PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 112(126) (2022), 83–93 DOI: https://doi.org/10.2298/PIM2226083A

DUHAMEL BANACH ALGEBRA STRUCTURE OF SOME SPACE AND RELATED TOPICS

Mevlut Altıntaş, Mehmet Gürdal, and Ramiz Tapdigoglu

Abstract. Let α be a fixed complex number, and let Ω be a simply connected region in complex plane $\mathbb{C}$ that is starlike with respect to $\alpha \in \Omega$. We define some Banach space of analytic functions on Ω and prove that it is a Banach algebra with respect to the α-Duhamel product defined by

$$\Big(f \circledast_\alpha g\Big)(z) := \frac{d}{dz}\int_\alpha^z f(z+\alpha-t)g(t)\,dt.$$

We prove that its maximal ideal space consists of the homomorphism h_α defined by $h_\alpha(f) = f(\alpha)$. Further, we characterize the lattice of invariant subspaces of the integration operator $J_\alpha f(z) = \int_\alpha^z f(t)\,dt$. Moreover, we describe in terms of α-Duhamel operators the extended eigenvectors of J_α.

1. Introduction

Let α be a fixed complex number. Let $\Omega \subset \mathbb{C}$ be a simply connected bounded region containing the point α such that $\mu z + (1-\mu)\alpha \in \Omega$ for all μ, $0 \leqslant \mu \leqslant 1$, i.e., Ω is starlike with respect to $\alpha \in \Omega$.

Let $C^{(n)}(\Omega)$ be the space of all single-valued and analytic functions on Ω with n^{th} derivative continuous on $\bar{\Omega}$. The space $C^{(n)}(\Omega)$ is a Banach space with the norm

$$\|f\|_n := \max\Big\{\max_{z\in\bar{\Omega}} |f^{(i)}(z)| : i = 0,1,2,\dots,n\Big\}.$$

The α-Duhamel product is defined in $C^{(n)}(\Omega)$ by the formula

$$\Big(f \circledast_\alpha g\Big)(z) := \frac{d}{dz}\int_\alpha^z f(z+\alpha-t)g(t)\,dt = \int_\alpha^z f'(z+\alpha-t)g(t)\,dt + f(\alpha)g(z), \tag{1.1}$$

2020 *Mathematics Subject Classification:* Primary 46E35; Secondary 46J15.

Key words and phrases: α-Duhamel product, Banach algebra, α-integration operator, maximal ideal space, extended eigenvector, invariant subspace.

Communicated by Stevan Pilipović.

where the integral is taken over the segment joining the points α and z ($z \in \Omega$). The α-integration operator J_α is defined on $C^{(n)}(\Omega)$ by the formula

$$(J_\alpha f)(z) := \int_\alpha^z f(t)\,dt, \quad z \in \Omega,$$

where the integration is performed as above over straight-line segments connecting the points α and z.

It is known (and easy to verify) that α-Duhamel product satisfies all the axioms of multiplication and the constant function $\mathbf{1}$ is the unit element with respect to this product. Note also that $J_\alpha f = (z-\alpha) \circledast_\alpha f$ for any $f \in C^{(n)}(\Omega)$. In general, for any $g \in C^{(n)}(\Omega)$, the α-Duhamel operator $D_g\colon C^{(n)}(\Omega) \to C^{(n)}(\Omega)$ is defined by

$$(D_{\alpha,g} f)(z) := g \circledast_\alpha f, \quad f \in C^{(n)}(\Omega).$$

Recall that the classical Duhamel product (i.e., $\alpha = 0$) was firstly introduced and investigated by Wigley in [**33**]. In the sequel, he elaborated in [**33**, **34**] at length on this product and used it to provide an algebra structure to the Frechet space $Hol(\mathbb{D})$ of all holomorphic functions, as well as to the Hardy spaces $H^p(\mathbb{D})$, $1 \leqslant p < +\infty$, and he characterized their maximal ideal spaces. Merryfield and Watson [**27**] extended the matter to the context of vector-valued Hardy spaces of the polydisc. Guediri and etg., proved in [**10**] that the Bergman space $L_a^2(\mathbb{D})$ of analytic functions on the unit disc $\mathbb{D} \subset \mathbb{C}$ is a Banach algebra with respect to the classical Duhamel product $\circledast$ defined by

$$(f \circledast g)(z) := \frac{d}{dz}\int_0^z f(z-t)g(t)\,dt = \int_0^z f'(z-t)g(t)\,dt + f(0)g(z). \tag{1.2}$$

In general, the Duhamel product and α-Duhamel product has been extensively explored on various spaces by several authors in [**8**–**10**, **13**, **24**, **27**, **34**]. Many applications of Duhamel products have been well investigated, see, for example, [**5**,**7**,**10**,**14**,**21**,**22**,**28**,**32**]. In particular, Ivanova and Melikhov used Duhamel product in their recent works [**15**–**18**] in investigation of the commutant of Pommiez operator. Also in [**29**–**31**], the author applied Duhamel product (1.2) in describing of nontrivial invariant subspaces of the classical Volterra integration operator W, $Wf(x,y) = \int_0^x \int_0^y f(t,\tau)\,d\tau dt$, on some W-invariant subspace.

