M} - M) - dy' E(\bar{N} - N),\\ \bar{A}_3 - A_3 &= \alpha x'(\bar{L} - L) + (\alpha y' - \beta x')(\bar{M} - M) - \beta y'(\bar{N} - N).\end{aligned}$$

Hence

$$\langle \vec{\bar{t}}, \bar{\xi} \rangle - \langle \vec{t}, \xi \rangle = (\alpha E + \beta F)(\alpha + dG - dF)(\bar{L} - L)x' + (\alpha x' + \beta y' dy' G - dx' F - dy' F + dx' E)(\bar{M} - M) + (\beta y' - dy' F + dy' E)(\bar{N} - N).$$

Thus (b) is proved. □

Now let $\eta(s)$ be a curve whose position vector lies in the plane generated by $\vec{N}$ and $\vec{T} \times \vec{N}$. Then the curve is given by

$$\begin{aligned} \eta(s) &= e\vec{N} + f\vec{T} \times \vec{N} = e(\psi_x \times \psi_y) + f\{(aF + bG)\psi_x - (aE + bF)\psi_y\} \\ &= f(aF + bG)\psi_x + f(aE - bF)\psi_y + e(\psi_x \times \psi_y), \end{aligned}$$

where e and f are real constants not simultaneously zero. Since the curve $\eta(s)$ is a linear combination of ψ_x, ψ_y and $\psi_x \times \psi_y$, it follows from (3.1) that the same conclusion of Theorem 3.1 and Theorem 3.2 holds for the curve $\eta(s)$ whose position vector is lying in the plane generated by ψ_x, ψ_y and $\psi_x \times \psi_y$.

Again if $\theta(s)$ is a curve whose position vector lies in the plane generated by $\vec{T}$ and $\vec{T} \times \vec{N}$. Then the curve is given by

$$\begin{aligned} \theta(s) &= g\vec{T} + h\vec{T} \times \vec{N} = g(a\psi_x + b\psi_y) + h\{(aF + bG)\psi_x - (aE + bF)\psi_y\} \\ &= (ga + haF + hbG)\psi_x + (gb - haE - hbF)\psi_y, \end{aligned}$$

where g and h are real constants not simultaneously zero. Since the curve $\theta(s)$ is the linear combination of ψ_x and ψ_y, it follows that the position vector of the curve $\theta(s)$ lies in the tangent plane. The properties of such type of a curve is described in [**11**] by Shaikh and Ghosh.

Acknowledgment. The second author greatly acknowledges The University Grants Commission, Government of India for the award of Junior Research Fellow.

References

1. C. Camci, L. Kula, K. Ilarslan, *Characterizations of the position vector of a surface curve in Euclidean 3-space*, An. Ştiinţ. Univ. “Ovidius” Constanţa, Ser. Mat. **19**(3) (2011), 59–70.
2. B.-Y. Chen, *What does the position vector of a space curve always lie in its rectifying plane*?, Am. Math. Mon. **110** (2003), 147–152.
3. B.-Y. Chen, F. Dillen, *Rectfying curve as centrode and extremal curve*, Bull. Inst. Math. Acad. Sinica **33**(2) (2005), 77–90.
4. S. Deshmukh, B.-Y. Chen, S. H. Alshammari, *On rectifying curves in Euclidean 3-space*, Turk. J. Math. **42** (2018), 609–620.
5. M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, New Jersey, 1976.
6. K. Ilarslan, E. Nesovic, *Some characterizations of null, pseudo null and partially null rectifying curves in Minkowski space-time*, Taiwan. J. Math. **12**(5) (2008), 1035–1044.
7. M. S. Lone, *Geometric invariants of normal curves under conformal transformation in* $\mathbb{E}^3$, arXiv:1908.03527, (2019).
8. A. Pressley, *Elementary Differential Geometry*, Springer-Verlag, 2001.
9. A. A. Shaikh, P. R. Ghosh, *Rectifying curves on a smooth surface immersed in the Euclidean space*, Indian J. Pure Appl. Math. **50**(4) (2019), 883–890.
10. A. A. Shaikh, P. R. Ghosh, *Rectifying and osculating curves on a smooth surface*, Indian J. Pure Appl. Math. **51**(1) (2020), 67–75.

11. ______, *Curves on a smooth surface with position vectors lie in the tangent plane*, Indian J. Pure Appl. Math. **51**(3) (2020), 1097–1104.
12. A. A. Shaikh, Y. H. Kim, P. R. Ghosh, *Some characterizations of rectifying and osculating curves on a smooth immersed surface*, J. Geom Phys. **171** (2022), article 104387.
13. A. A. Shaikh, M. S. Lone, P. R. Ghosh, *Normal curves on a smooth immersed surface*, Indian J. Pure Appl. Math. **51** (2020), 1343–1355.
14. ______, *Rectifying curves under conformal transformation*, J. Geom Phys. **163** (2021), article 104117.
15. ______, *Conformal image of an osculating curve on a smooth immersed surface*, J. Geom. Phys. **151** (2020), article 103625.

Department of Mathematics
University of Burdwan
Burdwan
West Bengal
India
aask2003@yahoo.co.in
aashaikh@math.buruniv.ac.in

mailtopinaki94@gmail.com

(Received 15 11 2020)
(Revised 01 02 2022)