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ON THE SEIDEL INTEGRAL GRAPHS WHICH BELONG TO THE CLASS $\alpha K_{a,a} \cup \beta K_{b,b}$

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ABSTRACT. We say that a simple graph G is Seidel integral if its Seidel spectrum consists entirely of integers. If $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral, we show that it belongs the class of Seidel integral graphs

$$\Big[\frac{kt}{\tau}x_0 + \frac{mt}{\tau}z\Big]K_{a,a} \cup \Big[\frac{kt}{\tau}y_0 + \frac{a}{\tau}z\Big]nK_{b,b},$$

where (i) $a = (t+\ell n)k+\ell m$ and $b = \ell m$; (ii) $t, k, \ell, m, n \in \mathbb{N}$ such that $(m, n) = 1$, $(n, t) = 1$ and $(\ell, t) = 1$; (iii) $\tau = (a, mt)$ such that $\tau \mid kt$; (iv) (x_0, y_0) is a particular solution of the linear Diophantine equation $ax - (mt)y = \tau$ and (v) $z \geqslant z_0$ where z_0 is the least integer such that $\left(\frac{kt}{\tau}x_0 + \frac{mt}{\tau}z_0\right) \geqslant 1$ and $\left(\frac{kt}{\tau}y_0 + \frac{a}{\tau}z_0\right) \geqslant 1$. In particular, we demonstrate that $\overline{\alpha K_a \cup \beta K_b}$ is integral in respect to its ordinary adjacency matrix if and only if $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral.

1. Introduction

Let G be a simple graph of order n and let $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of its (0,1) adjacency matrix of G. The spectrum of G is the set of its eigenvalues and is denoted by $\sigma(G)$. A graph G is said to be integral if its spectrum $\sigma(G)$ consists only of integers [**1**]. We say that $A^* = [s_{ij}]$ is the Seidel adjacency matrix of the graph G if $s_{ij} = -1$ for any two adjacent vertices i and j, $s_{ij} = 1$ for any two non-adjacent vertices i and j, and $s_{ij} = 0$ if $i = j$. The Seidel spectrum of G is the set of eigenvalues $\lambda_1^* \geqslant \lambda_2^* \geqslant \cdots \geqslant \lambda_n^*$ of its $(0, -1, 1)$ adjacency matrix $A^* = A^*(G)$ and is denoted by $\sigma^*(G)$. A graph G is said to be Seidel integral if its Seidel spectrum $\sigma^*(G)$ consists only of integers. We say that an eigenvalue μ is main if and only if $\langle \mathbf{j}, \mathbf{P}\mathbf{j}\rangle = n\cos^2\alpha > 0$, where $\mathbf{j}$ is the main vector (with coordinates equal to 1) and $\mathbf{P}$ is the orthogonal projection of the space $\mathbb{R}^n$ onto the eigenspace $\mathcal{E}_A(\mu)$. The quantity $\beta = |\cos\alpha|$ is called the main angle of μ. Similarly, we say that a Seidel eigenvalue μ^* is the Seidel main eigenvalue if and only if $\langle \mathbf{j}, \mathbf{P}^*\mathbf{j}\rangle = n\cos^2\alpha^* > 0$, where $\mathbf{P}^*$ is the orthogonal projection of the space $\mathbb{R}^n$

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onto the eigenspace $\mathcal{E}_{A^*}(\mu^*)$. The quantity $\beta^* = |\cos\alpha^*|$ is called the Seidel main angle of μ^*. In [**1**] was proved that the graph G and its complement $\overline{G}$ have the same number of main eigenvalues. We also know that $|\mathcal{M}(G)| = |\mathcal{M}^*(G)|$, where $\mathcal{M}(G)$ and $\mathcal{M}^*(G)$ denote the sets of all main and the Seidel main eigenvalues of G, respectively.

Let G be a graph of order n with exactly two main eigenvalues μ_1 and μ_2 and let $n_1 = n\beta_1^2$ and $n_2 = n\beta_2^2$.

THEOREM 1.1 (Lepović [**3**]). *Let G be a graph of order n with two main eigenvalues μ_1 and μ_2. Then*

$$\mu^*_{1,2} = \frac{n-2-2\mu_1-2\mu_2}{2} \pm \frac{\sqrt{(2\mu_1-2\mu_2+n)^2-8n_1(\mu_1-\mu_2)}}{2}. \tag{1.1}$$

Besides, we have

$$n^*_{1,2} = \frac{n}{2} \pm \frac{n^2+2(n-2n_1)(\mu_1-\mu_2)}{2\sqrt{(2\mu_1-2\mu_2+n)^2-8n_1(\mu_1-\mu_2)}}, \tag{1.2}$$

where $n_1^* = n(\beta_1^*)^2$ and $n_2^* = n(\beta_2^*)^2$.

