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ON TRANS-SASAKIAN 3-MANIFOLDS WITH $\mathcal{D}_a$-HOMOTETIC DEFORMATION WITH REGARD TO THE SCHOUTEN–VAN KAMPEN CONNECTION

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Abstract. We study some soliton types on trans-Sasakian 3-manifolds with $\mathcal{D}_a$-homotetic deformation with regard to the Schouten–van Kampen connection.

1. Introduction

In [**16**], Oubina defined a new class of almost contact metric structure, which is said to be trans-Sasakian structure of type (α, β). In [**7**], Chiena and Gonzales introduced two subclasses of trans-Sasakian structures which contain the Kenmotsu and Sasakian structures, respectively. Trans-Sasakian structures of type $(\alpha, 0)$, $(0, \beta)$ and $(0, 0)$ are α-Sasakian, β-Kenmotsu and cosymplectic, respectively [**2**, **13**].

The Schouten–van Kampen connection defined as adapted to a linear connection for studying nonholonomic manifolds and it is one of the most natural connections on a differentiable manifold [**3**, **12**, **20**]. Solov'ev studied hyperdistributions in Riemannian manifolds using the Schouten–van Kampen connection [**22**–**25**]. Then Olszak investigated the Schouten–van Kampen connection on almost (para) contact metric structures [**17**]. f-Kenmotsu manifolds admitting the Schouten–van Kampen connection were studied by Yıldız [**27**]. In recent times, Perktaş and Yıldız examined quasi-Sasakian manifolds and f-Kenmotsu manifolds with regard to the Schouten–van Kampen connection [**18**, **19**].

An almost Ricci soliton in a Riemannian manifold (M, g) is given by

$$L_X g + 2\,\mathrm{Ric} + 2\delta g = 0,$$

where L is the Lie derivative, Ric is the Ricci tensor, X is a complete vector field on M and δ is a smooth function. Then an almost Ricci soliton is said to be shrinking, steady and expanding according as δ is negative, zero and positive, respectively [**11**].

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An almost η-Ricci soliton in a Riemannian manifold (M, g) is defined by

$$L_X g + 2\,\mathrm{Ric} + 2\delta g + 2\mu\eta \otimes \eta = 0,$$

where μ is a smooth function [**8**].

In [**11**], Hamilton defined Yamabe flow to solve the Yamabe problem. The Yamabe soliton comes from the blow-up procedure along the Yamabe flow, so such solitons have been studied intensively [**1**, **5**, **6**, **10**, **15**].

An almost Yamabe soliton in a Riemannian manifold (M, g) is given by

$$\frac{1}{2}(L_X g) = (\mathrm{scal} - \lambda)g, \tag{1.1}$$

where scal is the scalar curvature of M. If λ is a constant, then an almost Yamabe soliton becomes a Yamabe soliton. Moreover, it is easy to see that Einstein manifolds are always almost Yamabe solitons. If (M, g) is of constant scalar curvature scal, then the Riemannian metric g is called a Yamabe metric [**1**].

In this paper we study some soliton types of trans-Sasakian 3-manifolds with D_a-homotetic deformation with regard to the Schouten–van Kampen connection. The paper is organized as follows: After preliminaries, we give some basic information about trans-Sasakian 3-manifolds with regard to the Schouten–van Kampen connection. In section 4, we adapt D_a-homothetic deformation on a trans-Sasakian 3-manifold. In section 5, we also adapt D_a-homothetic deformation on a trans-Sasakian 3-manifold with regard to the Schouten–van Kampen connection. In section 6, we study some soliton types on trans-Sasakian 3-manifold with D_a-homothetic deformation with regard to the Schouten–van Kampen connection. In the last section we give an example.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1, 1)$-tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\begin{gathered}\phi^2(U) = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0,\\ g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V),\\ g(U, \phi V) = -g(\phi U, V), \quad g(U, \xi) = \eta(U),\end{gathered}$$

for all $U, V \in \chi(M)$ [**2**]. The fundamental 2-form Φ of the manifold is defined by $\Phi(U, V) = g(U, \phi V)$. This may be expressed by the condition [**4**]

$$(\nabla_U \phi)V = \alpha(g(U, V)\xi - \eta(V)U) + \beta(g(\phi U, V)\xi - \eta(V)\phi U), \tag{2.1}$$

for smooth functions α and β on M. In this case we say that the trans-Sasakian structure is of type (α, β). From (2.1) it follows that

$$\nabla_U \xi = -\alpha\phi U + \beta(U - \eta(U)\xi), \tag{2.2}$$

$$(\nabla_U \eta)V = -\alpha g(\phi U, V) + \beta g(\phi U, \phi V). \tag{2.3}$$

An explicit example of proper trans-Sasakian 3-manifolds was constructed in [**14**]. In [**9**], the Ricci tensor and curvature tensor for trans-Sasakian 3-manifolds were studied and their explicit formulae were given.

