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ON $\oplus$-δ_{ss}-SUPPLEMENTED MODULES

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ABSTRACT. A module M is called $\oplus$-δ_{ss}-supplemented if every submodule X of M has a δ_{ss}-supplement Y in M which is a direct summand of M such that $X+Y=M$ and $X\cap Y\leqslant \mathrm{Soc}_\delta(Y)$ where $\mathrm{Soc}_\delta(Y)$ is the sum of simple and δ-small submodules of Y and $M=Y\oplus Y'$ for some $Y'\leqslant M$. Moreover, M is called a completely $\oplus$-δ_{ss}-supplemented module if every direct summand of M is $\oplus$-δ_{ss}-supplemented. Thus, we present two new types of algebraic structures which are stronger than δ-D_{11} and δ-D_{11}^{+}-modules, respectively. In this paper we investigate basic properties, decompositions and ring characterizations of these modules.

1. Introduction

Firstly, let us point that, R will indicate an associative ring with identity and M will indicate an R-module throughout this article. In addition to these, for a submodule X of M and for a direct summand X of M, the notations $X\leqslant M$ and $X\leqslant_\oplus M$ will be used respectively. A submodule X of M is called *small* in M, (denoted by $X\ll M$), if $X+P\neq M$ for any proper submodule P of M. Besides, the sum of all small submodules of M is denoted by $Rad(M)$. Dual to this term, the submodule X is called *essential*in M, if the submodule $\{0\}$ is the only one satisfying $X\cap Y=\{0\}$ for $Y\leqslant M$, denoted by $X\trianglelefteq M$. Besides, M is called an *essential extension* of X. A module M is called *closed* in M, denoted by $X\leqslant_c M$, if it has no proper essential extension in M. A submodule Y of M is called a *closure*of X in M if Y is closed and it is also an essential extension of X in M [**2**]. A module M is called *extending* (or *CS-module*) if every closed submodule of M is a direct summand of M [**20**]. In [**14**] a module M is called a *UC-module* if every submodule of M has a unique coclosure in M.

C1, C2, C3 conditions for a module M is given as follows. It is known that modules with the condition $C1$ are also known as *CS-modules* or extending modules.

(C1) Every submodule of M is essential in a direct summand of M.

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(C2) Every submodule which is isomorphic to a summand of M is a direct summand of M.

(C3) If X and Y are direct summands of M with $X \cap Y = 0$,then $X \oplus Y \leqslant_{\oplus} M$.

In [**12**], it is given that a module M with (C2)-condition also satisfies (C3)-condition. Moreover in [**13**, Proposition 1.22] it is declared that every quasi-injective module satisfies $C1$ and C2-conditions, directly.

A *supplement* submodule Y of a submodule X in M is the minimal element of the set of submodules of M satisfying $X + T = M$ for $T \leqslant M$, which is equivalent to $X + Y = M$ and $X \cap Y \ll Y$. If every submodule of M has a supplement in M, then M is called a *supplemented module* [**25**]. A module M is called $\oplus$*-supplemented* if every submodule X of M has a supplement which is a direct summand of M [**12**]. These modules are dual of (C_{11})-modules given in [**15**]. Recall that a module M satisfies (C_{11}) if every submodule of M has a complement which is a direct summand of M. A module M is called *lifting* if for any submodule X of M there exists a decomposition $M = A \oplus B$ provided that $A \leqslant X$ and $X \cap B \ll B$ [**2**]. Clearly, every lifting module is $\oplus$-supplemented.

The singular submodule $Z(M)$ of the module M is the set of elements m of M whose annihilators are essential in R. The module M is called *singular* (*nonsingular*) if $Z(M) = M$ $(Z(M) = 0)$. In [**24**], a generalization of small submodules and depend to this in [**10**], a generalization of supplemented modules are defined as follows. The submodule X of M is called δ*-small* in M, denoted by $X \ll_{\delta} M$, if $X + T \neq M$ for every proper submodule Y of M with $\frac{M}{T}$ is singular. Moreover, $\delta(M)$ denotes the sum of all δ-small submodules of M. A submodule Y is called a δ*-supplement* of X in M if $X + Y = M$ and $X \cap Y \ll_{\delta} Y$. M is called a δ*-supplemented* module if every submodule of M has a δ-supplement in M. Additionally, in [**19**], M is called a δ-D_{11}*-module* if every submodule of M has a δ-supplement which is a direct summand of M and, M is called a δ-D_{11}^{+}*-module* if every direct summand of M is a δ-D_{11}-module.