Our investigation is motivated by papers [**1**, **9**, **13**]. Namely, we prove that $(C^{(n)}(\Omega), \circledast_\alpha)$ is a commutative Banach algebra and describe its maximal ideal space and its J_α-invariant subspaces. We also study the extended eigenvectors in the sense of Biswas, Lambert and Petrović [**2**] for the α-integration operator J_α on $C^{(n)}(\Omega)$.

2. On the Banach algebra $(C^{(n)}(\Omega), \circledast_\alpha)$

In the present section, we study the Banach algebra structure of the space $C^{(n)}(\Omega)$ with respect to the α-Duhamel product (1.1) and describe its maximal ideal space. We begin with the following auxiliary lemmas.

LEMMA 2.1. *The space $C^{(n)}(\Omega)$ is a commutative Banach algebra with the α-Duhamel product with the identity $f = \mathbf{1}$.*

PROOF. Since $C^{(n)}(\Omega)$ is a Banach space, only the multiplicative norm product inequality needs to be shown. In fact, for any two functions $f, g \in C^{(n)}(\Omega)$, we have by induction that

$$\big(f \circledast_\alpha g\big)^{(k)}(z) := \int_\alpha^z f^{(k+1)}(z+\alpha-t)g(t)\,dt + \sum_{m=0}^{k} f^{(m)}(\alpha)g^{(k-m)}(z).$$

An integration by parts leads to

$$\big(f \circledast_\alpha g\big)^{(k)}(z) := \int_\alpha^z f^{(k)}(z+\alpha-t)g'(t)\,dt + \sum_{m=0}^{k-1} f^{(m)}(\alpha)g^{(k-m)}(z) + g(\alpha)f^{(k)}(z),$$

so we obtain that

$$\big|(f \circledast_\alpha g)^{(k)}(z)\big| \leqslant d\|f^{(k)}\|\|g'\| + \sum_{m=0}^{k-1} \|f^{(m)}\|\|g^{(k-m)}\| + \|g\|\|f^{(k)}\|,$$

where $d := \operatorname{diam}(\Omega)$, and hence $\big\|(f \circledast_\alpha g)^{(k)}(z)\big\| \leqslant (d+k+1)\|f\|_n\|g\|_n$, which implies that $\|f \circledast_\alpha g\|_n \leqslant (d+n+1)\|f\|_n\|g\|_n$, as desired. By considering the equivalent norm $\|f\|_{n,1} := (d+n+1)^{1/2}\|f\|_n$, we have from the latter inequality that $\|f \circledast_\alpha g\|_{n,1} \leqslant \|f\|_{n,1}\|g\|_{n,1}$. This proves that $(C^{(n)}(\Omega), \circledast_\alpha)$ is a Banach algebra. Since clearly $f \circledast_\alpha g = g \circledast_\alpha f$ and $f \circledast_\alpha \mathbf{1} = \mathbf{1} \circledast_\alpha f$ for all $f, g \in C^{(n)}(\Omega)$. □

Next lemma gives an invertibility criterion with respect to the α-Duhamel product.

LEMMA 2.2. *If $f \in (C^{(n)}(\Omega), \circledast_\alpha)$, then it is $\circledast_\alpha$-invertible if and only if $f(\alpha)\neq 0$.*

PROOF. In fact, if $g \in (C^{(n)}(\Omega), \circledast_\alpha)$ is the $\circledast_\alpha$-inverse of f, then we have from (1.1) that $1 = (f \circledast_\alpha g)(\alpha) = f(\alpha)g(\alpha)$, hence $f(\alpha) \neq 0$.

Conversely, if $f(\alpha) \neq 0$, we set $D_{\alpha,f} := f \circledast_\alpha g$. Now we prove that the α-Duhamel operator $D_{\alpha,f}$ is an invertible operator on the space $C^{(n)}(\Omega)$. For this purpose, let us write f as $f = F + f(\alpha)$, where $F := f - f(\alpha)$. Whence $D_{\alpha,f} = f(\alpha)I + D_{\alpha,F}$, where I is an identity operator on $C^{(n)}(\Omega)$. Note that for the proof of invertibility of operator $D_{\alpha,f}$, we can use, in fact, two methods. The first method is based on the classical Fredholm alternative for compact operators and the classical Titchmarsh convolution theorem (see, Karaev [**20**]). The second one uses the Gelfand formula for the spectral radius of elements of Banach algebra. Here we will apply the second method. The present proof is similar to one of the paper [**9**, Lemma 2.2]. For completeness of presentation, we provide here this proof. So, by considering that $f(\alpha) \neq 0$, it suffices to prove that $D_{\alpha,F}$ is a quasinilpotent operator, i.e., $\sigma(D_{\alpha,F}) = \{0\}$. For this, by using Gelfand formula [**9**, **22**], we prove that

$$\lim_{k\to\infty} \big\|D^k_{\alpha,F}\big\|^{\frac{1}{k}} = 0. \tag{2.1}$$

Before passing to the proof of (2.1), let us define the following convolution operator on $C^{(n)}(\Omega)$

$$*(\mathcal{K}_{\alpha,f}g)(z) := \Big(f *_\alpha g\Big)(z) := \int_\alpha^z f(z+\alpha-t)g(t)\,dt.$$

