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ON DEFINITIONS IN MATHEMATICS

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Dedicated to Žarko Mijajlović on the occasion of his 70th birthday.

Abstract. The concept of definition is usually not covered in mathematical logic textbooks. The definability of classes of structures is dealt with in model theory but the definability of concepts within a given structure is not. Our aim is to deal with these kind of definitions. We also address some of the implications for teaching and learning mathematics.

In Peter Smith's selection of the most famous textbooks on mathematical logic [**7**] there is almost no mention of definability of concepts (i.e. relations and operations) in formal mathematical theories. Our aim is to deal with this kind of definitions.

We will consider the introduction of definitions within a sufficiently formalized mathematical theories based on primitive non-logical concepts and their axioms (e.g. first order theories). The definitions are introduced sequentially. The first definition establishes the meaning of a new concept in terms of primitive concepts, the second one establishes the meaning of a second new concept in terms of primitive concepts and the first defined concept, etc.

1. Explicit definitions

The simplest kind of definitions are explicit definitions.

Explicit definitions of relations. An n-place predicate R is explicitly definable in a theory T if

$$R(x_1, \dots, x_n) \leftrightarrow A,$$

is provable in T, with the following restrictions:

1. A is a formula in which R does not occur,
2. variables $x_1, \dots, x_n$ are distinct,
3. A has no free variables other than $x_1, \dots, x_n$.

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In a special case, the theory T' does not contain the predicate R, and T is an extension of T' by the definition $R(x_1, \dots, x_n) \leftrightarrow A$. Restriction 1 prevents circularity. Restriction 2 guaranties that R is a n-place relation. If Restriction 3 is violated we would have, for example

$$R(x) \leftrightarrow P(x, y),$$

which is equivalent to

$$(P(x, y) \to R(x)) \land (R(x) \to P(x, y))$$

which is equivalent to

$$((\exists y)P(x, y) \to R(x)) \land (R(x) \to (\forall y)P(x, y))$$

which implies

$$(\exists y)P(x, y) \to (\forall y)P(x, y)$$

which could be false.

Note, that Restriction 3 does not prevent variables to be free in the definiendum although they are not free in the definiens. For example, $R(x, y) \leftrightarrow P(x)$ is permitted (and it could always be replaced with the equivalent $R(x, y) \leftrightarrow P(x) \land y = y$).

Concerning operations we have the analogous definition.

Explicit definitions of operations. An n-place operation o is explicitly definable in a theory T if

$$o(x_1, \dots, x_n) = t,$$

is provable in T, with the following restrictions:

1. t is a term in which o does not occur,
2. variables $x_1, \dots, x_n$ are distinct,
3. o has no free variables other than $x_1, \dots, x_n$.

The explicit definition of a constant is the special case of the 0-place operation.

In a special case, the theory T' does not contain the term o and T is an extension of T' by the definition $o(x_1, \dots, x_n) = t$.

As before, Restriction 1 prevents circularity. Restriction 2. guaranties that o is an n-place operation. If restriction 3. is violated we would have, for example,

$$o(x) = p(x, y)$$

which implies

$$p(x, y_1) = o(x) = p(x, y_2)$$

which could be false.

Note, that Restriction 3 does not prevent variables to be free in the definiendum although they are not free in the definiens. For example, $o(x, y) = p(x)$ is permitted.

2. Eliminability and non-creativity

It is easy to prove that explicit definitions of concepts (i.e. relations and operations) are always eliminable and non-creative. The more formal meaning of these notions is the following.

Eliminability. A definition D of a new concept is eliminable in a theory if and only if: whenever the new concept occurs in a formula F then there is a formula F', in which the new concept does not occur, such that $D \to (F \leftrightarrow F')$ is provable in the theory.

Non-creativity. A definition D of a new concept is non-creative in a theory if and only if: whenever $D \to F$ is provable in the theory and the new concept does not occur in the formula F, then F is provable in the theory.