In [**21**] the authors defined δ_{ss}-supplemented modules both as a restriction of δ-supplemented modules and as a generalization of ss-supplemented modules studied extensively in [**8**] as follows. A module M is called δ_{ss}*-supplemented* if for any submodule X of M there exists a δ_{ss}-supplement submodule of M, that is, $X + Y = M$ and $X \cap Y \leqslant \mathrm{Soc}_{\delta}(Y)$ where $\mathrm{Soc}_{\delta}(Y)$ is the sum of δ-small simple submodules of Y.

Combining the facts given above, we obtain two new types of algebraic structure which are stronger than δ-D_{11} and δ-D_{11}^{+} modules, respectively. They are $\oplus$-δ_{ss}-*supplemented* and *completely* $\oplus$-δ_{ss}*-supplemented modules* whose detailed concepts can be seen in the abstract. Afterwards, the following hierarchy can be reached for a module M:

$$\begin{tikzcd}
 & \text{completely } \oplus\text{-}\delta_{ss}\text{-supplemented} \arrow[d] & \\
 & \oplus\text{-}\delta_{ss}\text{-supplemented} \arrow[dl] \arrow[dr] & \\
\delta_{ss}\text{-supp.} \arrow[dr] & & \oplus\text{-}\delta\text{-supp.} \arrow[dl] \\
 & \delta\text{-supplemented} &
\end{tikzcd}$$

Let us give a summary of the data obtained in this article shortly. Every strongly δ-local module is $\oplus$-δ_{ss}-supplemented. Any finite direct sum of a (completely) $\oplus$-δ_{ss}-supplemented module is (completely) $\oplus$-δ_{ss}-supplemented and the converse is true whenever $M = \bigoplus_{i=1}^{n} M_i$ is the direct sum of the members of a family of (modules with the finite exchange property) relatively projective modules $\{M_i\}_{i\in I}$. It is clear that every completely $\oplus$-δ_{ss}-supplemented module is $\oplus$-δ_{ss}-supplemented. The converse is provided for modules with the property $(D3)$ or *UC-extending* modules. For a submodule X of $a\oplus$-δ_{ss}-supplemented module M with $\frac{M}{X}$ is projective, the case of being $\oplus$-δ_{ss}-supplemented is inherited. Any fully invariant δ-coclosed submodule of a $\oplus$-δ_{ss}-supplemented module is $\oplus$-δ_{ss}-supplemented. Any factor module $\frac{M}{X}$ of a $\oplus$-δ_{ss}-supplemented module M is $\oplus$-δ_{ss}-supplemented where $X \leqslant M$ is fully invariant. A $\oplus$-δ_{ss}-supplemented module M with $\delta(M) \leqslant \mathrm{Soc}(M)$ is $\oplus$-δ_{ss}-supplemented. A ring R is δ_{ss}-perfect if and only if every finitely generated free R-module is $\oplus$-δ_{ss}-supplemented.

2. Main results

DEFINITION 2.1. A module M is called a $\oplus$-δ_{ss}-supplemented module if any submodule of M has a δ_{ss}-supplement which is a direct summand of M.

It is clear that every semisimple module is a $\oplus$-δ_{ss}-supplemented module.

Recall that a module M is called *δ-local* if $\delta(M) \ll_\delta M$ and $\delta(M)$ is maximal [**1**]. Additionally, a module M is called *strongly δ-local* if it is δ-local and $\delta(M) \leqslant \mathrm{Soc}(M)$ [**21**]. Comparing the definitions, it is possible to say that every strongly δ-local module is δ-local; but the converse may not be true.

LEMMA 2.1. *Let M be a strongly δ-local module. Then, M is a $\oplus$-δ_{ss}-supplemented module.*

PROOF. Let X be any submodule of M.

Case 1. Let $X \leqslant \delta(M)$. Then, X is semisimple as a submodule of $\delta(M)$ as M is strongly δ-local. Thus, X is a δ-small submodule of M from [**21**, Lemma 2.2]. Hence, M is a δ_{ss}-supplement of X in M which is a direct summand of M.

Case 2. Let $X \nleqslant \delta(M)$. Then it can be written that $X + \delta(M) = M$ from the maximality of $\delta(M)$. Since $\delta(M) \ll_\delta M$, then there exists a projective semisimple submodule $Y \leqslant \delta(M)$ such that $X \oplus Y = M$. $\square$

Proposition 2.1. *For a δ-local module M the following statements are equivalent:* (i) *M is $\oplus$-δ_{ss}-supplemented.* (i) *M is strongly δ-local.*

Proof. (2) $\Rightarrow$ (1) is clear from Lemma 2.1.