Now we are ready to start the proof of (2.1). In fact, we have:

$$(D_{\alpha,F}g)(z) := \frac{d}{dz}\int_\alpha^z F(z+\alpha-t)g(t)\,dt = \int_\alpha^z F'(z+\alpha-t)g(t)\,dt = \mathcal{K}_{\alpha,F'},g(z),$$

that is $D_{\alpha,F}g = \mathcal{K}_{\alpha,F'},g$ for any $g \in C^{(n)}(\Omega)$, and therefore $D_{\alpha,F} = \mathcal{K}_{\alpha,F'}$. Thus we get

$$\begin{aligned}\big(\mathcal{K}^2_{\alpha,F'}g\big)(z) &= \mathcal{K}_{\alpha,F'}(\mathcal{K}_{\alpha,F'}g)(z) = \int_\alpha^z F'(z+\alpha-t)(\mathcal{K}_{\alpha,F'}g)(t)\,dt\\ &= \int_\alpha^z F'(z+\alpha-t)\bigg(\int_\alpha^t F'(t+\alpha-\tau)\,d\tau\bigg)\,dt.\end{aligned}$$

Hence, we see that

$$\big|\big(\mathcal{K}^2_{\alpha,F'}g\big)(z)\big| \leqslant \|F\|_n^2\|g\|_n\frac{|z-\alpha|^2}{2!}.$$

It can be easily obtained by induction that

$$\big|\big(\mathcal{K}^k_{\alpha,F'}g\big)(z)\big| \leqslant \|F\|_n^k\|g\|_n\frac{|z-\alpha|^k}{k!}.$$

On the other hand, we have

$$\begin{aligned}\big(\mathcal{K}^2_{\alpha,F'}g\big)'(z) &= \int_\alpha^z F''(z+\alpha-t)\bigg(\int_\alpha^t F'(z+\alpha-\tau)\,d\tau\bigg)\,dt\\ &\qquad + F'(\alpha)\int_\alpha^z F'(z+\alpha-\tau)g(\tau)\,d\tau.\end{aligned}$$

Thus, we obtain that

$$\begin{aligned}\big|\big(\mathcal{K}^2_{\alpha,F'}g\big)'(z)\big| &\leqslant \|F\|_n^2\|g\|_n\Big(\frac{|z-\alpha|^2}{2}+|z-\alpha|\Big)\\ &\leqslant \|F\|_n^2\|g\|_n\frac{(|z-\alpha|+1)^2}{2!}.\end{aligned}$$

Now, assume by induction that

$$\big|\big(\mathcal{K}^k_{\alpha,F'}g\big)'(z)\big| \leqslant \|F\|_n^k\|g\|_n\frac{(|z-\alpha|+1)^k}{k!}.$$

By differentiation we have

$$\big(\mathcal{K}^{k+1}_{\alpha,F'}g\big)'(z) = \int_\alpha^z F''(z+\alpha-t)\big(\mathcal{K}^k_{\alpha,F'}g\big)(t)\,dt + F''(\alpha)\big(\mathcal{K}^k_{\alpha,F'}g\big)(z),$$

hence we conclude that

$$\big|\big(\mathcal{K}_{\alpha,F'}^{k+1}g\big)'(z)\big| \leqslant \|F\|_n^{k+1}\|g\|_n\Big(\frac{|z-\alpha|^{k+1}}{(k+1)!}+\frac{|z-\alpha|^k}{k!}\Big) \leqslant \|F\|_n^{k+1}\|g\|_n\frac{(|z-\alpha|+1)^{k+1}}{(k+1)!}.$$

By considering that

$$\big(\mathcal{K}_{\alpha,F'}^2 g\big)'(z)=\int_\alpha^z F''(z+\alpha-t)(\mathcal{K}_{\alpha,F'}g)(t)\,dt+F'(\alpha)(\mathcal{K}_{\alpha,F'}g)(z),$$

we have

$$\big(\mathcal{K}_{\alpha,F'}^2 g\big)''(z)=\int_\alpha^z F'''(z+\alpha-t)(\mathcal{K}_{\alpha,F'}g)(t)\,dt + F''(\alpha)(\mathcal{K}_{\alpha,F'}g)(z)+F'(\alpha)(\mathcal{K}_{\alpha,F'}g)'(z),$$

which leads to

$$\big|\big(\mathcal{K}_{\alpha,F'}^2 g\big)''(z)\big| \leqslant \|F\|_n^2\|g\|_n\Big(\frac{|z-\alpha|^2}{2}+|z-\alpha|+\frac{(|z-\alpha|+1)^2}{2}\Big) \leqslant \|F\|_n^2\|g\|_n\frac{(|z-\alpha|+2)^2}{2}.$$

Thus, by induction we get

$$\big|\big(\mathcal{K}_{\alpha,F'}^k g\big)^{(j)}(z)\big| \leqslant \|F\|_n^k\|g\|_n\frac{(|z-\alpha|+j)^k}{k!}$$

for all $j=2,3,\dots,n$, which implies that

$$\big\|\mathcal{K}_{\alpha,F'}^k g\big\|_n \leqslant \|F\|_n^k\|g\|_n\frac{(n+1)^k}{(k!)^{1/k}}.$$

Hence,

$$\big\|\mathcal{K}_{\alpha,F'}^k\big\|^{1/k} \leqslant \|F\|_n\frac{n+1}{(k!)^{1/k}} \to 0 \quad \text{as} \quad k\to\infty.$$

This shows that $\mathcal{K}_{\alpha,F'}$ is a quasinilpotent operator. Consequently, $D_{\alpha,f}=f(\alpha)I+D_{\alpha,F}$ is invertible. □

Now we are ready to state our main result.

