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UNIQUELY EXCHANGE RINGS

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Abstract. An associative ring with unity is called exchange if every element is exchange, i.e., there exists an idempotent $e \in aR$ such that $1-e \in (1-a)R$; if this representation is unique for every element, we call the ring uniquely exchange. We give a complete description of uniquely exchange rings.

1. Introduction

Let R be an associative ring with identity. An element $a \in R$ is said to be exchange if there exists an idempotent $e \in aR$ such that $1-e \in (1-a)R$. The ring R is said to be exchange if all of its elements are exchange [**3**–**5**,**9**–**11**]. We say that, an element a in a ring R is said to be uniquely exchange if there exists a unique idempotent $e \in aR$ such that $1-e \in (1-a)R$. A ring R is said to be a uniquely exchange if every element is uniquely exchange. An element $a \in R$ is said to be clean if $x = e + u$ for some idempotent e and unit u in R. The ring R is said to be clean if all of its elements are clean. Clean rings were first introduced in a paper by Nicholson [**9**] as a class of exchange rings. It was shown by Nicholson [**9**, Proposition 1.8(1)] that if a is clean in the ring R, then there exists $e^2 = e \in aR$ such that $1-e \in (1-a)R$. R is said to be suitable if for each $a \in R$, there exists an idempotent $e \in aR$ such that $1-e \in (1-a)R$. This condition is left-right symmetric as shown in [**9**]. In the same paper, Nicholson [**9**] also showed that R is an exchange ring if and only if idempotents can be lifted modulo every left (right) ideal of R if and only if R is suitable. Hence, every clean ring is an exchange ring. The converse is known to be true in abelian rings (see [**9**, Proposition 1.8(2)]. An element $a \in R$ is uniquely clean provided that there exists a unique idempotent $e \in R$ such that $a - e \in R$ is invertible. A ring R is uniquely clean in case every element in R is uniquely clean. Many authors have studied such rings, see [**1**, **2**, **6**–**8**]. In this paper we investigate the uniquely exchange rings, and we show that every image of a uniquely exchange ring is again uniquely exchange and every Boolean ring is uniquely exchange. Finally, we prove that a local ring R is uniquely exchange if and only if $R/J(R) \cong \mathbb{Z}_2$.

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2. Main results

Definition 2.1. An element a in a ring R is called uniquely exchange if there exists a unique idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. A ring R is called a uniquely exchange if every element is uniquely exchange.

Proposition 2.1. *Central idempotents and central nilpotents are uniquely exchange in any ring R.*

Proof. By [**8**, Example 1], central idempotents and central nilpotents are uniquely clean. Hence there exists a unique idempotent $e \in R$ such that $e - x \in (x - x^2)R$, by [**9**, Proposition 1.8]. Therefore there exists a unique idempotent $e \in aR$ such that $1 - e \in (1 - a)R$, by [**9**, Proposition 1.1], as required. □

Corollary 2.1. *Every Boolean ring is uniquely exchange.*

A routine elementary argument establishes the following results.

Proposition 2.2. *Every homomorphic image of a uniquely exchange ring is uniquely exchange.*

Proposition 2.3. *A direct product $\prod_{i \in I} R_i$ of rings is uniquely exchange if and only if each R_i is uniquely exchange.*

Proposition 2.4. *A ring R is a uniquely exchange ring if and only if $R/J(R)$ is a uniquely exchange ring and idempotents left modulo $J(R)$.*

Proof. Follows from Proposition 2.2 and [**9**, Corollary 1.3]. □

Proposition 2.5. *Let R be a uniquely exchange ring,i.e; for every $a \in R$ there exists a unique idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Then $ea = ae$.*

Proof. Let $a \in R$. Then, if there exists a unique idempotent $e \in aR$ such that $1 - e \in (1 - a)R$, then $e + (ea - eae)$ is an idempotent. Hence

$$e + (ea - eae) \in aR, \qquad (1 - (e + (ea - eae)) \in (1 - a)R.$$

Since R is uniquely exchange, $e = e + (ea - eae)$. It follows that $ea = eae$, and similarly $ae = eae$. □