(1) $\Rightarrow$ (2) is clear from [**21**, Proposition 4.5] as every $\oplus$-δ_{ss}-supplemented module is δ_{ss}-supplemented. $\square$

Clearly, every $\oplus$-δ_{ss}-supplemented module is a δ-D_{11}-module. The following proposition shows that the converse may be true under a suitable condition.

Proposition 2.2. *Let M be a δ-D_{11}-module with $\delta(M) \leqslant \mathrm{Soc}(M)$. Then M is a $\oplus$-δ_{ss}-supplemented module.*

Proof. Let X be any submodule of M. By the assumption, there exists a submodule Y of M such that $X + Y = M$, $X \cap Y \ll_\delta Y$ and $M = Y \oplus Y'$ for a submodule $Y' \leqslant M$. As $X \cap Y \leqslant \delta(Y) \leqslant \delta(M) \leqslant \mathrm{Soc}(M)$, $X \cap Y$ is semisimple. Thus, Y is a δ_{ss}-supplement of X in M which is a direct summand of M. Hence, M is $\oplus$-δ_{ss}-supplemented. $\square$

Now, we need to have the following useful lemma for the completeness and the proof of Theorem 2.1.

Lemma 2.2. *Let M be a module and $X, Y \leqslant M$. If $X+Y$ has a δ_{ss}-supplement S in M and $X\cap(Y+S)$ has a δ_{ss}-supplement T in X, then $S+T$ is a δ_{ss}-supplement of Y in M.*

Proof. By the assumption, we have $(X+Y)+S = M$, $(X+Y)\cap S \leqslant \mathrm{Soc}_\delta(S)$. Additionally, we have

$$[X \cap (Y + S)] + T = X, \quad [X \cap (Y + S)] \cap T = (Y + S) \cap T \leqslant \mathrm{Soc}_\delta(T).$$

Therefore,

$$\begin{aligned} M = (X + Y) + S = ([X \cap (Y + S)] + T + Y) + S = Y + (S + T), \\ Y \cap (S + T) \leqslant [S \cap (Y + T)] + [T \cap (Y + S)] \\ \leqslant [S \cap (Y + X)] + [T \cap (Y + S)] \ll_\delta S + T. \end{aligned}$$

Moreover, $Y \cap (S + T)$ is semisimple as a submodule of a sum of two semisimple submodules from [**7**, Corollary 8.1.5]. Hence, $S + T$ is a δ_{ss}-supplement of Y in M. $\square$

Theorem 2.1. *Any finite direct sum of a $\oplus$-δ_{ss}-supplemented module module is $\oplus$-δ_{ss}-supplemented.*

Proof. It is enough to show that $M = M_1 \oplus M_2$ is a $\oplus$-δ_{ss}-supplemented module whenever each M_i is $\oplus$-δ_{ss}-supplemented for $i = 1, 2$. Let $X \leqslant M$. Then $M = M_1 + M_2 + X$ and $\{0\}$ is a δ_{ss}-supplement of M which is a trivial direct summand. On the other hand, $M_2 \cap (M_1 + X)$ has a δ_{ss}-supplement S in M which is a direct summand of M_2. By Lemma 2.2, S is a δ_{ss}-supplement of $M_1 + X$ in M. Since the module M_1 is $\oplus$-δ_{ss}-supplemented, then $M_1 \cap (X + S)$ has a δ_{ss}-supplement T in M_1 which is a direct summand of M_1. Therefore, $S + T$ is a

δ_{ss}-supplement of X in M. Further, $S+T$ is a direct summand of M as both S and T are direct summands of M_2 and M_1, respectively. Hence, $M = M_1 \oplus M_2$ is $\oplus$-δ_{ss}-supplemented. $\square$

Let $\{M_i\}_{i\in I}$ be a family of modules. The members of this family are called relatively projective if M_s is M_t-projective for all $1 \leqslant s \neq t \leqslant n$ for a given positive integer n.

THEOREM 2.2. *Let $\{M_i\}_{i\in I}$ be any finite family of relatively projective modules. Then the module $M = \bigoplus_{i=1}^{n} M_i$ is $\oplus$-δ_{ss}-supplemented if and only if each M_i is $\oplus$-δ_{ss}-supplemented.*

PROOF. ($\Rightarrow$) The sufficiency part is clear from Theorem 2.1.