From [**9**] we know that for a trans-Sasakian 3-manifold $2\alpha\beta + \xi\alpha = 0$, which implies that if α and β are constants. Then the manifold is either α-Sasakian or β-Kenmotsu or cosymplectic.

For constants α and β, we have

$$\begin{aligned} R(U,V)W = {} & \Big(\frac{\mathrm{scal}}{2} - 2(\alpha^2-\beta^2)\Big)(g(V,W)U - g(U,W)V) \\ & - \Big(\frac{\mathrm{scal}}{2} - 3(\alpha^2-\beta^2)\Big)(g(V,W)\eta(U)\xi - g(U,W)\eta(V)\xi \\ & + \eta(V)\eta(W)U - \eta(U)\eta(W)V), \end{aligned}$$

$$\mathrm{Ric}(U,V) = \Big(\frac{\mathrm{scal}}{2} - (\alpha^2-\beta^2)\Big)g(U,V) - \Big(\frac{\mathrm{scal}}{2} - 3(\alpha^2-\beta^2)\Big)\eta(U)\eta(V), \tag{2.4}$$

$$\begin{aligned} \mathrm{Ric}(U,\xi) &= 2(\alpha^2-\beta^2)\eta(U), \\ R(U,V)\xi &= (\alpha^2-\beta^2)(\eta(V)U - \eta(U)V), \\ R(\xi,U)V &= (\alpha^2-\beta^2)(g(U,V)\xi - \eta(V)U), \\ QU &= \Big(\frac{\mathrm{scal}}{2} - (\alpha^2-\beta^2)\Big)U - \Big(\frac{\mathrm{scal}}{2} - 3(\alpha^2-\beta^2)\Big)\eta(U)\xi, \end{aligned}$$

where Ric is the Ricci tensor, R is the curvature tensor and scal is the scalar curvature of the manifold M, respectively [9] Throughout the paper we consider trans-Sasakian 3-manifolds with α and β are constants.

On the other hand we have two naturally defined distributions in the tangent bundle TM of M as follows: $H = \ker\eta$, $V = \mathrm{span}\{\xi\}$. Then we have $TM = H \oplus V$, $H \cap V = \{0\}$ and $H \perp V$. This decomposition allows one to define the Schouten–van Kampen connection $\tilde\nabla$ over an almost contact metric structure. The Schouten–van Kampen connection $\tilde\nabla$ on an almost contact metric manifold with regard to Levi-Civita connection ∇ is defined by [**22**]

$$\tilde\nabla_U V = \nabla_U V - \eta(V)\nabla_U\xi + (\nabla_U\eta)(V)\xi. \tag{2.5}$$

Thus with the help of the Schouten–van Kampen connection (2.5), many properties of some geometric objects connected with the distributions H, V can be characterized [**22**–**24**]. For example g, ξ and η are parallel with regard to $\tilde\nabla$, that is, $\tilde\nabla\xi = 0$, $\tilde\nabla g = 0$, $\tilde\nabla\eta = 0$. Also the torsion $\tilde T$ of $\tilde\nabla$ is defined by

$$\tilde T(U,V) = \eta(U)\nabla_V\xi - \eta(V)\nabla_U\xi + 2d\eta(U,V)\xi.$$

3. Trans-Sasakian 3-manifolds with regard to the Schouten–van Kampen connection

Let M be a trans-Sasakian 3-manifold with α and β are constants with regard to the Schouten–van Kampen connection. Then using (2.2) and (2.3) in (2.5), we get

$$\tilde\nabla_U V = \nabla_U V + \alpha\{\eta(V)\phi U - g(\phi U,V)\xi\} + \beta\{g(U,V)\xi - \eta(V)U\}. \tag{3.1}$$

Let R and $\tilde{R}$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten–van Kampen connection $\tilde{\nabla}$ defined by

$$R(U,V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]}, \quad \tilde{R}(U,V) = [\tilde{\nabla}_U, \tilde{\nabla}_V] - \tilde{\nabla}_{[U,V]}.$$

Using (3.1), by direct calculations, we obtain the following formula connecting R and $\tilde{R}$ on a trans-Sasakian 3-manifold as follow:

$$\begin{aligned}\tilde{R}(U,V)W = R(U,V)W &+ \alpha^2\{g(\phi V,W)\phi U - g(\phi U,W)\phi V + \eta(U)\eta(W)V \\ &- \eta(V)\eta(W)U - g(V,W)\eta(U)\xi + g(U,W)\eta(V)\xi\} \\ &+ \beta^2\{g(V,W)U - g(U,W)V\}.\end{aligned}$$