THEOREM 2.1. $\big(C^{(n)}(\Omega),\circledast_\alpha\big)$ *is a unital commutative Banach algebra with maximal ideal space* $\mathcal{M}=\{\varphi_\alpha\}$*, where* $\varphi_\alpha\colon C^{(n)}(\Omega)\to\mathbb{C}$ *and* $\varphi_\alpha(f)=f(\alpha)$.

PROOF. We denote here by $\sigma(f)$ the spectrum of the element f in the Banach algebra $(C^{(n)}(\Omega),\circledast_\alpha)$ with respect to the α-Duhamel product $\circledast_\alpha$. It follows from Lemma 2.2 that $\sigma(f)=\{f(\alpha)\}$ and by Gelfand's theory we see that $\mathcal{M}=\{\varphi_\alpha\}$. In fact, the set $\big\{f\in C^{(n)}(\Omega) : f(\alpha)=0\big\}$ is a maximal ideal. Any other proper ideal cannot have an element which does not vanish at α, hence there is only one maximal ideal. Thus, the maximal ideal space $\mathcal{M}$ of the Banach algebra $(C^{(n)}(\Omega),\circledast_\alpha)$ consists of one homomorphism, namely, evaluation at α, and the Gelfand transform is trivial. □

Let J_α, $J_\alpha f(z) = \int_\alpha^z f(t)\,dt$, be the Volterra integration operator on $C^{(n)}(\Omega)$. It is easy to see from the definition of the α-Duhamel product (see formula (1.1)) that $J_\alpha f(z) = (z-\alpha) \circledast_\alpha f$, in general,

$$J_\alpha^m f(z) = \frac{(z-\alpha)^m}{m!} \circledast_\alpha f, \quad m \geqslant 0. \tag{2.2}$$

Recall that the function $g \in C^{(n)}(\Omega)$ is called a cyclic vector of operator J_α if $\operatorname{span}\left\{J_\alpha^m g : m \geqslant 0\right\} = C^{(n)}(\Omega)$. Since Ω is the star-like bounded region including the point α, it is known that (see Fage and Nagnibida [**7**]) $\left\{(z-\alpha)^m : m \geqslant 0\right\}$ is a complete system in $C^{(n)}(\Omega)$. Therefore, the following corollary is immediate from Lemma 2.2.

Corollary 2.1. *The nonzero function $f \in C^{(n)}(\Omega)$ is cyclic for operator J_α if and only if $f(\alpha) \neq 0$.*

Proof. Indeed, it follows from (2.2) that

$$\begin{aligned}
\operatorname{span}\left\{J_\alpha^m g : m \geqslant 0\right\} &= \operatorname{span}\left\{\frac{(z-\alpha)^m}{m!} \circledast_\alpha f : m \geqslant 0\right\}\\
&= \operatorname{span}\left\{D_{\alpha,f}\Big(\frac{(z-\alpha)^m}{m!}\Big) : m \geqslant 0\right\}\\
&= \overline{D_{\alpha,f}\operatorname{span}\left\{\frac{(z-\alpha)^m}{m!} : m \geqslant 0\right\}}\\
&= \overline{D_{\alpha,f} C^{(n)}(\Omega)},
\end{aligned}$$

hence

$$\operatorname{span}\left\{J_\alpha^m g : m \geqslant 0\right\} = \overline{D_{\alpha,f} C^{(n)}(\Omega)}. \tag{2.3}$$

Now the result follows from Lemma 2.2 and (2.3). □

The following corollary of Theorem 2.1 can be proved by the same arguments of the works Wigley [**33**], Fage and Nagnibida [**7**] and Tapdigoglu [**29**, **30**], and therefore omitted.

Corollary 2.2. *The lattice of nontrivial J_α-invariant subspaces is the set*

$$\operatorname{Lat}(J_\alpha) = \left\{E^{(k)} : k = 0,1,2,\dots,n\right\},$$
$$E^{(k)} := \left\{f \in C^{(n)}(\Omega) : f(\alpha) = f'(\alpha) = \dots = f^{(k)}(\alpha) = 0\right\},$$

that is, J_α is a unicellular operator on $C^{(n)}(\Omega)$.

3. Extended eigenvalues and extended eigenvectors of operator J_α

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be a unit disc and let $\alpha \in \mathbb{D}$ be a fixed point. In this section, we characterize extended eigenvectors in the sense of Malamud–Biswas–Lambert–Petrović [**2**, **25**, **26**] of the Volterra integration operator J_α on the space $C^{(n)}(\mathbb{D})$. Recall that if A is a nonzero bounded linear operator on $C^{(n)}(\mathbb{D})$ and λ is a complex number such that

$$J_\alpha A = \lambda A J_\alpha, \tag{3.1}$$

then λ is called an extended eigenvalue of J_α and operator A is called an extended eigenvector corresponding to λ. More detailed information about extended eigenvalues and extended eigenvectors can be found in [**2**–**4**, **10**, **14**, **19**, **23**, **31**].