($\Leftarrow$) For the necessity, it is enough to show that M_1 is $\oplus$-δ_{ss}-supplemented. Let $X \leqslant M_1$. Then $X \leqslant M$ and so there exists a δ_{ss}-supplement Y of X in M such that $M = Y \oplus Y'$ for a submodule Y' of M. Thus we have $X + Y = M$ and $X \cap Y \leqslant \mathrm{Soc}_\delta(M)$. From here, we get $M = X + Y = M_1 + Y$. Therefore, it can be written that $M = M_1 + Y_1$ for a submodule $Y_1 \leqslant Y$, from [**12**, 4.47]. Following, $Y = (M_1 + Y_1) \cap Y = Y_1 \oplus (M_1 \cap Y)$ is obtained. However, as $M = X + Y$,$M_1 = X + (M_1 \cap Y)$ is obtained by the modularity. Besides, $X \cap (M_1 \cap Y) = X \cap Y \leqslant \mathrm{Soc}_\delta(M)$. Therefore, $M_1 \cap Y$ is a δ_{ss}-supplement of X in M_1. Now, let us show that $M_1 \cap Y \leqslant_\oplus M_1$. As $M = Y \oplus Y'$, we have

$$\begin{aligned} M_1 &= (Y \oplus Y') \cap M_1 = \{[Y_1 \oplus (M_1 \cap Y)] \oplus Y'\} \cap M_1 \\ &= \{(M_1 \cap Y) \oplus (Y_1 \oplus Y')\} \cap M_1 = (M_1 \cap Y) \oplus \{(Y_1 \oplus Y') \cap M_1\}, \end{aligned}$$

by the modularity. Hence, M_1 is a $\oplus$-δ_{ss}-supplemented module. $\square$

Now, we state the following known fact to prove that a submodule of a $\oplus$-δ_{ss}-supplemented module is $\oplus$-δ_{ss}-supplemented module.

LEMMA 2.3. [**9**, Lemma 2.3] *Let $D \leqslant_\oplus M$, $X \leqslant M$ with $\frac{M}{X}$ is projective and $M = D + X$. Then, $D \cap X \leqslant_\oplus M$.*

THEOREM 2.3. *Let M be a $\oplus$-δ_{ss}-supplemented module and let $X \leqslant M$ such that $\frac{M}{X}$ is projective. Then the submodule X of M is a $\oplus$-δ_{ss}-supplemented module.*

PROOF. Let $N \leqslant M$ and let X be any submodule of N. By the assumption, there exists Y, Y' of M such that $X + Y = M$, $X \cap Y \leqslant \mathrm{Soc}_\delta(Y)$ and $M = Y \oplus Y'$. By the modularity, we have $N = X + (Y \cap N)$. Moreover, since $M = N + Y$, $Y \leqslant_\oplus M$ and $\frac{M}{N}$ is projective, then $N \cap Y \leqslant_\oplus M$. Besides, it can be seen that $N \cap Y \leqslant_\oplus N$ and $N \cap Y \leqslant_\oplus Y$. On the other hand, as $X \cap (Y \cap N) = X \cap Y \leqslant N \cap Y \leqslant Y$, $X \cap (Y \cap N) \ll_\delta Y \cap N$ and it is also semisimple. Hence, N is a $\oplus$-δ_{ss}-supplemented module. $\square$

Recall from [**19**] that a submodule X of M is called *weak δ-coclosed* in M if, given $Y \leqslant X$ such that $\frac{X}{Y}$ is singular and $\frac{X}{Y} \ll_\delta \frac{M}{Y}$, then $X = Y$. In addition to this, a module M has the *summand intersection property* if the intersection of two summands of M is a summand of M. A fully invariant submodule X of a module

M is a submodule provided that $f(X) \leqslant X$ for each $f \in End(M)$. Furthermore, if $M = M_1 \oplus M_2$, then $X = (X \cap M_1) \oplus (X \cap M_2)$.

In the light of these facts, we give the suitable conditions for some special submodules of a $\oplus$-δ_{ss}-supplemented module to be $\oplus$-δ_{ss}-supplemented.