Further, let K_n and $K_{m,n}$ denote the complete graph and the complete bipartite graph, respectively. We note that $\alpha K_{a,a} \cup \beta K_{b,b}$ is an integral graph with two main eigenvalues $\mu_1 = a$ and $\mu_2 = b$, for any $\alpha, \beta, a, b \in \mathbb{N}$ with $a > b$, where mG denotes the m-fold union of the graph G. As is pointed out [**3**], if G is an integral graph then G is Seidel integral if and only if the Seidel main spectrum of G contains integral values. Consequently, $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral if and only if its largest Seidel main eigenvalue $\mu_1^* \in \mathbb{N}$.

Next, we have established in [**4**] a characterization of integral graphs which belong to the class $\overline{\alpha K_a \cup \beta K_b}$, while in [**6**] we have established a characterization of Seidel integral graphs which belong to the class $\alpha K_a \cup \beta K_b$. Besides, we have established in [**5**] a characterization of integral graphs which belong to the class $\overline{\alpha K_{a,a} \cup \beta K_{b,b}}$. We now proceed to establish a characterization of Seidel integral graphs which belong to the class $\alpha K_{a,a} \cup \beta K_{b,b}$, as follows.

2. Main results

First, note that $o = 2\alpha a + 2\beta b$ is the order of $\alpha K_{a,a} \cup \beta K_{b,b}$. Then according to (1.1) we get implicitly

$$\mu_1^* = \alpha a + \beta b - 1 - (a+b) + \delta \quad \text{and} \quad \mu_2^* = \alpha a + \beta b - 1 - (a+b) - \delta, \tag{2.1}$$

where $\delta = \sqrt{\big((\alpha+1)a + (\beta-1)b\big)^2 - 4\alpha a(a-b)}$. Then $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral if and only if $(\alpha, \beta, a, b, \delta)$ represents a positive integral solution of the Diophantine equation

$$\big((\alpha+1)a + (\beta-1)b\big)^2 - 4\alpha a(a-b) = \delta^2. \tag{2.2}$$

Therefore, the characterization of Seidel integral graphs which are related to the class $\alpha K_{a,a} \cup \beta K_{b,b}$ is reduced to the problem of finding the most general positive solution of the equation (2.2).

Next, $\mu_1^*\mu_2^* = 4\mu_1\mu_2 - 2(n_1-1)\mu_2 - 2(n_2-1)\mu_1 - (n-1)$ for any G with two main eigenvalues (see [**3**]). In the case that $G = \alpha K_{a,a} \cup \beta K_{b,b}$ this relation is transformed into

$$(\mu_1^* + 1)(\mu_2^* + 1) = 4ab(1 - \alpha - \beta). \tag{2.3}$$

PROPOSITION 2.1. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is a Seidel integral graph then μ_1^* and μ_2^* are two odd numbers.*

PROOF. Using (2.1) we obtain $2\delta = \mu_1^* - \mu_2^*$, which provides that μ_1^* and μ_2^* are even or μ_1^* and μ_2^* are odd. If we assume that μ_1^* and μ_2^* are two even numbers then $(\mu_1^*+1)(\mu_1^*+1)$ is an odd number, a contradiction (see (2.3)). □

In the sequel (m, n) denotes the highest common divisor of integers $m, n \in \mathbb{N}$ while $m \mid n$ means that m divides n. With this notation, in order to demonstrate a method applied in this paper, we prove first the following result.

THEOREM 2.1. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 4ab - 1$ then it belongs to the class of Seidel integral graphs*

$$t(2m-1)\, K_{a,a} \cup \big((2s+1) - t\big)(2n-1)\, K_{b,b}, \tag{2.4}$$

where $a = (2s+1)n - (s+1)$ and $b = (2s+1)m - (s+1)$, $m, n \in \mathbb{N}$ and $n > m$, $t < 2s+1$ such that $(2s+1, t) = 1$.

PROOF. Let us assume that $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 4ab-1$. Using (2.3) we obtain $\mu_2^* = -(\alpha+\beta)$ and $2\delta = 4ab + \alpha + \beta - 1$. Then Diophantine equation (2.2) is reduced to

$$(2b+1)\big(2a - (\alpha+\beta-1)\big) = 2\alpha(a-b).$$

Let $2b+1 = r\alpha$ where $r = \frac{s}{t}$ such that $(s,t) = 1$. Then from the last relation we obtain $2(a-b) = r\big(2a - (\alpha+\beta-1)\big)$. In view of this, we get

$$\alpha = \frac{t}{s}\,(2b+1) \quad \text{and} \quad \beta = \frac{s-t}{s}\,(2a+1).$$

Since $(s,t) = 1$ it follows that $(s-t, s) = 1$. Then it must be $s \mid (2b+1)$ and $s \mid (2a+1)$, which provides that s is an odd number. Replacing s with $2s+1$ we find that $2b+1 = (2s+1)(2m-1)$ and $2a+1 = (2s+1)(2n-1)$. So we get $\alpha = t(2m-1)$, $\beta = ((2s+1)-t)(2n-1)$, $a = (2s+1)n - (s+1)$ and $b = (2s+1)m - (s+1)$, where $t < 2s+1$. □

REMARK 2.1. With the condition $a > b$ note that the parameters α, β, a, b determine the graph $\alpha K_{a,a} \cup \beta K_{b,b}$ up to isomorphism.