We will also consider the Riemann curvature $(0,4)$-tensors $\tilde{R}, R$, the Ricci tensors $\tilde{\mathrm{Ric}}, \mathrm{Ric}$, the Ricci operators $\tilde{Q}, Q$ and the scalar curvatures $\tilde{\mathrm{scal}}, \mathrm{scal}$ of the connections $\tilde{\nabla}$ and ∇ are given by

$$\begin{aligned}\tilde{R}(U,V,W,Z) = R(U,V,W,Z) &+ \alpha^2\{g(\phi V,W)g(\phi U,Z) - g(\phi U,W)g(\phi V,Z) \\ &+ g(V,Z)\eta(U)\eta(W) - g(U,Z)\eta(V)\eta(W) \\ &- g(V,W)\eta(U)\eta(Z) + g(U,W)\eta(V)\eta(Z)\} \\ &+ \beta^2\{g(V,W)g(U,Z) - g(U,W)g(V,Z)\},\end{aligned}$$

$$\tilde{\mathrm{Ric}}(V,W) = \mathrm{Ric}(V,W) + 2\beta^2 g(V,W) - 2\alpha^2\eta(V)\eta(W),$$
$$\tilde{Q}U = QU + 2\beta^2 U - 2\alpha^2\eta(U)\xi,$$
$$\tilde{\mathrm{scal}} = \mathrm{scal} - 2\alpha^2 + 6\beta^2,$$

respectively, where

$$\tilde{R}(U,V,W,Z) = g(\tilde{R}(U,V)W,Z), \quad R(U,V,W,Z) = g(R(U,V)W,Z).$$

4. Trans-Sasakian 3-manifolds and $\mathcal{D}_a$-homothetic deformations

In this section, we will present how a $\mathcal{D}_a$-homothetic deformation affects the curvature tensor of a trans-Sasakian 3-manifold.

Let (M,ϕ,ξ,η,g) be an almost contact metric manifold with $dim\ M = 2n+1$. The equation $\eta = 0$ defines an $2n$-dimensional distribution $\mathcal{D}_a$ on M [**26**]. An $2n$-homothetic deformation or $\mathcal{D}_a$-homothetic deformation [**26**] is defined as a change of structure tensors of the form

$$\eta^a = a\eta, \quad \xi^a = \frac{1}{a}\xi, \quad \phi^a = \phi, \quad g^a = ag + a(a-1)\eta\otimes\eta, \tag{4.1}$$

where a is a positive constant. If (M,ϕ,ξ,η,g) is an almost contact metric structure with constant form η, then $(M,\phi^a,\xi^a,\eta^a,g^a)$ is also an almost contact metric structure [**26**].

Now by direct computations we give

LEMMA 4.1. *Let M be a trans-Sasakian 3-manifold. For a $\mathcal{D}_a$-homothetic deformation on M, the Levi-Civita connections ∇^a and ∇ are related by*

$$\begin{aligned}\nabla^a_U V = \nabla_U V &- (a-1)\alpha\{\eta(U)\phi V + \eta(V)\phi U\} \\ &+ (a-1)\beta\{g(U,V)\xi - \eta(U)\eta(V)\xi\},\end{aligned} \tag{4.2}$$

for any vector fields U, V on M.

Using (4.2) we have the following relation between R^a and R:

PROPOSITION 4.1. *Let M be a trans-Sasakian 3-manifold. For a $\mathcal{D}_a$-homothetic deformation on M, the Riemannian curvature tensors R^a and R are related by*

$$
\begin{aligned}
R^a(U,V)Z = R(U,V)Z &+ (a-1)\alpha^2\{2g(\phi U,V)\phi Z - g(U,\phi Z)\phi V \\
&+ g(V,\phi Z)\phi U - g(U,Z)\eta(V)\xi \\
&+ g(V,Z)\eta(U)\xi - 2\eta(U)\eta(Z)V + 2\eta(V)\eta(Z)U\} \\
&+ (a-1)\beta^2\{2g(V,Z)U - 2g(U,Z)V \\
&+ 2\eta(U)\eta(Z)V - 2\eta(V)\eta(Z)U \\
&+ g(U,Z)\eta(V)\xi - g(V,Z)\eta(U)\xi\} \\
&- (a-1)^2\alpha^2\{\eta(U)\eta(Z)V - \eta(V)\eta(Z)U\} \\
&- (a-1)\alpha\beta\{3\eta(U)\eta(Z)\phi V - 3\eta(V)\eta(Z)\phi U \\
&+ g(V,Z)\phi U - g(U,Z)\phi V\} \\
&- (a-1)^2\alpha\beta\{g(\phi V,\phi Z)\phi U - g(\phi U,\phi Z)\phi V \\
&+ 2g(U,\phi V)\eta(Z)\xi + g(U,\phi Z)\eta(V)\xi - g(V,\phi Z)\eta(U)\xi\}.
\end{aligned}
\tag{4.3}
$$