Proposition 2.6. *Let R be a uniquely exchange ring and $e^2 = e \in R$. Then eRe is uniquely exchange.*

Proof. If $a \in eRe$ choose $f^2 = f \in aR$ such that $1 - f \in (1 - a)R$. Since $a \in eRe$ and $f \in aR$, we see that $a = exe$ for some $x \in R$ and $f = ay$ for some $y \in R$, so $f = exey$. Therefore $ef = f$, and fe is an idempotent. Hence $e - fe = e(1 - f)e \in (e - a)eRe$. Therefore eRe is exchange. To check uniqueness, let $a \in eRe$ and there exist two idempotents $f, f' \in aR$ such that

$$e - fe \in (e - a)eRe, \qquad e - f'e \in (e - a)eRe.$$

Hence $e(1 - f)e \in e(1 - a)Re$ and $e(1 - f')e \in e(1 - a)Re$, and so $1 - f \in (1 - a)R$ and $1 - f' \in (1 - a)R$, a contradiction. □

Let R be a ring and let ${}_RM_R$ be an R-R-bimodule which is a general ring (possibly with no unity) in which $(mn)a = m(na) = m(an)$ and $(am)n = a(mn)$ hold for all $m, n \in M$ and $a \in R$. Then the ideal-extension $I(R; M)$ of R by M is defined to be the additive abelian group $I(R; M) = R \oplus M$ with multiplication $(a, m)(b, n) = (ab, an + mb + mn)$. Note that if S is a ring and $S = R \oplus A$, where R is a subring and $A \triangleleft S$, then $S \cong I(R; A)$.

Proposition 2.7. *An ideal-extension $S = I(R; M)$ is uniquely exchange if the following conditions are satisfied:*

(1) *R is uniquely exchange.*
(2) *If $e \in \mathrm{Id}(R)$, then $em = me$ for all $m \in M$.*
(3) *If $m \in M$, then $m + n + mn = 0$ for some $n \in M$.*

Proof. Let $s = (a, m) \in S$ and by (1) there exists a unique idempotent $e \in aR$ such that $1 - e \in (1 - a)R$. Since S is a clean, by [**8**, Proposition 7], and so S is a exchange ring. Now suppose that there exist idempotents $(e, x), (e', x') \in sS$ such that

$$1_S - (e, x) \in (1_S - s)S, \qquad 1_S - (e', x') \in (1_S - s)S.$$

Hence $(e, x) = (e', x')$ by the following result.

We show that, if $(e, x)^2 = (e, x)$, then $e^2 = e$ and $x = 0$. $(e, x)^2 = (e, x)$ gives $e^2 = e$ and $x = 2ex + x^2$ using (2). Then multiplying by e gives $ex + ex^2 = 0$, and multiplying by x gives $x^2 = 2ex^2 + x^3$. Hence adding this latter equation to $x = 2ex + x^2$ yields $x = x^3$, and so x^2 is an idempotent in M. By (3), $-x^2 + y + (-x^2)y = 0$, for some $y \in M$, so that $x^2 + n = x^2n$ where $n = -y$. Multiplying by x^2 yields $x^2 = 0$, whence $x = x^3 = 0$, as required. □

Proposition 2.8. *Suppose that the ideal-extension $S = I(R; M)$ is uniquely exchange. Then the following statements hold:*

(1) *R is uniquely exchange.*
(2) *If $e \in \mathrm{Id}(R)$ and $(e, 0) \in (a, m)S$ such that $1_S - (e, 0) \in (1_S - (a, m))S$, then then $em = me$.*

Proof. Suppose that S is uniquely exchange. It is routine to see that (1) holds. If $e \in \mathrm{Id}(R)$, then $(e, 0)$ is an idempotent in S and $(e, 0) \in (e, m)S$ such that $1_S - (e, 0) \in (1_S - (e, m))S$. There $(e, 0)$ commutes with (e, m) for every $m \in M$, by Proposition 2.5, and (2) follows. □

In the following, we characterize the local uniquely exchange rings.