In what follows, we show that there exists an one-to-one correspondence between the Seidel integral graphs $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = 4ab-1$ and the parameters m, n, s, t.

PROPOSITION 2.2. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is a Seidel integral graph with $\mu_1^* = 4ab-1$ then it uniquely determines the parameters m, n, s, t.*

Proof. Let us assume that m_1, n_1, s_1, t_1 and m_2, n_2, s_2, t_2 determine the same Seidel integral graph $\alpha K_{a,a} \cup \beta K_{b,b}$ with the largest Seidel main eigenvalue $\mu_1^* = 4ab - 1$. Then according to Remark 2.1 and relation (2.4) we have: (i) $t_1(2m_1 - 1) = t_2(2m_2 - 1)$; *(ii)* $((2s_1+1) - t_1)(2n_1 - 1) = ((2s_2+1) - t_2)(2n_2 - 1)$; *(iii)* $(2s_1+1)(2n_1-1) = (2s_2+1)(2n_2-1)$ and *(iv)* $(2s_1+1)(2m_1-1) = (2s_2+1)(2m_2-1)$. Using (i) and *(iv)* we get $\frac{2s_1+1}{t_1} = \frac{2s_2+1}{t_2}$. Since $(2s+1, t) = 1$ it follows that $s_1 = s_2$ and $t_1 = t_2$. Consequently, using (i) and *(ii)* we obtain $m_1 = m_2$ and $n_1 = n_2$. $\square$

Further, using a procedure similar to the proof of Theorem 2.1, we proceed to establish a characterization of Seidel integral graphs for the class $\alpha K_{a,a} \cup \beta K_{b,b}$. The proof is based on the following statement [**2**].

Theorem 2.2. *The linear Diophantine equation $ax + by = c$ has at least one solution if and only if $d \mid c$ where $d = (a, b)$. In that case the most general solution of this equation is given in the form*

$$x = \frac{c}{d}\,x_0 - \frac{b}{d}\,z \quad \textit{and} \quad y = \frac{c}{d}\,y_0 + \frac{a}{d}\,z \qquad (z \in \mathbb{Z}),$$

where (x_0, y_0) represents a particular solution[1] *of the equation $ax + by = d$.*

Theorem 2.3. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral then it belongs to the class of Seidel integral graphs*

$$\Big[\frac{kt}{\tau}\,x_0 + \frac{mt}{\tau}\,z\Big]K_{a,a} \cup \Big[\frac{kt}{\tau}\,y_0 + \frac{a}{\tau}\,z\Big]nK_{b,b}, \tag{2.5}$$

where (i) $a = (t + \ell n)k + \ell m$ and $b = \ell m$; (ii) $t, k, \ell, m, n \in \mathbb{N}$ such that $(m, n) = 1$, $(n, t) = 1$ and $(\ell, t) = 1$; (iii) $\tau = (a, mt)$ such that $\tau \mid kt$; (iv) (x_0, y_0) is a particular solution of the linear Diophantine equation $ax - (mt)y = \tau$ and (v) $z \geqslant z_0$ where z_0 is the least integer such that $\big(\frac{kt}{\tau}\,x_0 + \frac{mt}{\tau}\,z_0\big) \geqslant 1$ and $\big(\frac{kt}{\tau}\,y_0 + \frac{a}{\tau}\,z_0\big) \geqslant 1$.

Proof. Let us assume that $\mu_1^* \in \mathbb{N}$ and let $\theta = \frac{\rho}{\varphi}$ so that $\mu_1^* + 1 = 2\theta a$ and $(\rho, \varphi) = 1$. Using Proposition 2.1 note that θa is an integer. Using (2.1) and (2.3) we obtain

$$\mu_2^* = -\frac{2b(\alpha+\beta-1)}{\theta} - 1 \quad \text{and} \quad \delta = \theta a + \frac{b(\alpha+\beta-1)}{\theta}.$$

Then by a straightforward calculation it is not difficult to see that (2.2) may be transformed in the form $\frac{\theta+1}{\theta} = \frac{\alpha(a-b)}{\theta a - b(\alpha+\beta-1)}$. Let c be a constant such that $(\overline{1})$ $\alpha(a-b) = c(\theta+1)$ and $(\overline{2})$ $\theta a - b(\alpha+\beta-1) = c\theta$. Combining $(\overline{1})$ and $(\overline{2})$ we find that $c = (\alpha-\theta)a + (\beta-1)b$. Observe that c is an integer because $\theta a = \frac{\mu_1^*+1}{2} \in \mathbb{N}$. Consequently, using $(\overline{1})$ or $(\overline{2})$ we arrive at $\alpha(a-b) = \big((\alpha-\theta)a + (\beta-1)b\big)(\theta+1)$. Hence,

$$(a-b) = r\big((\alpha-\theta)a + (\beta-1)b\big) \quad \text{and} \quad (\theta+1) = r\alpha, \tag{2.6}$$

where $r = \frac{s}{t}$ such that $(s, t) = 1$. Making use of (2.6), by an easy calculation we obtain $(\overline{3})$ $r\beta b = (r-1)\big(r\alpha a - (a-b)\big)$.