Now taking the inner product in (4.3) with a vector field W, we write

$$
\begin{aligned}
g(R^a(U,V)Z,W) = g(R(U,V)Z,W &+ (a-1)\alpha^2\{2g(\phi U,V)g(\phi Z,W) \\
&- g(U,\phi Z)g(\phi V,W) + g(V,\phi Z)g(\phi U,W) \\
&- g(U,Z)\eta(V)\eta(W) + g(V,Z)\eta(U)\eta(W) \\
&- 2\eta(U)\eta(Z)g(V,W) + 2\eta(V)\eta(Z)g(U,W)\} \\
&+ (a-1)\beta^2\{2g(V,Z)g(U,W) - 2g(U,Z)g(V,W) \\
&+ 2\eta(U)\eta(Z)g(V,W) - 2\eta(V)\eta(Z)g(U,W) \\
&+ g(U,Z)\eta(V)\eta(W) - g(V,Z)\eta(U)\eta(W)\} \\
&- (a-1)^2\alpha^2\{\eta(U)\eta(Z)g(V,W) - \eta(V)\eta(Z)g(U,W)\} \\
&- (a-1)\alpha\beta\{3\eta(U)\eta(Z)g(\phi V,W) - 3\eta(V)\eta(Z)g(\phi U,W) \\
&+ g(V,Z)g(\phi U,W) - g(U,Z)g(\phi V,W)\} \\
&- (a-1)^2\alpha\beta\{g(\phi V,\phi Z)g(\phi U,W) - g(\phi U,\phi Z)g(\phi V,W) \\
&+ 2g(U,\phi V)\eta(Z)\eta(W) + g(U,\phi Z)\eta(V)\eta(W) \\
&- g(V,\phi Z)\eta(U)\eta(W)\}.
\end{aligned}
\tag{4.4}
$$

If we take $U = W = e_i$ in (4.4), $\{i = 1,2,3\}$, where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold, we get

$$
\begin{aligned}
\mathrm{Ric}^a(V,Z) = \mathrm{Ric}(V,Z) &+ \{2(1-a)\alpha^2 + 3(a-1)\beta^2\}g(V,Z) \\
&+ \{(2a^2+2a-4)\alpha^2 - 3(a-1)\beta^2\}\eta(V)\eta(Z) \\
&+ (a-1)\alpha\beta g(\phi V,Z),
\end{aligned}
\tag{4.5}
$$

where Ric^a and Ric denote the Ricci tensors with regard to the connections ∇^a and ∇, respectively. Also if we take $V = Z = e_i$ in (4.5), we get

$$\mathrm{scal}^a = \mathrm{scal} + 2(a-1)^2\alpha^2 + 6(a-1)\beta^2,$$

where scal^a and scal denote the scalar curvatures with regard to the connections ∇^a and ∇, respectively.

5. Trans-Sasakian 3-manifolds with $\mathcal{D}_a$-homothetic deformation with regard to the Schouten–van Kampen connection

In this section, we study how a $\mathcal{D}_a$-homothetic deformation affects a trans-Sasakian 3-manifold M with the Schouten–van Kampen connection.

Lemma 5.1. *Let M be a trans-Sasakian 3-manifold with the Schouten–van Kampen connection. For a $\mathcal{D}_a$-homothetic deformation on M, the connections $\widetilde{\nabla}^a$ and ∇ are related by*

$$\widetilde{\nabla}^a_U V = \nabla_U V - (a-1)\alpha\eta(U)\phi V + a\beta\{g(U,V)\xi - \eta(V)U\} + \alpha\{g(U,\phi V)\xi + \eta(V)\phi U\}, \tag{5.1}$$

for any U, V on M.