The two criteria are usually attributed to Lesniewski, as the criteria for definitions[1]:

> *If a formula introducing a new concept is eliminable and non-creative, then it is a definition of the concept.*

For example, in a semigroup $(G, +)$ we could try to define constant 0 and operation $-$ with the following implicit "definitions":

$$x + 0 = x$$

and

$$x - y = z \leftrightarrow x = y + z.$$

But, the proposed "definitions" are creative.

From $x + 0 = x$ it follows that $(\exists y)(x + y = x)$ which is not provable for semigroups.

Similarly, $x - y = z$ implies $(\exists z)(x - y = z)$, which together with $x - y = z \leftrightarrow x = y + z$ implies $(\exists z)(x = y + z)$ and this is not provable for semigroups.

Implicit definitions should be carefully formulated to deal with this problem.

3. Implicit definitions

Implicit definitions of constants. A constant c is explicitly definable in a theory T if

$$x = c \leftrightarrow C(x),$$

is provable in T, with the following restrictions:

1. $C(x)$ is a formula in which c does not occur,
2. $C(x)$ has no free variables other than x,
3. $(\exists! x)C(x)$ is provable in the theory.

In a special case, the theory T' does not contain the term c and T is an extension of T' by the definition $x = c \leftrightarrow C(x)$.

Note that the formula $x = c \leftrightarrow C(x)$ is equivalent to $C(c) \wedge (C(x) \to x = c)$. Given $(\exists! x)C(x)$, this is equivalent to $C(c)$. Hence, in presence of Restriction 3, $C(c)$ is an implicit definition of c.

For example, in the theory of fields, formula

$$x = 1/2 \leftrightarrow x + x = 1$$

successfully defines $1/2$ because $(\exists! x)(x + x = 1)$ is provable in the theory. Hence, $1/2 + 1/2 = 1$ defines $1/2$.

[1] According to [**9**, p. 161] this is a myth. It was Lukasiewicz who brought the issue up and Ajdukiewicz who was the first to formulate the criteria.

On the other hand, formula

$$x = \sqrt{2} \leftrightarrow x \cdot x = 2$$

does not define $\sqrt{2}$ because $(\exists! x)(x \cdot x = 2)$ is not provable in the theory. Hence $\sqrt{2} \cdot \sqrt{2} = 2$ does not define $\sqrt{2}$.

It is easy to prove that implicit definitions of constants satisfy Lesniewski's criteria.

THEOREM 3.1. *Implicit definitions of constants are eliminable and non-creative.*

PROOF. Let c be implicitly defined by $x = c \leftrightarrow C(x)$ and remember that this is equivalent to $C(c) \wedge (C(x) \to x = c)$.

Eliminability:

Every formula $A(c)$ which contains c is logically equivalent to the formula $(\forall x)(x = c \to A(x))$. From $x = c \leftrightarrow C(x)$ it follows that this is equivalent to $(\forall x)(C(x) \to A(x))$. Hence

$$(x = c \leftrightarrow C(x)) \to (A(c) \leftrightarrow (\forall x)(C(x) \to A(x)))$$

which was to be proved (cf. the above criterion of eliminability; D is now $x = c \leftrightarrow C(x)$, F is $A(c)$ and F' is $(\forall x)(C(x) \to A(x))$).

Non-creativity:

Suppose that $(C(c) \wedge (C(x) \to x = c)) \to F$. From $(\exists! x)C(x)$ it follows that $(C(c) \wedge (C(x) \to x = c)) \leftrightarrow C(c)$ and hence $C(c) \to F$. If c does not occur in F then $(\exists x)C(x) \to F$. This is a theorem of first order logic. It follows that $(\exists! x)C(x) \to F$. But $(\exists! x)C(x)$ is provable in the theory (by 3. restriction), hence F is provable in the theory. This was to be proved (cf. the above criterion of non-creativity; D is now $C(c) \wedge (C(x) \to x = c)$ i.e. $x = c \leftrightarrow C(x)$). □

Now we discuss the more general implicit definitions of operations (constants are special 0-place operations).