[1]A particular solution of the equation $ax + by = d$ may be obtained by using the EUCLID algorithm. In that case the coefficients a and b uniquely determine x_0 and y_0.

Using now the right-hand side of relation (2.6), note that $r\alpha a = \frac{\mu_1^*+1}{2} + a$, which shows that $(r\alpha a)$ is integral and $r-1 = \frac{s-t}{t} > 0$. Since $\beta b = (1-\frac{1}{r})\big(r\alpha a - (a-b)\big)$ (see $\overline{3}$)) it turns out that $r \mid (a-b)$. Let $(\overline{4})$ $(a-b) = \gamma r$ and let $(\overline{5})$ $\gamma = kt$. Then $(\overline{3})$ is reduced to the form

$$\beta = \frac{(s-t)}{b}\,\frac{(\alpha a - kt)}{t}. \tag{2.7}$$

Further, let $(s-t, b) = \ell$ and let $m, n \in \mathbb{N}$ such that $(\overline{6})$ $(s-t) = \ell n$ and $(\overline{7})$ $b = \ell m$, where $(m, n) = 1$. Since $(s-t, t) = 1$ according to $(\overline{6})$ we obtain $(n, t) = 1$ and $(\ell, t) = 1$. Consequently, using (2.7) we have $\beta = \frac{(\alpha a - kt)\, n}{mt}$. Since $(n, mt) = 1$ it follows that $(mt) \mid (\alpha a - kt)$. Therefore, setting $(\overline{8})$ $\alpha a - kt = \eta(mt)$ we get $(\overline{9})$ $\beta = \eta n$. We note that $(\overline{8})$ represents a linear Diophantine equation in variables α and η. Of course, if $(a, mt) = \tau$ then $(\overline{8})$ has at least one solution if and only if $\tau \mid kt$. In that case, according to Theorem 2.2 we obtain that

$$\alpha = \frac{kt}{\tau}\, x_0 + \frac{mt}{\tau}\, z \quad \text{and} \quad \eta = \frac{kt}{\tau}\, y_0 + \frac{a}{\tau}\, z,$$

where $ax_0 - (mt)y_0 = \tau$. Finally, from $(\overline{4})$ through $(\overline{7})$, and according to $(\overline{9})$ and the last relation, we get easily that $a = (t+\ell n)k + \ell m$ and $\beta = \big[\frac{kt}{\tau}\, y_0 + \frac{a}{\tau}\, z\big] n$. □

Proposition 2.3. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is a Seidel integral graph then it uniquely determines the parameters τ, t, k, ℓ, m, n.*

Proof. Let assume that $\tau_1, t_1, k_1, \ell_1, m_1, n_1$ and $\tau_2, t_2, k_2, \ell_2, m_2, n_2$ determine the same Seidel integral graph $\alpha K_{a,a} \cup \beta K_{b,b}$. Since the parameters α, β, a, b determine the graph $\alpha K_a \cup \beta K_b$ up to isomorphism, using the second equality of (2.6) we have $r\alpha a = \frac{\mu_1^*+1}{2} + a$, which shows that $s_1 = s_2$ and $t_1 = t_2$ because $(s, t) = 1$. Next, using $(\overline{4})$ and $(\overline{5})$ we get $k_1 = k_2$. Since $(s-t, b) = \ell$ we also have $\ell_1 = \ell_2$. Since $b = \ell m$ and $s - t = \ell n$, we find that $m_1 = m_2$ and $n_1 = n_2$. Finally, since $(a, mt) = \tau$ it follows that $\tau_1 = \tau_2$. □

Remark 2.2. If (x_0, y_0) is obtained by using the EUCLID algorithm then a fixed Seidel integral graph $\alpha K_{a,a} \cup \beta K_{b,b}$ also uniquely determines the parameters x_0, y_0, z_0, z.

Remark 2.3. We have proved in [**4**] that the characterization of integral graphs which are related to the class $\overline{\alpha K_a \cup \beta K_b}$ (in respect to its ordinary adjacency matrix) is reduced to the problem of finding the most general positive solution of the equation (2.2). More precisely, we have proved the following result.

Theorem 2.4 (Lepović [**4**]). *If $\overline{\alpha K_a \cup \beta K_b}$ is integral then it belongs to the class of integral graphs*

$$\overline{\Big[\frac{kt}{\tau}\, x_0 + \frac{mt}{\tau}\, z\Big] K_a \cup \Big[\frac{kt}{\tau}\, y_0 + \frac{a}{\tau}\, z\Big] n K_b},$$

where (i) $a = (t+\ell n)k + \ell m$ and $b = \ell m$; (ii) $t, k, \ell, m, n \in \mathbb{N}$ such that $(m, n) = 1$, $(n, t) = 1$ and $(\ell, t) = 1$; (iii) $\tau = (a, mt)$ such that $\tau \mid kt$; (iv) (x_0, y_0) is a *particular solution of the linear Diophantine equation* $ax-(mt)y=\tau$ *and* (v) $z \geqslant z_0$ *where* z_0 *is the least integer such that* $\big(\frac{kt}{\tau}\,x_0+\frac{mt}{\tau}\,z_0\big)\geqslant 1$ *and* $\big(\frac{kt}{\tau}\,y_0+\frac{a}{\tau}\,z_0\big)\geqslant 1$.