Proposition 5.1. *Let M be a trans-Sasakian 3-manifold with the Schouten–van Kampen connection. For a $\mathcal{D}_a$-homothetic deformation on M, the Riemannian curvature tensors $\widetilde{R}^a$ and R are related by*

$$\begin{aligned}\widetilde{R}^a(U,V)Z = R(U,V)Z &+ \alpha^2\{2(a-1)g(\phi U,V)\phi Z + g(U,\phi Z)\phi V \\ &- g(V,\phi Z)\phi U + g(U,Z)\eta(V)\xi \\ &- g(V,Z)\eta(U)\xi + \eta(U)\eta(Z)V - \eta(V)\eta(Z)U\} \\ &+ \beta^2\{(3a-2)\{g(V,Z)U - g(U,Z)V\} \\ &- (a^2-3a+2)\{\eta(U)\eta(Z)V - \eta(V)\eta(Z)U\} \\ &+ g(U,Z)\eta(V)\xi - g(V,Z)\eta(U)\xi\} \\ &- \alpha\beta\{3(a-1)\{\eta(U)\eta(Z)\phi V - \eta(V)\eta(Z)\phi U\} \\ &+ (a-1)\{g(V,Z)\phi U - g(U,Z)\phi V\} \\ &- (a-1)^2\{g(\phi V,\phi Z)\phi U - g(\phi U,\phi Z)\phi V \\ &+ 2g(U,\phi V)\eta(Z)\xi + g(U,\phi Z)\eta(V)\xi \\ &- g(V,\phi Z)\eta(U)\xi\}\},\end{aligned} \tag{5.2}$$

for any U, V, Z on M.

Now taking the inner product with a vector field W in (5.2), we write

$$\begin{aligned}g(\widetilde{R}^a(U,V)Z,W) = g(R(U,V)Z,W) &+ \alpha^2\{2(a-1)g(\phi U,V)g(\phi Z,W) \\ &+ g(U,\phi Z)g(\phi V,W) - g(V,\phi Z)g(\phi U,W) \\ &+ g(U,Z)\eta(V)\eta(W) - g(V,Z)\eta(U)\eta(W) \\ &+ \eta(U)\eta(Z)g(V,W) - \eta(V)\eta(Z)g(U,W)\}\end{aligned} \tag{5.3}$$

$$
\begin{aligned}
&+ \beta^2\{(3a-2)\{g(V,Z)g(U,W) - g(U,Z)g(V,W)\}\\
&- (a^2-3a+2)\{\eta(U)\eta(Z)g(V,W)\\
&- \eta(V)\eta(Z)g(U,W)\} + g(U,Z)\eta(V)\eta(W)\\
&- g(V,Z)\eta(U)\eta(W)\} - \alpha\beta\{3(a-1)\{\eta(U)\eta(Z)g(\phi V,W)\\
&- \eta(V)\eta(Z)g(\phi U,W)\} + (a-1)\{g(V,Z)g(\phi U,W)\\
&- g(U,Z)g(\phi V,W)\} - (a-1)^2\{g(\phi V,\phi Z)g(\phi U,W)\\
&- g(\phi U,\phi Z)g(\phi V,W) + 2g(U,\phi Y)\eta(Z)\eta(W)\\
&+ g(U,\phi Z)\eta(V)\eta(W) - g(V,\phi Z)\eta(U)\eta(W)\}\}.
\end{aligned}
$$

If we take $U = W = e_i$ in (5.3), where $\{ei\}$ is an orthonormal basis of the tangent space at each point of the manifold, $\{i = 1, 2, 3\}$, we get

$$
\begin{aligned}
\tilde{\mathrm{Ric}}^a(V,Z) = \mathrm{Ric}(V,Z) &+ 2\{(1-a)\alpha^2 + (2a^2-3a+2)\beta^2\}g(V,Z)\\
&+ 2\{(a-2)\alpha^2 - (a^2-3a+2)\beta^2\}\eta(V)\eta(Z)\\
&+ (a-1)g(V,\phi Z),
\end{aligned}
\tag{5.4}
$$

where $\tilde{\mathrm{Ric}}^a$ and Ric denote the Ricci tensors with respect to the connections $\widetilde{\nabla}^a$ and ∇, respectively. If we use (2.4) in (5.4), we have

$$
\begin{aligned}
\tilde{\mathrm{Ric}}^a(V,Z) = &\left\{\frac{\mathrm{scal}}{2} + (1-2a)\alpha^2 + (4a^2-6a+5)\beta^2\right\}g(V,Z)\\
&- \left\{\frac{\mathrm{scal}}{2} + (1-2a)\alpha^2 + (2a^2-6a+7)\beta^2\right\}\eta(Z)\eta(V)\\
&+ (a-1)g(V,\phi Z).
\end{aligned}
\tag{5.5}
$$

Also if we take $V = Z = e_i$ in (5.5), we get

$$
\tilde{\mathrm{scal}}^a = \mathrm{scal} + (-4a+2)\alpha^2 + (10a^2-12a+8)\beta^2,
$$

where $\tilde{\mathrm{scal}}^a$ and scal denote the scalar curvatures with respect to the connections $\widetilde{\nabla}^a$ and ∇, respectively.