Implicit definitions of operations. An n-place term o is implicitly definable in a theory T if

$$x = o(x_1, \dots, x_n) \leftrightarrow O(x),$$

is provable in T, with the following restrictions:

1. $O(x)$ is a formula in which o does not occur,
2. $O(x)$ has no free variables other than x, x_1, ..., x_n,
3. variables x_1, ..., x_n are distinct,
4. $(\exists! x)O(x)$ is provable in the theory.

In a special case, the theory T' does not contain the term o and T is an extension of T' by the definition $x = o(x_1, \dots, x_n) \leftrightarrow O(x)$.

For example, in the theory of fields, formula

$$x = x_1 - x_2 \leftrightarrow x_1 = x + x_2$$

successfully defines $-$ because $(\exists! x)(x_1 = x + x_2)$ is provable in the theory. On the other hand, formula

$$x = \sqrt{x_1} \leftrightarrow x_1 \cdot x_1 = x$$

does not define $\sqrt{x_1}$ because $(\exists! x_1)(x_1 \cdot x_1 = 2)$ is not provable in the theory.

It is easy to prove that implicit definitions of operations satisfy Lesniewski's criteria.

THEOREM 3.2. *Implicit definitions of operations are eliminable and non-creative.*

PROOF. Proof is almost the same as the preceding one.

Let o be implicitly defined by $x = o(x_1, \ldots, x_n) \leftrightarrow O(x)$ and note that this is logically equivalent to $O(o(x_1, \ldots, x_n)) \wedge (O(x) \to x = o(x_1, \ldots, x_n))$.

Eliminability:

Every formula $A(o(x_1, \ldots, x_n))$ which contains $o(x_1, \ldots, x_n)$ is logically equivalent to the formula $(\forall x)(x = o(x_1, \ldots, x_n) \to A(x))$. From $x = o(x_1, \ldots, x_n) \leftrightarrow O(x)$ it follows that this is equivalent to $(\forall x)(O(x) \to A(x))$. Hence

$$(x = o(x_1, \ldots, x_n) \leftrightarrow O(x)) \to (A(o(x_1, \ldots, x_n)) \leftrightarrow (\forall x)(O(x) \to A(x)))$$

which was to be proved (cf. the above criterion of eliminability; D is now $x = o(x_1, \ldots, x_n) \leftrightarrow C(x)$, F is $A(o(x_1, \ldots, x_n))$ and F' is $(\forall x)(O(x) \to A(x)))$.

Non-creativity:

Suppose that $O(o(x_1, \ldots, x_n)) \wedge (O(x) \to x = o(x_1, \ldots, x_n)) \to F$. From $(\exists! x)O(x)$ it follows that $O(o(x_1, \ldots, x_n)) \wedge (O(x) \to x = o(x_1, \ldots, x_n)) \leftrightarrow O(o(x_1, \ldots, x_n))$ and hence $O(o(x_1, \ldots, x_n)) \to F$. If o does not occur in F then $(\exists x)O(x) \to F$. This is a theorem of first order logic. It follows that $(\exists! x)O(x) \to F$. But $(\exists! x)O(x)$ is provable in the theory (by 4. restriction), hence F is provable in the theory. This was to be proved (cf. the above criterion of non-creativity; D is now $O(o(x_1, \ldots, x_n)) \wedge (O(x) \to x = o(x_1, \ldots, x_n))$ i.e. $x = o(x_1, \ldots, x_n) \leftrightarrow O(x)$). □

4. Conditional definitions

Sometimes it is necessary to introduce conditional implicit definitions. For example, we cannot define the operation of division / with the implicit "definition"

$$x = x_1/x_2 \leftrightarrow x_2 \cdot x = x_1$$

because for $x_2 = 0$ every x satisfies $x_2 \cdot x = x_1$. But this is the only troublesome case and if we exclude it, by condition $x_2 \neq 0$, we'll have the correct conditional definition

$$x_2 \neq 0 \to (x = x_1/x_2 \leftrightarrow x_2 \cdot x = x_1).$$

Generally, we have the following:

Conditional implicit definitions of operations. An n-place term o is, under condition $C(x_1, \ldots, x_n)$, implicitly definable in a theory T if

$$C(x_1, \ldots, x_n) \to (x = o(x_1, \ldots, x_n) \leftrightarrow O(x)),$$

is provable in T, with the following restrictions:

1. $C(x_1, \ldots, x_n)$ and $O(x)$ are formulas in which o does not occur,

2. $O(x)$ has no free variables other than $x, x_1, \dots, x_n$ and x does not occur free in $C(x_1, \dots, x_n)$,
3. variables $x_1, \dots, x_n$ are distinct,
4. $C(x_1, \dots, x_n) \to (\exists! x) O(x)$ is provable in the theory.