Using Theorems 2.3 and 2.4, we obtain an unexpected result that gives a connection between the Seidel integral graphs of the class $\alpha K_{a,a}\cup\beta K_{b,b}$ and the integral graphs of the class $\overline{\alpha K_a\cup\beta K_b}$, as follows.

THEOREM 2.5. *We have that* $\alpha K_{a,a}\cup\beta K_{b,b}$ *is Seidel integral if and only if* $\overline{\alpha K_a\cup\beta K_b}$ *is integral in respect to its ordinary adjacency matrix.*

PROPOSITION 2.4. *If* $\alpha K_{a,a}\cup\beta K_{b,b}$ *is Seidel integral with main eigenvalues* μ_1^* *and* μ_2^*, *then* $\overline{\alpha K_a\cup\beta K_b}$ *is integral with main eigenvalues* $\overline{\mu}_1=\frac{\mu_1^*+1}{2}$ *and* $\overline{\mu}_2=\frac{\mu_2^*+1}{2}$.

PROOF. Let us assume that $\alpha K_{a,a}\cup\beta K_{b,b}$ is Seidel integral for some α,β,a,b,δ. Then according to [**4**], we have

$$\overline{\mu}_1=\frac{\alpha a+\beta b-(a+b)+\delta}{2}\quad\text{and}\quad\overline{\mu}_2=\frac{\alpha a+\beta b-(a+b)-\delta}{2},$$

from which we obtain the statement using (2.1). □

THEOREM 2.6. *If* $\alpha K_{a,a}\cup\beta K_{b,b}$ *is Seidel integral with* $\mu_1^*=3ab-1$, *then it belongs to one of the following classes of Seidel integral graphs*

$$(2t-1)m\,K_{a,a}\cup\big(2s-(2t-1)\big)n\,K_{b,b}, \tag{2.8}$$

where (i) $a=2y_0+(4s)z^+$ *and* $b=2y_0+(4s)z^-$; (ii) $m=2x_0+3z^-$ *and* $n=2x_0+3z^+$ (iii) $s,t\in\mathbb{N}$ *such that* $(2s,2t-1)=1$, $(s,3)=1$ *and* $s\geqslant t$; (iv) (x_0,y_0) *is a particular solution of the linear Diophantine equation* $(4s)x-3y=1$ *and* (v) $z^+>z^-\geqslant z_0$ *where* z_0 *is the least integer such that* $\big(2x_0+3z_0\big)\geqslant 1$ *and* $\big(2y_0+(4s)z_0\big)\geqslant 1$;

$$tm\,K_{a,a}\cup\big((2s+1)-2t\big)n\,K_{b,b}, \tag{2.9}$$

where (i) $a=2y_0^++2(2s+1)z^+$, $b=2y_0^-+(2s+1)z^-$ *and* $a>b$; (ii) $m=2x_0^-+3z^-$ *and* $n=2x_0^++3z^+$ (iii) $s,t\in\mathbb{N}$ *such that* $(2s+1,2t)=1$, $(2s+1,3)=1$ *and* $s\geqslant t$; (iv) (x_0^+,y_0^+) *is a particular solution of the linear Diophantine equation* $2(2s+1)x-3y=1$; (v) $z^+\geqslant z_0^+$ *where* z_0^+ *is the least integer such that* $\big(2x_0^++3z_0^+\big)\geqslant 1$ *and* $\big(2y_0^++2(2s+1)z_0^+\big)\geqslant 1$; (vi) (x_0^-,y_0^-) *is a particular solution of the linear Diophantine equation* $(2s+1)x-3y=1$ *and* (vii) $z^-\geqslant z_0^-$ *where* z_0^- *is the least integer such that* $\big(2x_0^-+3z_0^-\big)\geqslant 1$ *and* $\big(2y_0^-+(2s+1)z_0^-\big)\geqslant 1$;

$$(2t-1)m\,K_{a,a}\cup(s-t+1)n\,K_{b,b}, \tag{2.10}$$

where (i) $a=2y_0^++(2s+1)z^+$, $b=2y_0^-+2(2s+1)z^-$ *and* $a>b$; (ii) $m=2x_0^-+3z^-$ *and* $n=2x_0^++3z^+$ (iii) $s,t\in\mathbb{N}$ *such that* $(2s+1,2t-1)=1$, $(2s+1,3)=1$ *and* $s\geqslant t$; (iv) (x_0^+,y_0^+) *is a particular solution of the linear Diophantine equation* $(2s+1)x-3y=1$; (v) $z^+\geqslant z_0^+$ *where* z_0^+ *is the least integer such that* $\big(2x_0^++3z_0^+\big)\geqslant 1$ *and* $\big(2y_0^++(2s+1)z_0^+\big)\geqslant 1$; (vi) (x_0^-,y_0^-) *is a particular solution of the linear Diophantine equation* $2(2s+1)x-3y=1$ *and* (vii) $z^-\geqslant z_0^-$ *where* z_0^- *is the least integer such that* $\big(2x_0^-+3z_0^-\big)\geqslant 1$ *and* $\big(2y_0^-+2(2s+1)z_0^-\big)\geqslant 1$.