THEOREM 2.4. *Let M be a $\oplus$-δ_{ss}-supplemented module, $X \leqslant M$ be weak δ-coclosed and $Y \leqslant_{\oplus} M$. If $X \cap Y \leqslant_{\oplus} M$, then X is a $\oplus$-δ_{ss}-supplemented module.*

PROOF. Let $X' \leqslant X$. By the assumption, there exists two submodules $S, X'' \leqslant M$ such that $X' + S = M$, $X' \cap S \leqslant \mathrm{Soc}_\delta(S)$ and $M = X' \oplus X''$. Therefore, we have that $X = X' + (S \cap X)$ and $X' \cap (S \cap X) = X' \cap S \leqslant \mathrm{Soc}_\delta(S)$. Following this, we have $X' \cap S \ll_\delta M$ and so $X' \cap S \ll_\delta X$ as $X' \cap S \leqslant X \leqslant M$ and X is weak δ-coclosed by [**19**, Lemma 2.2]. It can be easily shown that $X \cap S \leqslant_{\oplus} X$ as $M = X' \oplus X''$. Thus, $X' \cap (S \cap X) = X' \cap S \ll_\delta X \cap S$. Hence, $S \cap X$ is a δ_{ss}-supplement of X which is a direct summand of M, that is, X is $\oplus$-δ_{ss}-supplemented. □

COROLLARY 2.1. *Let M be a $\oplus$-δ_{ss}-supplemented module that has the summand intersection property. Then every direct summand of M is $\oplus$-δ_{ss}-supplemented.*

PROOF. Since every direct summand of M is also weak δ-coclosed in M from [**19**, Lemma 2.1], the proof is clear. □

COROLLARY 2.2. *Let M be a $\oplus$-δ_{ss}-supplemented module and $X \leqslant M$ be δ-coclosed. If $X \cap D \leqslant_{\oplus} X$ where $D \leqslant_{\oplus} M$, then X is a $\oplus$-δ_{ss}-supplemented module.*

PROOF. Since any δ-coclosed submodule is also weak δ-coclosed, then the proof is clear from Theorem 2.4. □

COROLLARY 2.3. *Let M be a $\oplus$-δ_{ss}-supplemented module and $X \leqslant M$ be weak δ-coclosed. If $f(X) \leqslant X$ for all $f = f^2 \in End(M)$,then X is a $\oplus$-δ_{ss}-supplemented module. In particular, any fully invariant δ-coclosed submodule (or direct summand) of M is $\oplus$-δ_{ss}-supplemented.*

PROOF. Let M_1 be any direct summand of M. So M has a decomposition such that $M = M_1 \oplus M_2$ for a submodule $M_2 \leqslant M$. For the projection map $f \colon M_1 \oplus M_2 \to M_1$, we have $f(X) = X \cap M_1 \leqslant_{\oplus} X$. Then, X is a $\oplus$-δ_{ss}-supplemented module from Theorem 2.4. □

Here we give a theorem related with the decomposition of a $\oplus$-δ_{ss}-supplemented module.

THEOREM 2.5. *Let M be a $\oplus$-δ_{ss}-supplemented module. Then M has a decomposition such that $M = M_1 \oplus M_2$ where $\delta(M_1) \leqslant \mathrm{Soc}_\delta(M_1)$ and $\delta(M_2) = M_2$.*

PROOF. Since M is $\oplus$-δ_{ss}-supplemented module, $\delta(M)$ has a δ_{ss}-supplement M_1 in M which is a direct summand of M such that $M = \delta(M) + M_!, \delta(M) \cap M_1 = \delta(M_1) \leqslant \mathrm{Soc}_\delta(M_1)$ and $M = M_1 \oplus M_2$. Additionally, $M = \delta(M) + M_1 = [\delta(M_1) \oplus \delta(M_2)] + M_1 = M_1 + \delta(M_2)$ and so, $\frac{M}{M_1} \cong M_2 = \delta(M_2)$ is obtained. □

Let us note that $\delta^*(M)$ is defined as a submodule of a module M in [**19**] as follows. $\delta^*(M) = \{m \in M \mid Rm \ll_\delta E(Rm)\}$ where $E(Rm)$ is the injective hull of Rm.

Corollary 2.4. *Let M be a $\oplus$-δ_{ss}-supplemented module. Then M has a decomposition $M = M_1 \oplus M_2$ such that $\delta^*(M_1) \leqslant \mathrm{Soc}_\delta(M_1)$ and $\delta^*(M_2) = M_2$.*

Proof. The proof is clear from Theorem 2.5. □

In the following theorem, we show that the case of being $\oplus$-δ_{ss}-supplemented is preserved under factor modules.