Since $\ker(J_\alpha) = \{0\}$, it follows from (3.1) that the point $\lambda = 0$ is not an extended eigenvalue of J_α, i.e., $0 \notin \mathrm{ext}_p(J_\alpha)$ (the set of all extended eigenvalues of J_α). Consequently, $\mathrm{ext}_p(J_\alpha) \subset \mathbb{C} \setminus \{0\}$. Our next results shows that the set $\mathbb{C} \setminus \{0\}$ is the extended point spectrum of J_α, that is $\mathrm{ext}_p(J_\alpha) = \mathbb{C} \setminus \{0\}$. For the related results, see [**6**, **11**, **12**, **23**, **31**].

THEOREM 3.1. *Let $\lambda \in \mathbb{C} \setminus \{0\}$ be any fixed number. Let $J_\alpha \colon C^{(n)}(\mathbb{D}) \to C^{(n)}(\mathbb{D})$ be an integration operator.*

(i) *If $|\lambda| > 1$, then for every operator $B \in \mathcal{B}(C^{(n)}(\mathbb{D}))$ with $B\mathbf{1} \neq 0$, the operator $T = D_{\alpha,B\mathbf{1}} C_{1/\lambda}$ satisfies the equation*

$$J_\alpha T = \lambda T J_\alpha, \tag{3.2}$$

and conversely, every nonzero operator $T \in \mathcal{B}(C^{(n)}(\mathbb{D}))$ satisfying (3.2) *must be of the form $T = D_{\alpha,B\mathbf{1}} C_{1/\lambda}$, here $D_{\alpha,B\mathbf{1}}$ is the α-Duhamel operator on $C^{(n)}(\mathbb{D})$ defined by $D_{\alpha,B\mathbf{1}} g = B\mathbf{1} \circledast_\alpha g$ and $(C_{1/\lambda} f)(z) = f\big(\frac{1}{\lambda} z\big)$.*

(ii) *If $|\lambda| \leqslant 1$, then $J_\alpha B = \lambda B J_\alpha$ if and only if B satisfies $BC_\lambda = D_{\alpha,B\mathbf{1}}$, where $(C_\lambda f)(z) = f(\lambda z)$.*

PROOF. (i) Let $B \in \mathcal{B}(C^{(n)}(\mathbb{D}))$ be any operator such that $B\mathbf{1} \neq 0$, where $\mathbf{1}$ is the unit element of the Banach algebra $(C^{(n)}(\mathbb{D}), \circledast_\alpha)$. We consider the operator $D_{\alpha,B\mathbf{1}} C_{1/\lambda}$, where $D_{\alpha,B\mathbf{1}}$ is the α-Duhamel operator on $C^{(n)}(\mathbb{D})$ with symbol $B\mathbf{1} \in C^{(n)}(\mathbb{D})$, and $C_{1/\lambda}$ is the simple composition operator defined on $C^{(n)}(\mathbb{D})$ by $(C_{1/\lambda} f)(z) = f\big(\frac{1}{\lambda} z\big)$. Then it is easy to see that this operator satisfies equation (3.2). In fact, we have for each $f \in C^{(n)}(\mathbb{D})$ that

$$\begin{aligned}
D_{\alpha,B\mathbf{1}} C_{1/\lambda} J_\alpha f(z) &= D_{\alpha,B\mathbf{1}} (J_\alpha f)\Big(\frac{1}{\lambda} z\Big) = B\mathbf{1} \circledast_\alpha (J_\alpha f)\Big(\frac{1}{\lambda} z\Big) \\
&= B\mathbf{1} \circledast_\alpha \Big(\frac{z-\alpha}{\lambda} \circledast_\alpha f\Big(\frac{z}{\lambda}\Big)\Big) \\
&= \frac{z-\alpha}{\lambda} \circledast_\alpha \Big(B\mathbf{1} \circledast_\alpha f\Big(\frac{z}{\lambda}\Big)\Big) = \frac{z-\alpha}{\lambda} \circledast_\alpha D_{\alpha,B\mathbf{1}} C_{1/\lambda} f(z) \\
&= \frac{1}{\lambda}\Big((z-\alpha) \circledast_\alpha D_{\alpha,B\mathbf{1}} C_{1/\lambda} f(z)\Big) = \frac{1}{\lambda} J_\alpha D_{\alpha,B\mathbf{1}} C_{1/\lambda} f(z) \\
&= \frac{1}{\lambda} J_\alpha D_{\alpha,B\mathbf{1}} C_{1/\lambda} f(z),
\end{aligned}$$

and hence, $(\mathcal{D}_{\alpha B\mathbf{1}} C_{1/\lambda}) J_\alpha = \frac{1}{\lambda} J_\alpha (D_{\alpha,B\mathbf{1}} C_{1/\lambda})$ which implies $\lambda (D_{\alpha,B\mathbf{1}} C_{1/\lambda}) J_\alpha = J_\alpha (D_{\alpha,B\mathbf{1}} C_{1/\lambda})$, as desired.