6. Soliton types on trans-Sasakian 3-manifolds with a $\mathcal{D}_a$-homothetic deformation with regard to the Schouten–van Kampen connection

In this section we study almost Ricci solitons, almost η-Ricci solitons and Yamabe solitons on a trans-Sasakian 3-manifold M with a $\mathcal{D}_a$-homothetic deformation with regard to the Schouten–van Kampen connection.

In a trans-Sasakian 3-manifold M with a $\mathcal{D}_a$-homothetic deformation endowed with the Schouten–van Kampen connection bearing an almost Ricci soliton, we can write

$$
(\tilde{L}^a_X g^a + 2\tilde{\mathrm{Ric}}^a + 2\delta g^a)(U,V) = 0.
\tag{6.1}
$$

Putting $X = \xi^a$ in (6.1), we obtain

$$
\begin{aligned}
g^a(\tilde{\nabla}^a_U\xi^a, V) &+ g^a(U, \tilde{\nabla}^a_V\xi^a) + g^a(\nabla^a_U\xi^a, V)\\
&+ g^a(U, \nabla^a_V\xi^a) + 2\tilde{\mathrm{Ric}}^a(U,V) + 2\delta g^a(U,V) = 0.
\end{aligned}
\tag{6.2}
$$

Now using (2.2), (2.5), (4.2), (5.1) and (5.5), in (6.2), we get

$$\begin{aligned} &2\beta(2-a)\{g(U,V)-\eta(U)\eta(V)\} \\ &+2\Big[\Big\{\frac{\mathrm{scal}}{2}+(1-2a)\alpha^2+(4a^2-6a+5)\beta^2\Big\}g(U,V) \\ &-\Big\{\frac{\mathrm{scal}}{2}+(1-2a)\alpha^2+(2a^2-6a+7)\beta^2\Big\}\eta(U)\eta(V) \\ &-(a-1)g(\phi U,V)\Big]+2\delta\{ag(U,V)+a(a-1)\eta(U)\eta(V)\}=0. \end{aligned} \tag{6.3}$$

Taking $U=V=\xi$ in (6.3), we have $2(a^2-1)\beta^2+a^2\delta=0$, i.e., $\delta=\frac{2(1-a^2)}{a^2}\beta^2$. Thus we give the following:

Theorem 6.1. *Let M be a trans-Sasakian 3-manifold bearing an almost Ricci soliton (ξ^a,δ,g^a) with a $\mathcal{D}_a$-homothetic deformation with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$. Then we get:* (i) *If $1-a^2>0$, then the soliton is expanding,* (ii) *If $1-a^2<0$, then the soliton is shrinking,* (iii) *If $1-a^2=0$, then the soliton is steady.*

Also we can say the following:

Corollary 6.1. *An almost Ricci soliton (ξ^a,δ,g^a) on a 3-dimensional α-Sasakian manifold with a $\mathcal{D}_a$-homothetic deformation with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$ is always steady.*

Again considering (6.3), we obtain

$$\tilde{\mathrm{Ric}}^a(U,V)=\Big(\frac{\beta}{a}(a-2)-\delta\Big)g^a(U,V)+\frac{\beta}{a}(2-a)\eta^a(U)\eta^a(V).$$

Thus we have the following:

Theorem 6.2. *A trans-Sasakian 3-manifold M with a $\mathcal{D}_a$-homothetic deformation bearing an almost Ricci soliton (ξ^a,δ,g^a) with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$ is an η^a-Einstein manifold.*

Now we consider almost η-Ricci soliton on a trans-Sasakian 3-manifold with a $\mathcal{D}_a$-homothetic deformation with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$. Then we write

$$(\tilde{L}^a_X g^a+2\tilde{\mathrm{Ric}}^a+2\delta g^a+2\mu\eta^a\otimes\eta^a)(U,V)=0. \tag{6.4}$$

Putting $X=\xi^a$ in (6.4) and using (2.2), (2.5), (4.2), (5.1) and (5.5) in (6.4), we have

$$\tilde{\mathrm{Ric}}^a(U,V)=\Big(\frac{\beta}{a}(a-2)-\delta\Big)g^a(U,V)-\Big(\frac{\beta}{a}(a-2)+\mu\Big)\eta^a(U)\eta^a(V). \tag{6.5}$$