Theorem 2.6. *Let M be a module and $X \leqslant M$ be fully invariant. If M is a $\oplus$-δ_{ss}-supplemented module, then $\frac{M}{X}$ is a $\oplus$-δ_{ss}-supplemented module.*

Proof. Let us assume that M is $\oplus$-δ_{ss}-supplemented and $\frac{A}{X} \leqslant \frac{M}{X}$ be any submodule of M. Then, there exists a decomposition of M such that $M = B \oplus B'$, $A + B = M$ and $A \cap B \leqslant \mathrm{Soc}_\delta(B)$. Therefore, $\frac{B+X}{X}$ is a δ_{ss}-supplement of $\frac{A}{X}$ in $\frac{M}{X}$ [**21**]. Now, it remains to show that $\frac{B+X}{X} \leqslant_\oplus \frac{M}{X}$. Let $\pi\colon B \oplus B' \to B$ be the projection map with the kernel $(1-\pi)M = B'$. Then $\pi^2 = \pi \in End(M)$ and $\pi M = B$. From assumption $\pi X \leqslant X$ and $(1-\pi)X \leqslant X$ is obtained. Thus, we have $\pi X = X \cap B$ and $(1-\pi)X = X \cap B'$. Therefore we have $X = \pi X \oplus (1-\pi)X = (X \cap B) \oplus (X \cap B')$. Then it is clear that $\frac{B+X}{X} = \frac{B \oplus (X\cap B')}{X}$ and $\frac{B'+X}{X} = \frac{B' \oplus (X \cap B)}{X}$ which implies $\frac{M}{X} = \frac{B\oplus(X\cap B')}{X} + \frac{B'\oplus(X\cap B)}{X}$. In addition to these,

$$\begin{aligned}[B \oplus (X \cap B')] \cap [B' \oplus (X \cap B)] &= \{[B \oplus (X \cap B')] \cap B'\} \oplus (X \cap B)\\ &= (X \cap B') \oplus (B \cap B') \oplus (X \cap B) = (X \cap B) \oplus (X \cap A) = X.\end{aligned}$$

This verifies that $\frac{B+X}{X} \leqslant_\oplus \frac{M}{X}$. Hence, $\frac{M}{X}$ is $\oplus$-δ_{ss}-supplemented. □

Recall that a module M is called *δ-radical* if $\delta(M) = M$ [**16**].

Corollary 2.5. *Let M be a $\oplus$-δ_{ss}-supplemented module, then $\frac{M}{P_\delta(M)}$ is a $\oplus$-δ_{ss}-supplemented module where $P_\delta(M)$ is the sum of all δ-radical submodules of M.*

Proof. Since $P_\delta(M) \leqslant M$ is fully invariant, then the proof is clear from Theorem 2.6. □

2.1. Rings whose modules are $\oplus$-δ_{ss}-supplemented.

Theorem 2.7. *Let M be a finitely generated module whose direct summands are $\oplus$-δ_{ss}-supplemented. Then M is a direct sum of cyclic modules.*

Proof. It is clear from [**18**, Theorem 3.1] as every $\oplus$-δ_{ss}-supplemented module is $\oplus$-δ_{ss}-supplemented. □

Corollary 2.6. *Every two-generated $\oplus$-δ_{ss}-supplemented module is a direct sum of cyclic modules.*

Proof. It is clear from Theorem 2.7. □

Corollary 2.7. *An n-generated module is $\oplus$-δ_{ss}-supplemented if and only if every cyclic module is $\oplus$-δ_{ss}-supplemented and every n-generated module is a direct sum of cyclic modules.*

Proof. ($\Rightarrow$) Assume that every n-generated module is $\oplus$-δ_{ss}-supplemented. Then, so is every cyclic module as a one-generated module. Let M be any n-generated module. As every direct summand of an n-generated module is n-generated, M is a direct sum of cyclic modules by Theorem 2.7.

($\Leftarrow$) Let M be any n-generated module. By the assumption, M is a finite direct sum of cyclic modules. Since each summand is $\oplus$-δ_{ss}-supplemented, then M is $\oplus$-δ_{ss}-supplemented from Theorem 2.1. $\square$

Theorem 2.8. *Let R be a ring. Then, $_RR$ is $\oplus$-δ_{ss}-supplemented if and only if every finitely generated free R-module is $\oplus$-δ_{ss}-supplemented.*

Proof. ($\Rightarrow$) By the assumption, $_RR$ is $\oplus$-δ_{ss}-supplemented as a finitely generated free R-module.