On the other hand, it is not difficult to show that $D_{\alpha,B\mathbf{1}} C_{1/\lambda}$ is a nonzero operator on $C^{(n)}(\mathbb{D})$. In fact, if not, then $D_{\alpha,B\mathbf{1}} C_{1/\lambda} = 0$, and hence $D_{\alpha,B\mathbf{1}} C_{1/\lambda} f = 0$ for all $f \in C^{(n)}(\mathbb{D})$, or equivalently,

$$\frac{d}{dz} \int_\alpha^z (B\mathbf{1})(z + \alpha - t) f(t)\, dt = 0$$

for all $z \in \mathbb{D}$. From this we get that

$$\int_{\alpha}^{z} (B\mathbf{1})(z+\alpha-t)f(t)\,dt = \text{const}$$

for all $z \in \mathbb{D}$. In particular, for $z = \alpha$ we have that const $= 0$, and hence

$$\int_{\alpha}^{z} (B\mathbf{1})(z+\alpha-t)f(t)\,dt = 0 \tag{3.3}$$

for all $z \in \mathbb{D}$. Since f is arbitrary, it follows from (3.3) by commutativity property of the convolution $\underset{\alpha}{*}$ for $f = \mathbf{1}$ that $\int_{\alpha}^{z}(B\mathbf{1})(t)\,dt = 0$, which easily implies that $B\mathbf{1} = 0$. This contradicts to our assumption that $B\mathbf{1} \neq 0$. This shows that actually $D_{\alpha,B\mathbf{1}}C_{1/\lambda}$ is a nonzero operator, so we deduce that $\lambda \in \mathbb{C}$ with $|\lambda| > 1$ is an extended eigenvalue of the operator J_α, that is $\lambda \in \operatorname{ext}_p(J_\alpha)$.

Now, let us prove that every extended eigenvector $B \in \mathcal{B}(C^{(n)}(\mathbb{D}))$ corresponding to this extended eigenvalue λ has the form $\mathcal{D}_{\alpha,B\mathbf{1}}C_{1/\lambda}$. Indeed, let

$$\lambda B J_\alpha = J_\alpha B.$$

Then $\frac{1}{\lambda}J_\alpha B = BJ_\alpha$, and hence $(\frac{1}{\lambda})^m J_\alpha^m B = BJ_\alpha^m$ for all $m \geqslant 0$. Then $\left(\frac{1}{\lambda}\right)^m J_\alpha^m B\mathbf{1}$ $= BJ_\alpha^m\mathbf{1}$, therefore, using formula (2.2), we have

$$\frac{\left(\frac{z-\alpha}{\lambda}\right)^m}{m!} \circledast_\alpha B\mathbf{1} = B\left(\frac{\left(\frac{z-\alpha}{\lambda}\right)^m}{m!} \circledast_\alpha \mathbf{1}\right) = B\left(\frac{\left(\frac{z-\alpha}{\lambda}\right)^m}{m!}\right),$$

hence

$$B\left(\frac{\left(\frac{z-\alpha}{\lambda}\right)^m}{m!}\right) = B\mathbf{1} \circledast_\alpha \frac{\left(\frac{z-\alpha}{\lambda}\right)^m}{m!},$$

thus

$$B((z-\alpha)^m) = B\mathbf{1} \circledast_\alpha \left(\frac{z-\alpha}{\lambda}\right)^m, \quad m \geqslant 0.$$

Therefore, $Bp(z-\alpha) = B\mathbf{1} \circledast_\alpha p\left(\frac{z-\alpha}{\lambda}\right)$ for all polynomials p. Since the polynomials from $(z-\alpha)$ are dense in $C^{(n)}(\mathbb{D})$, we conclude that $(Bf)(z) = \mathcal{D}_{\alpha,B\mathbf{1}}C_{1/\lambda}f(z)$ for all $f \in C^{(n)}(\mathbb{D})$. Consequently, $B = \mathcal{D}_{\alpha,B\mathbf{1}}C_{1/\lambda}$, as required.

(ii) We mention that (see (2.2))

$$\frac{(z-\alpha)^m}{m!} \circledast_\alpha f(z) = J_\alpha^m f(z), \quad \forall f \in C^{(n)}(\mathbb{D}).$$

Let $J_\alpha B = \lambda B J_\alpha$. Then $\lambda^m BJ_\alpha^m = J_\alpha^m B$ for any integer $m \geqslant 0$, in other words, $\lambda^m BJ_\alpha^m f = J_\alpha^m Bf$ for all $f \in C^{(n)}(\mathbb{D})$. In particular, $\lambda^m BJ_\alpha^m\mathbf{1} = J_\alpha^m B\mathbf{1}$, and hence, by virtue of the above identity, we have

$$B\Big(\frac{(\lambda(z-\alpha))^m}{m!} \circledast_\alpha \mathbf{1}\Big) = \Big(\frac{(z-\alpha)^m}{m!} \circledast_\alpha B\mathbf{1}),$$

or equivalently,

$$B(\lambda(z-\alpha))^m = (z-\alpha)^m \circledast_\alpha B\mathbf{1}, \quad m \geqslant 0.$$