PROOF. Let us assume that $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 3ab - 1$. Using (2.3) we obtain $\mu_2^* = -\frac{4}{3}(\alpha+\beta) + \frac{1}{3}$ and $2\delta = 3ab + \frac{4}{3}(\alpha+\beta-1)$. Then Diophantine equation (2.2) is reduced to

$$(3b+2)\Big(2a - \frac{4}{3}(\alpha+\beta-1)\Big) = 4\alpha(a-b).$$

Let $3b+2 = 2r\alpha$ where $r = \frac{s}{t}$ such that $(s,t) = 1$. Then from the last relation we obtain $2(a-b) = r\big(2a - \frac{4}{3}(\alpha+\beta-1)\big)$. In view of this, we get

$$\alpha = \frac{t}{2s}\,(3b+2) \quad \text{and} \quad \beta = \frac{s-t}{2s}\,(3a+2). \tag{2.11}$$

CASE 1. (s is even and t is odd). Let $s \to 2s$ and $t \to 2t-1$ where $p \to q$ means that 'p is replaced with q'. Then

$$\alpha = \frac{2t-1}{4s}\,(3b+2) \quad \text{and} \quad \beta = \frac{2s-(2t-1)}{4s}\,(3a+2).$$

Since $(4s, 2t-1) = 1$ it follows that $4s \mid (3b+2)$. Setting $3b+2 = (4s)m$ we obtain (1.1) $(4s)m - 3b = 2$. We note that (1.1) represents a linear Diophantine equation in variables m and b. Of course, this equation has at least one solution if and only if $(s,3) = 1$. In that case, according to Theorem 2.2 we obtain that $m = 2x_0 + 3z^-$ and $b = 2y_0 + (4s)z^-$, where $(4s)x_0 - 3y_0 = 1$.

Next, since $\big(4s, 2s-(2t-1)\big) = 1$ it follows that $4s \mid (3a+2)$. Setting $3a+2 = (4s)n$ we obtain (1.2) $(4s)n - 3a = 2$. We note that (1.2) represents a linear Diophantine equation in variables n and a. Of course, this equation has at least one solution if and only if $(s,3) = 1$. In that case, according to Theorem 2.2 we obtain that $n = 2x_0 + 3z^+$ and $a = 2y_0 + (4s)z^+$, where $(4s)x_0 - 3y_0 = 1$. So we arrive at the corresponding class of Seidel integral graphs displayed in (2.8).

CASE 2. (s is odd and t is even). Let $s \to 2s+1$ and $t \to 2t$. In this case relation (2.11) is transformed into

$$\alpha = \frac{t}{2s+1}\,(3b+2) \quad \text{and} \quad \beta = \frac{(2s+1)-2t}{2(2s+1)}\,(3a+2).$$

Since $(2s+1, t) = 1$ it follows that $(2s+1) \mid (3b+2)$. Setting $3b+2 = (2s+1)m$ we obtain (2.1) $(2s+1)m - 3b = 2$. We note that (2.1) represents a linear Diophantine equation in variables m and b. Of course, this equation has at least one solution if and only if $(2s+1, 3) = 1$. In that case, according to Theorem 2.2 we obtain that $m = 2x_0^- + 3z^-$ and $b = 2y_0^- + (2s+1)z^-$, where $(2s+1)x_0^- - 3y_0^- = 1$.

Next, since $\big(2(2s+1), (2s+1) - 2t\big) = 1$ it follows that $2(2s+1) \mid (3a+2)$. Setting $3a+2 = 2(2s+1)n$ we obtain (2.2) $2(2s+1)n - 3a = 2$. We note that (2.2) represents a linear Diophantine equation in variables n and a. Of course, this equation has at least one solution if and only if $(2s+1, 3) = 1$. In that case, according to Theorem 2.2 we obtain that $n = 2x_0^+ + 3z^+$ and $a = 2y_0^+ + 2(2s+1)z^+$, where $2(2s+1)x_0^+ - 3y_0^+ = 1$. So we arrive at the corresponding class of Seidel integral graphs displayed in (2.9).