($\Leftarrow$) Let M be a finitely generated free R-module such that $M = Rx_1 + Rx_2 + \cdots + Rx_n \cong R^{(n)} = R \oplus R \oplus \cdots \oplus R$ where each $Rx_i \cong R$. Since the R-module R is $\oplus$-δ_{ss}-supplemented and so is Rx_i,for each i. Hence M is $\oplus$-δ_{ss}-supplemented as a finite direct sum of $\oplus$-δ_{ss}-supplemented modules from Theorem 2.1. $\square$

Theorem 2.9. *For a ring R the following statements are equivalent:*

(1) *R is δ_{ss}-perfect.*
(2) *$_RR$ is δ_{ss}-supplemented.*
(3) *$_RR$ is $\oplus$-δ_{ss}-supplemented.*
(4) *Every finitely generated free R-module is $\oplus$-δ_{ss}-supplemented.*

Proof. (1) $\Leftrightarrow$ (2) is clear from [**21**, Theorem 5.3]
(2) $\Leftrightarrow$ (3) is clear from [**21**, Theorem 5.6]
(3) $\Leftrightarrow$ (4) is clear from Theorem 2.8. $\square$

In the following example, we show that the containing relation is proper between the class of $\oplus$-δ_{ss}-supplemented modules and the class $\oplus$-δ-supplemented modules.

Example 2.1. Let $R = \frac{F[x_1,x_2,\dots]}{\langle\{x_1^2,x_2^2-x_1,x_3^2-x_2,\dots\}\rangle}$ be the ring of polynomials over a field F where $x_1, x_2, \dots$ are countably many indeterminates. From [**24**, Example 4.4], it can be seen that $_RR$ is a δ-semiperfect ring which is not δ_{ss}-perfect. Hence the R-module R is a $\oplus$-δ-supplemented module which is not $\oplus$-δ_{ss}-supplemented from Theorem 2.9 and [**18**, Lemma 3.5].

3. Completely $\oplus$-δ_{ss}-supplemented modules

Definition 3.1. A module M is called a *completely $\oplus$-δ_{ss}-supplemented module* if every direct summand of M is $\oplus$-δ_{ss}-supplemented.

Recall from [**12**] that a module M is called a *$(D3)$-module*, if for the submodules $M_1, M_2 \leqslant_\oplus M$ with $M = M_1 + M_2$, satisfy that $M_1 \cap M_2 \leqslant_\oplus M$.

According to the definitions, it is clear that every completely $\oplus$-δ_{ss}-supplemented module is $\oplus$-δ_{ss}-supplemented. Now, we investigate the conditions when the converse is true.

THEOREM 3.1. *Let M be a $\oplus$-δ_{ss}-supplemented module with $(D3)$. Then, M is a completely $\oplus$-δ_{ss}-supplemented module.*

PROOF. Let $X \leqslant_\oplus M$ and $Y \leqslant X$. As $Y \leqslant M$ and M is $\oplus$-δ_{ss}-supplemented, there exists a δ_{ss}-supplement T of Y in M such that $Y + T = M$, $Y \cap T \leqslant \mathrm{Soc}_\delta(T)$ and $M = T \oplus T'$. By the modularity, $X = (Y + T) \cap X = Y + (T \cap X)$ and $Y \cap (T \cap X) = Y \cap T \leqslant \mathrm{Soc}_\delta(T)$ is obtained. On the other hand, $X \cap T \leqslant_\oplus M$ as M has the property $(D3)$. Therefore, it can be easily verified that $X \cap T$ is also a direct summand of X by the modularity. In addition to these, $Y \cap (T \cap X) \leqslant \mathrm{Soc}_\delta(T \cap X)$ by [**17**]. Hence, every direct summand X of M is $\oplus$-δ_{ss}-supplemented, that is, M is completely $\oplus$-δ_{ss}-supplemented. □

A module M is said to have the *exchange property* if for any module X and a decomposition $X = M' \oplus Y = \bigoplus_{i \in I} A_i$ where $M' \cong M$, there exists submodules A'_i of A_i for each i such that $X = M' \oplus (\bigoplus A'_i)$. The module M is said to have the *finite exchange property* whenever this condition holds for a finite set. And this property is preserved by summands and finite direct sums [**12**].