In a special case, the theory T' does not contain the term o and T is an extension of T' by the definition $C(x_1, \dots, x_n) \to (x = o(x_1, \dots, x_n) \leftrightarrow O(x))$. If the condition $C(x_1, \dots, x_n)$ is satisfied, it is easy to prove that a conditional implicit definition of an operation satisfies Lesniewski's criteria. The proof is completely the same as in the previous theorem. But if $C(x_1, \dots, x_n)$ is not satisfied the meaning of $o(x_1, \dots, x_n)$ is not clear. For example, what is the meaning of $1/0$ in arithmetic of rational numbers? Does it designate anything? Is $1/0 = 2/0$ true, false or neither?

Mathematicians answer:

> *Term* $1/0$ *does not designate anything and* $1/0 = 2/0$ *is neither true nor false, it is meaningless.*

Logicians answer:

> *Every term designates and every sentence is either true or false. Furthermore, for every* n*-place operation* o *and every constants* $c_1, \dots, c_n$*, there is a designating term* $o(c_1, \dots, c_n)$ *and for every terms* t_1, t_2 *there is the meaningful (i.e., true or false) sentence* $t_1 = t_2$*. These are the presumptions of classical logic. Hence, in arithmetic of rational numbers,* $1/0$ *should designate something and* $1/0 = 2/0$ *should be true or false.*

(Mathematicians are usually not aware that they revise classical logic when they declare that there are terms which do not designate and sentences which are neither true nor false.)
After that logicians split.

(i) Sometimes they declare that e.g. $x/0 = 0$. This is designation by convention. In that case $1/0 = 2/0$ is true by convention. The advantage of this approach is that there are no formal changes in the logical theory.
(ii) Sometimes they introduce the new object ∞ and declare that $x/0 = \infty$. The disadvantage of this approach is that there should be formal changes in the theory. Now the theory has two types of objects: rational numbers and ∞. We should extend the theory with the predicate Q (being a rational number) and introduce the new axiom $-Q(\infty)$. We should also prefix any sentence about rational numbers $x, \dots$ with the prefix $Q(x, \dots) \to$. Alternatively, we introduce two types of variables.
(iii) Sometimes logicians declare that $1/0$ designates, but that it is not decidable what is designated. It implies that sentences containing $1/0$ are true or false but not decidable.
(iv) Sometimes, they reduce operations to relations, which is always possible, but very cumbersome.

We will explore the last approach a bit more.

Instead of constants (0-place predicates) 0, 1 we introduce 1-place predicates Z, O and instead of formulas $A(0)$, $A(1)$ we use formulas $Z(x) \to A(x)$, $O(x) \to A(x)$. Etc.

Instead of 2-place operations $+$, $\cdot$ we introduce 3-place predicates $S(x, y, z)$, $M(x, y, z)$ and instead of formulas $A(x + y)$, $A(x \cdot y)$ we use formulas $S(x, y, z) \to A(z)$, $M(x, y, z) \to A(z)$. Etc.

We can now define division in the following way

$$D(x, y, z) \leftrightarrow -Z(y) \wedge M(y, z, x).$$

Some wonder why there is no problem with this relational definition, but there is a problem with the analogous operational definition

$$x/y = z \leftrightarrow y \neq 0 \wedge y \cdot z = x.$$

The difference is that (operationally constructed) closed terms always designate, while (relationally constructed) closed formulas need not always be true. For example,

$$D(1, 0, z) \leftrightarrow -Z(0) \wedge M(0, z, 1)$$

implies that $D(1, 0, z)$ is false, for any z, and there is no problem with that.