Again, since the polynomials $p(z-\alpha)$ are dense in $C^{(n)}(\mathbb{D})$, we obtain

$$(BC_\lambda)f(z) = (Bf)(\lambda z) = B\mathbf{1} \circledast_\alpha f(z) = \mathcal{D}_{\alpha,B\mathbf{1}}f(z)$$

for every $f \in C^{(n)}(\mathbb{D})$. So, $(BC_\lambda)f(z) = \mathcal{D}_{\alpha,B\mathbf{1}}f(z)$ for all $f \in C^{(n)}(\mathbb{D})$. This implies that $BC_\lambda = \mathcal{D}_{\alpha,B\mathbf{1}}$, where $C_\lambda f(z) = f(\lambda z)$, $f \in C^{(n)}(\mathbb{D})$, since C_λ with $|\lambda| \leqslant 1$ is a bounded operator on $C^{(n)}(\mathbb{D})$.

Conversely, if $BC_\lambda = \mathcal{D}_{\alpha,B\mathbf{1}}$, then for every polynomial $p \in C^{(n)}(\mathbb{D})$, we have

$$\begin{aligned} J_\alpha Bp(z) &= J_\alpha BC_\lambda p\Big(\frac{z}{\lambda}\Big) = J_\alpha \mathcal{D}_{\alpha,B\mathbf{1}} p\Big(\frac{z}{\lambda}\Big) \\ &= \mathcal{D}_{\alpha,B\mathbf{1}} J_\alpha p\Big(\frac{z}{\lambda}\Big) = BC_\lambda J_\alpha p\Big(\frac{z}{\lambda}\Big) \\ &= BC_\lambda\Big((z-\alpha) \circledast_\alpha p\Big(\frac{z}{\lambda}\Big)\Big) \\ &= \lambda BC_\lambda\Big(\frac{z-\alpha}{\lambda} \circledast_\alpha p\Big(\frac{z}{\lambda}\Big)\Big) \\ &= \lambda BC_\lambda (J_\alpha p)\Big(\frac{z}{\lambda}\Big) = \lambda B J_\alpha p(z), \end{aligned}$$

hence, $J_\alpha B = \lambda B J_\alpha$, as desired. □

Note that the composition operator C_φ on $C^{(n)}(\mathbb{D})$ in general is the operator defined by $(C_\varphi f)(z) = (f \circ \varphi)(z) = f(\varphi(z))$ for a suitable function $\varphi \colon \mathbb{D} \to \mathbb{D}$. The proof of the following corollary is similar to the proof in [**10**, Corollary 3.1], however only for completeness we provide it here.

COROLLARY 3.1. *The composition operator C_φ satisfies $C_\varphi J_\alpha = \lambda J_\alpha C_\varphi$, where λ is the number such that $|\lambda| > 1$, if and only if $\varphi(z) = \frac{z}{\lambda}$.*

PROOF. Since C_φ is a composition operator, it is obvious that $C_\varphi \mathbf{1} = \mathbf{1}(\varphi(z)) = \mathbf{1}$. Then we have from (i) of Theorem 3.1 that $C_\varphi J_\alpha = \lambda J_\alpha C_\varphi$ if and only if $C_\varphi = \mathcal{D}_{\alpha, C_\varphi \mathbf{1}} C_{1/\lambda} = \mathcal{D}_{\alpha,\mathbf{1}} C_{1/\lambda} = C_{1/\lambda}$, which shows that $\varphi(z) = \frac{z}{\lambda}$ for all $z \in \mathbb{D}$. □

COROLLARY 3.2. $\{J_\alpha\}' = \big\{D_{\alpha,f} : f \in C^{(n)}(\mathbb{D})\big\}$, *i.e., the commutant of operator J_α consists from α-Duhamel operators $D_{\alpha,f}$ with $f \in C^{(n)}(\mathbb{D})$.*

In conclusion, we remark that Theorem 3.1 also implies that $\mathrm{ext}_p(J) = \mathbb{C} \setminus \{0\}$, since it can be shown that the equation $BC_\lambda = \mathcal{D}_{\alpha,B\mathbf{1}}$ has a nonzero solution B for any $\lambda \in \mathbb{C}$ such that $|\lambda| \leqslant 1$.

References

1. A. Aleman, B. Korenblum, *Volterra invariant subspaces of H^p*, Bull. Sci. Math. **132** (2008), 510–528.
2. A. Biswas, A. Lambert, S. Petrović, *Extended eigenvalues and the Volterra operator*, Glasg. Math. J. **44** (2002), 521–534.
3. G. Cassier, H. Alkanjo, *Extended spectrum, extended eigenspaces and normal operators*, J. Appl. Math. Anal. Appl. **418** (2014), 305–316.
4. ———, *Extended spectrum and extended eigenspaces of quasinormal operators*, Banach J. Math. Anal. **11**(2) (2017), 266–281.
5. I. Dimovski, *Convolutional Calculus*, Kluwer, Dordrecht, 1990.
6. I. Yu. Domanov, M. M. Malamud, *On the spectral analysis of direct sums of Riemann-Liouville operators in Sobolev spaces of vector functions*, Integral Equations Oper. Theory **63** (2009), 181–215.