THEOREM 3.2. *Let $\{X_i\}_{i=1}^n$ be a family of completely $\oplus$-δ_{ss}-supplemented modules with the finite exchange property. Then $\bigoplus_{i=1}^n X_i$ is completely $\oplus$-δ_{ss}-supplemented.*

PROOF. Let $X \leqslant_\oplus \bigoplus_{i=1}^n X_i$. Then, it can be written that $\bigoplus_{i=1}^n X_i = X \oplus Y$ for a submodule Y of $\bigoplus_{i=1}^n X_i$. We will show that X is a $\oplus$-δ_{ss}-supplemented module. By [**12**, Lemma 3.20], $\bigoplus_{i=1}^n X_i$ and X have the finite exchange property. Therefore, $X \oplus Y = (\bigoplus_{i=1}^n X'_i) \oplus Y$,where $X'_i \leqslant_\oplus X_i$ for each i. Since each X_i is completely $\oplus$-δ_{ss}-supplemented module, then X'_i is $\oplus$-δ_{ss}-supplemented for every $i = 1, 2, \ldots, n$. Thus, $\bigoplus_{i=1}^n X'_i \cong X$ is $\oplus$-δ_{ss}-supplemented, from Theorem 2.4. □

THEOREM 3.3. *Let M be a UC extending module. If M is $\oplus$-δ_{ss}-supplemented, then M is completely $\oplus$-δ_{ss}-supplemented.*

PROOF. By the assumption, M has the property (D3) from [**5**, Lemma 2.4]. Then M is completely $\oplus$-δ_{ss}-supplemented from Theorem 3.2. □

A partial endomorphism of M is a homomorphism from a submodule of M into M. If every nonzero partial endomorphism of M is one to one, then M is called *monoform*. Furthermore, if every partial endomorphism of M satisfies $\ker(f) \leqslant_c M$, then M is called *polyform*. It is clear that every monoform module is polyform. Let $X_1 \leqslant X_2 \leqslant \ldots$ be any ascending chain of submodules of a module M. If there exists an integer n such that $X_n \trianglelefteq X_k$ for every $k \geqslant n$, then n is called the *finite uniform dimension* of M [**22**] and [**23**]. If every finitely generated submodule of M has finite uniform dimension, then M is called a locally finite dimensional module.

COROLLARY 3.1. *A polyform (monoform) extending module M is $\oplus$-δ_{ss}-supplemented if and only if M is completely $\oplus$-δ_{ss}-supplemented.*

PROOF. The sufficiency is clear. For the necessity, let us assume that M is $\oplus$-δ_{ss}-supplemented. As M is a UC-module from [**22**, Proposition 2.2], M is a completely $\oplus$-δ_{ss}-supplemented module from Theorem 3.3. □

Recall from [**3**] that a module M is said to be quasi-injective if M is M-injective, that is, every homomorphism $f\colon X \to M$ can be extended to an endomorphism of M where $X \leqslant M$. Semisimple modules and injective modules are quasi-injective.

THEOREM 3.4. *Let M be a locally finite dimensional polyform module. If M is quasi-injective, then for any index set I, $M^{(I)}$ is $\oplus$-δ_{ss}-supplemented if and only if $M^{(I)}$ is completely $\oplus$-δ_{ss}-supplemented.*

PROOF. Let $M^{(I)}$ be a $\oplus$-δ_{ss}-supplemented module. By the assumption, $M^{(I)}$ is a polyform module from [**23**, Proposition 3.3] and also $M^{(I)}$ is quasi-injective from [**22**, Corollary 3.4]. Thus, $M^{(I)}$ is a $(C1)$-module (or extending). Hence, $M^{(I)}$ is completely $\oplus$-δ_{ss}-supplemented from Corollary 3.1. $\square$

THEOREM 3.5. *Let M be a module with $(D3)$ condition. Then the following statements are equivalent:*

(1) *M is completely $\oplus$-δ_{ss}-supplemented.*
(2) *M is $\oplus$-δ_{ss}-supplemented.*
(3) *$M = X \oplus Y$, such that X and Y are $\oplus$-δ_{ss}-supplemented, $\delta(X) \leqslant \mathrm{Soc}_\delta(X)$ and $\delta(Y) = Y$.*
(4) *$M = X \oplus Y$, such that X and Y are $\oplus$-δ_{ss}-supplemented, $\delta^*(X) \leqslant \mathrm{Soc}_\delta(X)$ and $\delta^*(Y) = Y$.*

PROOF. (1) $\Rightarrow$ (2) is clear from the definitions.
(2) $\Rightarrow$ (1) is clear from Theorem 3.1.
(1) $\Rightarrow$ (3) is clear from Theorem 5 and Theorem 3.1 as M is a $(D3)$-module.
(1) $\Rightarrow$ (4) is clear from Theorem 5 and Theorem 3.1.
(3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (2) are clear from Theorem 2.1. $\square$

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