On the other hand,

$$1/0 = z \leftrightarrow 0 \neq 0 \wedge 0 \cdot z = 1$$

also implies that $1/0 = z$ is false, for any z, and there is a problem with that. Namely, it means that $1/0$ does not designate. But every closed term should designate.

If you think this is not a big problem just think about the logical rule of existential introduction $A(t) \to (\exists z)A(z)$. It implies $t = t \to (\exists z)(z = t)$ which together with the logical truth $t = t$ implies $(\exists z)(z = t)$. Hence, if you accept that there exists a non-designating term t, e.g., $-(\exists z)(z = 1/0)$, then you should reject the logical rule of existential introduction.

5. Padoa's method

Let us turn to implicit definitions of relations which we have not introduced so far.

Implicit definitions of relations. An n-place predicate R is implicitly definable in a theory T if

$$D_R \wedge D_{R'} \to (\forall x_1) \cdots (\forall x_n) \left(R(x_1, \ldots, x_n) \Leftrightarrow R'(x_1, \ldots, x_n)\right)$$

is provable in theory T', with the following restrictions:

1. D_R is a formula in which $R(x_1, \ldots, x_n)$ occurs,
2. variables $x_1, \ldots, x_n$ are distinct and D_R has no free variables other than $x_1, \ldots, x_n$,
3. D'_R is the formula in which R' is substituted for every R in D_R
4. T' is the extension of T with R' which is not in T.

Beth's famous theorem claims that implicit definitions of relations are superfluous.

Theorem 5.1 (Beth). *A relation $R(x_1, \ldots, x_n)$ is implicitly defined in a theory if and only if an explicit definition of $R(x_1, \ldots, x_n)$ can be derived in the theory.*

We introduced definitions as formulae which establish the meaning of concepts by relating the *definienda* (concepts they define) to *definientia* (other concepts already available). They are true "*per definitionem*" i.e. they are postulated truths. What differentiates them from axioms (another kind of postulated truths) is that they are eliminable and non-creative.

So, we can say that *definitions are eliminable, non-creative axioms.*

It is possible that a primitive concept is implicitly definable by other primitive concepts i.e., that the axioms in which it occurs are eliminable and non-creative. If it is not, then we say that it is independent of the other concepts. It is not easy to prove that directly but it could be done indirectly by using Padoa's method.

Padoa's method. To prove that a given primitive concept is independent of the other primitive concepts, construct two interpretations of the axioms of the theory such that the given concept is extensionally different in the interpretations while the other concepts are extensionally the same. (It follows that the domain of the interpretations must be the same.)

For example, in the theory of rings, multiplication $\cdot$ is independent of other primitives $+$, 0 and 1. Padoa's two interpretations (which prove that) are the ordinary field of complex numbers $(\mathbb{C}, +, \cdot, 0, 1)$ and non-ordinary $(\mathbb{C}, +, *, 0, 1)$ in which $x*y = \bar{x}\cdot\bar{y}$, where $\bar{x}$ and $\bar{y}$ are conjugates of x and y. Padoa [**5**] has not proved the correctness of his method[2]). For him and many others it is evidently correct. Of course, we can use Beth's theorem to prove it for relations. If $R(x_1, \ldots, x_n)$ is defined in a theory then it is explicitly defined by

$$R(x_1, \ldots, x_n) \leftrightarrow D,$$

where D does not contain R, but only other primitive concepts. Because of the equivalence, if there were two interpretations coinciding on the other primitive concepts they must coincide on R.

6. Definitions in models vs. of models

In every interpretation of the theory, definitions uniquely determine extensions of defined concepts. On the other hand, axioms of the theory can have different interpretations and it is common to think about the axioms as definitions of these interpretations (usually called models).

So, we should clearly distinguish between the definitions *in* models and the definitions *of* models. In this paper we are mainly concerned with the in-type, while standard logic (especially model theory) is mainly concerned with the of-type.