CASE 3. (s is odd and t is odd). Let $s \to 2s+1$ and $t \to 2t-1$. In this case relation (2.11) is transformed into

$$\alpha = \frac{2t-1}{2(2s+1)}\,(3b+2) \quad \text{and} \quad \beta = \frac{s-t+1}{2s+1}\,(3a+2).$$

Since $\big(2(2s+1), 2t-1\big) = 1$ it follows that $2(2s+1) \mid (3b+2)$. Setting $3b+2 = 2(2s+1)m$ we obtain (3.1) $2(2s+1)m - 3b = 2$. We note that (3.1) represents a linear Diophantine equation in variables m and b. Of course, this equation has at least one solution if and only if $(2s+1, 3) = 1$. In that case, according to Theorem 2.2 we obtain that $m = 2x_0^- + 3z^-$ and $b = 2y_0^- + 2(2s+1)z^-$, where $2(2s+1)x_0^- - 3y_0^- = 1$.

Next, since $\big(2s+1, 2s+1-(2t-1)\big) = 1$ and $2s+1-(2t-1) = 2(s-t+1)$ it follows that $(2s+1, s-t+1) = 1$. In view of this fact, we find that $(2s+1) \mid (3a+2)$. Setting $3a+2 = (2s+1)n$ we obtain (3.2) $(2s+1)n - 3a = 2$. We note that (3.2) represents a linear Diophantine equation in variables n and a. Of course, this equation has at least one solution if and only if $(2s+1, 3) = 1$. In that case, according to Theorem 2.2 we obtain that $n = 2x_0^+ + 3z^+$ and $a = 2y_0^+ + (2s+1)z^+$, where $(2s+1)x_0^+ - 3y_0^+ = 1$. So we arrive at the corresponding class of Seidel integral graphs displayed in (2.10). □

PROPOSITION 2.5. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is a Seidel integral graph with $\mu_1^* = 3ab-1$ then it uniquely determines the parameters m, n, s, t.*

PROOF. Let us assume that m_1, n_1, s_1, t_1 and m_2, n_2, s_2, t_2 determine the same Seidel integral graph $\alpha K_{a,a} \cup \beta K_{b,b}$ with the largest Seidel main eigenvalue $\mu_1^* = 3ab-1$. Since the parameters α, β, a, b determine the graph $\alpha K_{a,a} \cup \beta K_{b,b}$ up to isomorphism, using the first equality of (2.11) we have $2r\alpha = (3b+2)$, which shows that $s_1 = s_2$ and $t_1 = t_2$ because $(s,t) = 1$. In view of this, we note that the classes represented by relations (2.8), (2.9), (2.10) are mutually disjoint. Consequently, without loss of generality, we can assume that the corresponding Seidel integral graph determined by the parameters m_1, n_1, s_1, t_1 and m_2, n_2, s_2, t_2 belong to the class of Seidel integral graphs displayed in relation (2.8). Hence, using (2.8) we have $(2t_1-1)m_1 = (2t_2-1)m_2$ and $(2s_1-(2t_1-1))n_1 = (2s_2-(2t_2-1))n_2$, which provides that $m_1 = m_2$ and $n_1 = n_2$. □

THEOREM 2.7. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 2ab-1$ then it belongs to the class of Seidel integral graphs*

$$tm\, K_{a,a} \cup (s-t)n\, K_{b,b}, \tag{2.12}$$

where $a = sn-1$ and $b = sm-1$, $m, n \in \mathbb{N}$ and $n > m$, $s > t$ such that $(s,t) = 1$.

PROOF. Let us assume that $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 2ab-1$. Using (2.3) we obtain $\mu_2^* = -2(\alpha+\beta)+1$ and $2\delta = 2ab + 2(\alpha+\beta-1)$. Then Diophantine equation (2.2) is reduced to

$$(b+1)\big(a-(\alpha+\beta-1)\big) = \alpha(a-b).$$

Let $b+1 = r\alpha$ where $r = \frac{s}{t}$ such that $(s,t) = 1$. Then from the last relation we obtain $a - b = r\big(a - (\alpha+\beta-1)\big)$. In view of this, we get

$$\alpha = \frac{t}{s}\,(b+1) \quad \text{and} \quad \beta = \frac{s-t}{s}\,(a+1).$$

Since $(s,t) = 1$ and $(s-t, s) = 1$ it follows that $s \mid (b+1)$ and $s \mid (a+1)$. Setting $(b+1) = sm$ and $(a+1) = sn$, we find that $\alpha = tm$ and $\beta = (s-t)n$. □

Theorem 2.8. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = ab - 1$ then it belongs to one of the following classes of Seidel integral graphs*

$$(2t-1)m\,K_{a,a} \cup \big(2s-(2t-1)\big)n\,K_{b,b}, \tag{2.13}$$

where $a = 4sn - 2$ and $b = 4sm - 2$, $m, n \in \mathbb{N}$ and $n > m$, $s \geqslant t$ such that $(2s, 2t-1) = 1$;

$$tm\,K_{a,a} \cup \big((2s+1) - 2t\big)n\,K_{b,b}, \tag{2.14}$$

where $a = 2(2s+1)n - 2$ and $b = (2s+1)m - 2$, $m, n \in \mathbb{N}$ and $a > b$, $s \geqslant t$ such that $(2s+1, 2t) = 1$;

$$(2t-1)m\,K_{a,a} \cup (s-t+1\,K_{b,b}, \tag{2.15}$$

where $a = (2s+1)n - 2$ and $b = 2(2s+1)m - 2$, $m, n \in \mathbb{N}$ and $a > b$, $s \geqslant t$ such that $(2s+1, 2t-1) = 1$.