Hence we give

Theorem 6.3. *A trans-Sasakian 3-manifold M with a $\mathcal{D}_a$-homothetic deformation bearing an almost η-Ricci soliton (ξ^a,δ,μ,g^a) with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$ is an η^a-Einstein manifold.*

Again taking $U = V = \xi$ in (6.5), we have

$$a^2(\delta + \mu) + 2(a^2 - 1)\beta^2 = 0. \tag{6.6}$$

Thus we get $\delta + \mu = \frac{2(1-a^2)}{a^2}\beta^2$. Then we can say the following:

Corollary 6.2. *An almost η-Ricci soliton on a trans-Sasakian 3-manifold with a $\mathcal{D}_a$-homothetic deformation with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$ is always $\big(\xi^a, \delta, \frac{2(1-a^2)}{a^2}\beta^2 - \delta, g^a\big)$.*

Finally we study almost Yamabe solitons on a trans-Sasakian 3-manifold with a $\mathcal{D}_a$-homothetic deformation with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$. Assume that (M, ξ^a, λ, g^a) is an almost Yamabe soliton on a trans-Sasakian 3-manifold with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$. Then, from (1.1), we write

$$\frac{1}{2}(\tilde{L}^a_{\xi^a} g^a)(U, V) = (\tilde{\mathrm{scal}}^a - \lambda) g^a(U, V),$$

that is,

$$\beta(2 - a)\{g(U, V) - \eta(U)\eta(V)\} = (\tilde{\mathrm{scal}}^a - \lambda) g^a(U, V). \tag{6.7}$$

Putting $X = \xi^a$ in (6.7), we obtain $\tilde{\mathrm{scal}}^a = \lambda$, which implies the following:

Theorem 6.4. *The scalar curvature $\tilde{\mathrm{scal}}^a$ of a trans-Sasakian 3-manifold with a $\mathcal{D}_a$-homothetic deformation bearing an almost Yamabe soliton (M, ξ^a, λ, g^a) with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$ is equal to λ.*

So we give the followings:

Corollary 6.3. *A trans-Sasakian 3-manifold with a $\mathcal{D}_a$-homothetic deformation bearing a Yamabe soliton (M, ξ^a, λ, g^a) with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$ is of constant scalar curvature with respect to the Schouten–van Kampen connection.*

Corollary 6.4. *If a trans-Sasakian 3-manifold with a $\mathcal{D}_a$-homothetic deformation bearing a Yamabe soliton (M, ξ^a, λ, g^a) with regard to the Schouten–van Kampen connection $\widetilde{\nabla}^a$, then the Riemannian metric g^a is a Yamabe metric.*

7. An example

We consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = e^{2z}\frac{\partial}{\partial x}, \quad e_2 = e^{2z}\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M^3)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = 0.$$

Then using the linearity property of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2(Z) = -Z + \eta(Z)\xi, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct calculations, we have

$$[e_1, e_3] = -2e_1, \quad [e_2, e_3] = -2e_2, \quad [e_1, e_2] = 0.$$

$$\nabla_{e_1} e_1 = 2e_3, \quad \nabla_{e_2} e_2 = 2e_3, \quad \nabla_{e_1} e_3 = -2e_1, \quad \nabla_{e_2} e_3 = -2e_2, \tag{7.1}$$

where ∇ is the the Levi-Civita connection on M. So, it can be easily shown that $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold of type $(0, -2)$ with the non-zero following components of the curvature tensor R [**21**]

$$\begin{aligned} &R(e_1, e_2)e_2 = -4e_1, \quad R(e_2, e_3)e_2 = 4e_3, \quad R(e_1, e_3)e_3 = -4e_1, \\ &R(e_2, e_3)e_3 = -4e_2, \quad R(e_1, e_3)e_1 = 4e_3, \quad R(e_1, e_2)e_1 = 4e_2. \end{aligned} \tag{7.2}$$

Now with help of (3.1) and (7.1), we obtain that M is flat with regard to the Schouten–van Kampen connection $\tilde{\nabla}$.