For example, the first order Peano axions:

$(A0)$ $\quad x' \neq 0$
(A') $\quad x \neq y \to x' \neq y'$
$(A+0)$ $\quad x + 0 = x$

[2]A thorough discussion can be found in [**8**].

$(A+')$	$x + y' = (x + y)'$
$(A \cdot 0)$	$x \cdot 0 = 0$
$(A \cdot ')$	$x \cdot y' = (x \cdot y) + x$
$(SA\infty)$	$P(0) \wedge (\forall x)(P(x) \to P(x')) \to (\forall x)P(x)$

define the class *of* its models. It includes the standard model, but also all the nonstandard models.

On the other hand, it is sometimes thought that $(A + 0)$ and $(A+')$ define addition in every model of $(A0)$, (A') and $(SA\infty)$. This is not true, because from $(A0)$, (A') and $(SA\infty)$ it does not follow that there is a unique function $+$ satisfying $(A + 0)$ and $(A+')$. We may prove this by using Padoa's method. Namely, in nonstandard models of $(A0)$, (A') and $(SA\infty)$ it is possible to give two different interpretations of $+$, satisfying $(A+0)$ and $(A+')$: the interpretations must coincide on the standard part (which is the beginning of every nonstandard model) and on the nonstandard part, $x + y$ could be changed to $x' + y$ without violating $(A+')$.

In the same way we may prove that $(A{\cdot}0)$ and $(A \cdot ')$ do not define multiplication in every model of $(A0)$, (A'), $(A + 0)$, $(A+')$ and $(SA\infty)$.

Of course, $(A + 0)$ and $(A+')$ define addition in every model of $(A0)$, (A') and $(A\infty)$,

$$(A\infty) \qquad (\forall M)(0 \in M \wedge (\forall x)(x \in M \to x' \in M)) \to M = \mathbb{N},$$

within some usual set theory (e.g. ZFC). Namely, given a model $(\mathbb{N}, 0, ')$ of these three axioms (and of the usual set theoretical axioms) we can define

$$A_{\min} = \bigcap \{A : A \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \wedge (x, 0, x) \in A \wedge (x, y, z) \in A \to (x, y', z') \in A\}.$$

Using $(A0)$, (A') and $(A\infty)$ it is easy to prove that

$$(\forall x, y \in \mathbb{N})(\exists! z \in \mathbb{N})(x, y, z) \in A_{\min},$$

i.e. $A_{\min}$ is the unique function $x, y \to z$ that satisfies $(A + 0)$ and $(A+')$.

In the same way we may prove that $(A \cdot 0)$ and $(A \cdot ')$ do define multiplication in every model of $(A0)$, (A'), $(A + 0)$, $(A+')$ and $(A\infty)$, within a given set theory (e.g. ZFC).

So, the recursive definitions of addition (i.e. $(A+0)$ and $(A+')$) and multiplication (i.e., $(A \cdot 0)$ and $(A{\cdot}')$) are not definitions in the first-order arithmetic (the one with the first-order axiom schema of induction $(SA\infty)$), but they are definitions in a set theory (the one with the second-order axiom of induction $(A\infty)$ and the appropriate set theoretical axioms).

Concerning definitions *in* models vs. definitions *of* models, we may finally add that every definition *of* models could be understood as a definition *in* models, if we understand it as a definition in an axiomatic set theory (e.g., in ZFC), because models are objects of such theory. Anyway, this is the way we understand mathematical theories. For example, when we think of group theory, we are not thinking of the first-order theory with the axioms $(ab)c = a(bc)$, $ae = ea = a$, $aa^{-1} = a^{-1}a = e$, but of these axioms added to some set theory (it is only on the first few pages of a group theory textbook that we use only these axioms without any set theory).

7. Mathematical vs. everyday definitions

We should warn that formal mathematical definitions, we were discussing until now, are very different from everyday definitions. Mathematical definitions are stipulated, prescriptive and trivially true. Everyday definitions are extracted, descriptive and could be true or false.

Everyday definitions are extracted from instances of actual usage, mathematical definitions stipulate the usage. Everyday definitions describe what is commonly meant by the defined concept, mathematical definitions prescribe what is going to be meant by it. Everyday definitions are true if they truly report the usage they are extracted from and they are false if they do not; mathematical definitions are always, *per definitionem*, true.