Lemma 2.1. *Let $R \neq 0$ be a ring. Then the following are equivalent:*

(1) *R is local.*
(2) *R is clean and 0 and 1 are the only idempotents in R.*
(3) *R is exchange and 0 and 1 are the only idempotents in R.*

Proof. (1)$\Longleftrightarrow$ (2) follows from [**8**, Lemma 14].
(2)$\Longrightarrow$ (3) follows from [**9**, Proposition 1.8].
(3)$\Longrightarrow$ (1) Suppose that $a \notin J(R)$. So, $1 - ar$ is not invertible for some $r \in R$. Since the ring R is exchange, the element $1 - ar$ is exchange, so there exists an

idempotent $e \in (1-ar)R$ such that $1-e \in (1-(1-ar))R = arR$. Since the only idempotents in R are 0 and 1, and $1-ar$ is not invertible, it follows that $e=0$, so $1 = 1-0 \in arR$ and it follows that there exists $s \in R$ such that $ars = 1$. Analogously, there exists $r_1, t \in R$ such that $tr_1a = 1$, so a is invertible. This proves that R is local. □

THEOREM 2.1. *Let $R \neq 0$ be a ring. Then the following are equivalent:*

(1) *R is local and uniquely clean.*
(2) *R is uniquely clean and 0 and 1 are the only idempotents in R.*
(3) *R is uniquely exchange and 0 and 1 are the only idempotents in R.*
(4) *$R/J(R) \cong \mathbb{Z}_2$.*

PROOF. (1) $\iff$ (2) $\iff$ (4) follows from [**8**, Theorem 15].
(2) $\implies$ (3) follows from [**9**, Proposition 1.8].
(3) $\implies$ (4) If $\bar{a} \neq \bar{0}$ in $\bar{R} = R/J(R)$, we show that $\bar{a} = \bar{1}$. If not then both a and $1-a$ are units because R is local by Lemma 2.1. Hence $aR = (1-a)R$. Therefore $-a = 1-a$, which implies that $0 = 1$, a contradiction. □

LEMMA 2.2. *Let R be a exchange ring and $I \nsubseteq J(R)$ is a right (or left) ideal of R. Then there exists $0 \neq e^2 = e \in I$.*

PROOF. Suppose that $I \nsubseteq J(R)$ is a right ideal containing no nonzero idempotent. If $a \in I$, then there exists $e^2 = e \in aR$ such that $1-e \in (1-a)R$. Hence $e = 0$, and so $1 \in (1-a)R$. Therefore $1-a$ is a unit. Thus $I \subseteq J(R)$, a contradiction. A similar argument works if I is a left ideal. □

COROLLARY 2.2. *Let R be a uniquely exchange ring. Then $R/J(R)$ has characteristic 2.*

PROOF. We must show that $2 = 1+1 \in J(R)$. If $2 \notin J(R)$, then there exists $0 \neq e^2 = e \in 2R$ by Lemma 2.2. Hence $e = 2b$, where $b \in R$. We may assume that $eb = b = be$. Then $u = (1-e) - 2e$ is a unit with inverse $(1-e) - b$. Hence $1, 1-e \in uR$ and $0, e \in (1-u)R$. Since R is uniquely exchange, $1 = 1-e$, and so $e = 0$, a contradiction. □

Suppose RG is now the group ring of G over R defined as usual.

PROPOSITION 2.9. *Let R be a commutative uniquely exchange ring. Then $R((C_2)^n)$ is uniquely exchange for all $n \geqslant 0$.*

PROOF. It is easy to see that $R((C_2)^n) \cong (R(C_2)^{n-1})C_2$, so it suffices to show that if RC_2 is uniquely exchange. Since R is commutative uniquely exchange, R is a clean ring, by [**9**], and so RC_2 is clean, by [**8**, Proposition 24]. Hence $RC_2$is exchange. To check uniqueness, let $a \in RC_2$. Then there exists an idempotent $e \in aRC_2$ such that $1-e \in (1-a)RC_2$. If $e = r+sg$, then $r^2+s^2 = r$ and $2rs = s$, so $s = 0$ (as $2 \in J(R)$). Hence $e^2 = e = r \in R$. Since R is uniquely exchange, this shows that e is uniquely determined by a. □

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