7. M. K. Fage, N. I. Nagnibida, *The Problem of Equivalence of Ordinary Linear Differential Operators*, Novosibirsk, Russia: Nauka Sibirsk Otdel, 1987.
8. M. T. Garayev, *Some properties and applications of convolution algebras*, Adv. Math. Models Appl. **4**(3) (2019), 188–197.
9. M. T. Garayev, H. Guediri, H. Sadraoui, *On some problems in the space* $C^n[0,1]$, Sci. Bull., Ser. A, Appl. Math. Phys., Politeh. Univ. Buchar **78**(1) (2016), 147–156.
10. H. Guediri, M. T. Garayev, H. Sadraoui, *The Bergman space as a Banach algebra*, New York J. Math. **21** (2015), 339–350.
11. M. Gürdal, *Description of extended eigenvalues and extended eigenvectors of integration operators on the Wiener algebra*, Expo. Math. **27** (2009), 153–160.
12. ______, *On the extended eigenvalues and extended eigenvectors of shift operator on the Wiener algebra*, Appl. Math. Lett. **22**(11) (2009), 1727–1729.
13. M. Gürdal, F. Şöhret, *On some operator equation in the space of analytic functions and related questions*, Proc. Est. Acad. Sci. **62** (2013), 81–87.
14. M. Gürdal, M. T. Garayev, S. Saltan, *Some concrete operators and their properties*, Turk. J. Math. **39** (2015), 970–989.
15. O. A. Ivanona, S. N. Melikhov, *On the commutant of the generalized backward shift operator in weighted spaces of entire functions*; in: A. G. Kusraev, Z. D. Totieva (eds), *Operator Theory and Differential Equations*, Birkhauser, Basel, 2021, 37–48.
16. ______, *On operators which commute with the Pommiez type operator in weighted spaces of entire functions*, St. Petersbg. Math. J. **28**(2) (2017), 209–224.
17. ______, *On invariant subspaces of the Pommiez operator in the spaces of entire functions of exponential type*, J. Math. Sci. **241** (2019), 760–769.
18. ______, *On the completeness of orbits of a Pommiez operator in weighted* (LF)*-spaces of entire functions*, Complex Anal. Oper. Theory **11** (2017), 1407–1424.
19. M. T. Karaev, *On extended eigenvalues and extended eigenvectors of some operator classes*, Proc. Am. Math. Soc. **134** (2006), 2383–2392.
20. ______, *On some applications of the ordinary and extended Duhamel products*, Sib. Math. J. **46** (2005), 431–442; translated from Sibirsk. Math. Zh. **46** (2005), 553–566.
21. ______, *Duhamel Algebras and Applications*, Funct. Anal. Appl. **52** (2018), 1–8.
22. M. T. Karaev, H. Tuna, *Description of maximal ideal space of some Banach algebra with multiplication as Duhamel product*, Complex Variables, Theory Appl. **49** (2004), 449–457.
23. M. T. Karaev, M. Gürdal, S. Saltan, *Some applications of Banach algebras techniqus*, Math. Nachr. **284** (2011), 1678–1689.
24. Y. Linchuk, *On derivation operators with respect to the Duhamel convolution in the space of analytic functions*, Math. Commun. **20** (2015), 17–22.
25. M. M. Malamud, *Similarity of Volterra operators and related problems in the theory of differential equations of fractional orders*, Trans. Mosc. Math. Soc. **56** (1995), 57–122l translated from Trudy Moskov. Mat. Obsch. **55** (1994), 73–148.
26. ______, *Invariant and hyperinvariant subspaces of direct sums of simple Volterra operators*, Oper. Theory: Adv. Appl. **102** (1998), 143–167.
27. K. G. Merryfield, S. Watson, *A local algebra structure for* H^p *of the polydisc*, Colloq. Math. **62** (1991), 73–79.
28. S. Saltan, Y. Özel, *Maximal ideal space of some Banach algebras and related problems*, Banach J. Math. Anal. **8** (2014), 16–29.
29. R. Tapdigoglu, *On the description of invariant subspaces in the space* $C^{(n)}[0,1]$, Houston J. Math. **39** (2013), 169–176.
30. ______, *Invariant subspaces of Volterra integration operator: Axiomatical approach*, Bull. Sci. Math. **136** (2012), 574–578.
31. ______, *On the Banach algebra structure for* $C^{(n)}$ *of the bidisc and related topics*, Ill. J. Math. **64**(2) (2020), 185–197.
32. V. A. Tkachenko, *Operators that commute with generalized integration in spaces of analytic functionals*, Math. Notes **22**(2) (1979), 141–146.

33. N. M. Wigley, *The Duhamel product of analytic functions*, Duke Math. J. **41**(1) (1974), 211–217.
34. ______, *A Banach algebra structure for H^p*, Can. Math. Bull. **18**(4) (1975), 597–603.

(Received 28 10 2021)

Department of Mathematics
Suleyman Demirel University
Isparta
Turkey
mevlut_altintas95@hotmail.com
gurdalmehmet@sdu.edu.tr

Azerbaijan State University of Economics (UNEC)
Baku
Azerbaijan
and
Department of Mathematics
Khazar University
Baku
Azerbaijan
tapdigoglu@gmail.com