It is a simple cognitive fact that (in the most common cases) we learn concepts by usage and not by definitions. We know what a chair is, although we do not know how to define it. If we ever try to define it, it is only after we are well acquainted with the concept by using it. Stipulated concepts cannot be accurately acquired in that way.

This is the main reason why definitions create problems for mathematics students. Their teachers assume that mathematical concepts are acquired by means of their definitions and that students will use the definitions to solve problems and prove theorems. But definitions of concepts will be ignored by many students.

Their everyday understanding of concepts is based on experiences of paradigmatic cases and their usage, not on the concepts definitions. For example, the concept of a chair evokes a picture of a typical chair, experiences of sitting on chairs, etc. Such a concept image is necessary and sufficient for the successful use of the concept.

Students tend to transfer this successful cognitive strategy to mathematical concepts. For example, the concept of a closed planar curve may evoke a picture of a typical high school curve (e.g., a circle, a parabola, etc.), experiences of solving problems with these typical examples, etc. Such a concept image is necessary to understand the concept (we may even say that having such a concept image is the main part of the understanding of the concept).

But, this concept image is not sufficient to prove Jordan theorem. To prove it you have to use a definition of the closed planar curve.

Hence, mathematical situations requires mathematical students to acquire cognitive strategies which are completely different from those that are successful in everyday situations. It is reasonable to expect that the everyday cognitive strategies will take over the "unnatural" mathematical strategies. Especially in the beginning of mathematical studies and even more if students are not alerted to the difference.

A common experience of many mathematics teachers is that "their students ignore theory and pay attention only to exercises". It is a clear manifestation of preferring the everyday strategy to the mathematical strategy. To promote the mathematical strategy we ought to give explicit instruction on this difference to our students.

8. Is there a difference

But, is it really true that mathematical definitions are stipulated, prescriptive and trivially true? A very short answer is: in the formal presentation of a mathematical theory they are, in the real history of the mathematical theory they are not.

Namely, stipulative and prescriptive character of mathematical definitions cannot account for the historical fact that mathematicians use concepts even centuries before they finally define them. For example, Fermat implicitly used our concept of the derivative, Newton and Leibniz discovered it, Euler developed it, Lagrange named it and at the end of this long period Cauchy and Weierstrass defined it; cf. [**2**]. And the final definition, among other things, was an attempt to describe the concept image developed by the earlier periods.

Another example is Lakatos' history [**3**], of the mathematician's search for the definition of polyhedra.

Similarly, truthful character of mathematical definitions cannot account for the historical fact that mathematicians often propose definitions which afterwards turn out to be false. For example, Jordan [**1**] in 1887. defined a continuous curve as a continuous image of the unit interval (which is, by the way, a description of the orbit of a moving point). But when Peano [**6**] in 1890. proved that there is a Jordan's continuous curve that goes through every point of a square, Jordan's definition was discarded as false. The reason is that the 1-dimensionality of curves is the crucial part of the concept image of curves which should be respected by its definition.

On the other hand, when mathematicians finally decide what the definition of a concept should be, it is never completely true to the concept image produced by rich history of the concept. As Nietzsche [**4**] pointed out in 1887., definable is only what has no history (*definierbar ist nur Das, was keine Geschichte hat*).

For example, mathematicians define continuity of a function at a point in the usual ε, δ—way. Continuity at a set of points is then defined as continuity at all points of the set. According to the definition, a constant function defined on rational numbers is continuous on its domain. Of course, in our concept image of continuity this function is totally discontinuous, because the set of rational numbers on which it is defined is totally discontinuous. But our stipulated definition of function continuity does not include continuity of the domain of the function as a precondition for the function continuity (although it is a part of our concept image of continuity).

A more common example is $f(x) = 1/x$ which is continuous on its domain of definition, although "we see" the clear discontinuity at 0.

That conflict between concept definitions and concepts images is another obstacle for math students and they need to be warned about this, so they can overcome it.

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