Let us consider a $\mathcal{D}_a$-homothetic deformation on M. Thus from (4.2), we have the non-zero components of ∇^a on M are

$$\nabla^a_{e_1} e_1 = (4-2a)e_3, \quad \nabla^a_{e_1} e_3 = -2e_1, \quad \nabla^a_{e_2} e_2 = (4-2a)e_3, \quad \nabla^a_{e_2} e_3 = -2e_2. \tag{7.3}$$

Using (7.3), we obtain that the non-zero components of the R^a on M are

$$\begin{aligned} &R^a(e_1, e_2)e_1 = 4(2-a)e_2, \quad R^a(e_1, e_2)e_2 = 4(a-2)e_1, \\ &R^a(e_1, e_3)e_1 = 4(2-a)e_3, \quad R^a(e_1, e_3)e_3 = -4e_1, \\ &R^a(e_2, e_3)e_2 = 4(2-a)e_3, \quad R^a(e_2, e_3)e_3 = -4e_2. \end{aligned} \tag{7.4}$$

Also using (5.1) and (7.3), we get that the non-zero components of $\tilde{\nabla}^a$ on M are

$$\begin{aligned} &\tilde{\nabla}^a_{e_1} e_3 = 2(a-1)e_1, \quad \tilde{\nabla}^a_{e_2} e_3 = 2(a-1)e_2, \\ &\tilde{\nabla}^a_{e_2} e_2 = 2(1-a)e_3, \quad \tilde{\nabla}^a_{e_1} e_1 = 2(1-a)e_3. \end{aligned} \tag{7.5}$$

Hence using (7.5), we have that the non-zero components of the curvature tensor $\tilde{R}^a$ on M are

$$\begin{aligned} &\tilde{R}^a(e_1, e_2)e_1 = 4(a-1)^2 e_2, \quad \tilde{R}^a(e_1, e_2)e_2 = -4(a-1)^2 e_1, \\ &\tilde{R}^a(e_1, e_2)e_3 = 4(a-1)^2 e_3, \quad \tilde{R}^a(e_1, e_3)e_1 = 4(1-a)e_3, \\ &\tilde{R}^a(e_1, e_3)e_3 = 4(a-1)e_1, \quad \tilde{R}^a(e_2, e_3)e_2 = 4(1-a)e_3, \\ &\tilde{R}^a(e_2, e_3)e_3 = 4(a-1)e_2. \end{aligned} \tag{7.6}$$

Now using (7.2), (7.4) and (7.6), we have that the non-zero components of the Ricci tensors Ric, $\tilde{\mathrm{Ric}}$, Ric^a, and $\tilde{\mathrm{Ric}}^a$

$$\mathrm{Ric}(e_1, e_1) = -8, \quad \mathrm{Ric}(e_2, e_2) = -8, \quad \mathrm{Ric}(e_3, e_3) = -8,$$

$$\tilde{\mathrm{Ric}}(e_1,e_1)=\alpha^2-\beta^2-6\beta-8,\quad \tilde{\mathrm{Ric}}(e_2,e_2)=\alpha^2-\beta^2-6\beta-8,$$
$$\tilde{\mathrm{Ric}}(e_3,e_3)=-4\beta-8,$$
$$\mathrm{Ric}^a(e_1,e_1)=-16+8a,\quad \mathrm{Ric}^a(e_2,e_2)=-16+8a,\quad \mathrm{Ric}^a(e_3,e_3)=-8,$$

and

$$\begin{aligned}\tilde{\mathrm{Ric}}^a(e_1,e_1)=-4a^2+12a-8,\quad \tilde{\mathrm{Ric}}^a(e_2,e_2)=-4a^2+12a-8,\\ \tilde{\mathrm{Ric}}^a(e_3,e_3)=8(a^2-1),\end{aligned}\tag{7.7}$$

respectively. For any $U,V\in\chi(M)$, we write

$$U=a_1e_1+a_2e_2+a_3e_3,\quad V=b_1e_1+b_2e_2+b_3e_3.$$

Using (4.1) and (7.7), we have

$$\begin{aligned}(\tilde{L}^a_{\xi^a}g^a)(U,V)&+2\tilde{\mathrm{Ric}}^a(U,V)+2\delta g^a(U,V)+2\mu\eta^a(U)\eta^a(V)\\ &=\big\{2\beta(2-a)+2(-4a^2+12a-8)+2a\delta\big\}a_1b_1\\ &\quad+\big\{2\beta(2-a)+2(-4a^2+12a-8)+2a\delta\big\}a_2b_2\\ &\quad+\big\{16(a^2-1)+2a^2(\delta+\mu)\big\}a_3b_3.\end{aligned}$$

Since $\beta=-2$, we have $4a^2-14a+12-a\delta=0$ and $8(a^2-1)+a^2(\delta+\mu)=0$. Thus $\delta=-4a-7+\frac{12}{a}$ and $\mu=4a-1-\frac{12}{a}+\frac{8}{a^2}$, which imply $\delta+\mu=\frac{8}{a^2}-8$. Hence M admits an η-Ricci soliton $\big(\xi^a,-4a-7+\frac{12}{a},4a-1-\frac{12}{a}+\frac{8}{a^2},g^a\big)$. Thus Corollary 6.2 is verified.

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