Proof. Let us assume that $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = ab - 1$. Using (2.3) we obtain $\mu_2^* = -4(\alpha+\beta) + 3$ and $2\delta = ab + 4(\alpha+\beta-1)$. Then Diophantine equation (2.2) is reduced to

$$(b+2)\big(a - 2(\alpha+\beta-1)\big) = 2\alpha(a-b).$$

Let $b+2 = 2r\alpha$ where $r = \frac{s}{t}$ such that $(s,t) = 1$. Then from the last relation we obtain $(a-b) = r\big(a - 2(\alpha+\beta-1)\big)$. In view of this, we get

$$\alpha = \frac{t}{2s}\,(b+2) \quad \text{and} \quad \beta = \frac{s-t}{2s}\,(a+2). \tag{2.16}$$

Case 1. (s is even and t is odd). Let $s \to 2s$ and $t \to 2t-1$. In this case relation (2.16) is transformed into

$$\alpha = \frac{2t-1}{4s}\,(b+2) \quad \text{and} \quad \beta = \frac{2s-(2t-1)}{4s}\,(a+2).$$

Since $(4s, 2t-1) = 1$ and $(4s, 2s-(2t-1)) = 1$ it follows that $4s \mid (b+2)$ and $4s \mid (a+2)$. Setting $b+2 = 4sm$ and $a+2 = 4sn$ we obtain $\alpha = (2t-1)m$ and $\beta = (2s-(2t-1))n$. So we arrive at the corresponding class of Seidel integral graphs displayed in (2.13).

Case 2. (s is odd and t is even). Let $s \to 2s+1$ and $t \to 2t$. In this case relation (2.16) is transformed into

$$\alpha = \frac{t}{2s+1}\,(b+2) \quad \text{and} \quad \beta = \frac{(2s+1)-2t}{2(2s+1)}\,(a+2).$$

Since $(2s+1,t)=1$ and $\big(2(2s+1),(2s+1)-2t\big)=1$ it follows that $(2s+1)\mid(b+2)$ and $2(2s+1)\mid(a+2)$. Setting $b+2=(2s+1)m$ and $a+2=2(2s+1)n$ we obtain $\alpha=tm$ and $\beta=((2s+1)-2t)n$. So we arrive at the corresponding class of Seidel integral graphs displayed in (2.14).

Case 3. (s is odd and t is odd). Let $s\to 2s+1$ and $t\to 2t-1$. In this case relation (2.16) is transformed into

$$\alpha=\frac{2t-1}{2(2s+1)}\,(b+2)\quad\text{and}\quad \beta=\frac{s-t+1}{2s+1}\,(a+2).$$

Since $\big(2(2s+1),2t-1\big)=1$ and $(2s+1,s-t+1)=1$ it follows that $2(2s+1)\mid(b+2)$ and $(2s+1)\mid(a+2)$. Setting $b+2=2(2s+1)m$ and $a+2=(2s+1)n$ we obtain $\alpha=(2t-1)m$ and $\beta=(s-t+1)n$. So we arrive at the corresponding class of Seidel integral graphs displayed in (2.15). □

Using Remark 2.1 and using the proof of Propositions 2.2 and 2.5, in a quite analogous manner we can obtain the following two results.

Proposition 2.6. *If $\alpha K_{a,a}\cup\beta K_{b,b}$ is a Seidel integral graph with $\mu_1^*=2ab-1$ then it uniquely determines the parameters m,n,s,t.*

Proposition 2.7. *If $\alpha K_{a,a}\cup\beta K_{b,b}$ is a Seidel integral graph with $\mu_1^*=ab-1$ then it uniquely determines the parameters m,n,s,t.*

Theorem 2.9. *If $\alpha K_{a,a}\cup\beta K_{b,b}$ is Seidel integral with $\mu_1^*=4a-1$, then it belongs to one of the following classes of Seidel integral graphs:* (1^0) $K_{a,a}\cup 2(\beta+1)K_{b,b}$ *where* $a=(3\beta+2)(2m-1)$ *and* $b=2(2m-1)$ *or* (2^0) $K_{a,a}\cup(\beta+2)K_{b,b}$ *where* $a=(3\beta+4)m$ *and* $b=4m$ *or* (3^0) $K_{a,a}\cup 2(2\beta+1)K_{b,b}$ *where* $a=(3\beta+1)(2m-1)$ *and* $b=2m-1$ *or* (4^0) $2K_{a,a}\cup(2\beta+1)K_{b,b}$ *where* $a=(3\beta+1)(2m-1)$ *and* $b=2m-1$ *or* (5^0) $2K_{a,a}\cup(\beta+1)K_{b,b}$ *where* $a=(3\beta+2)m$ *and* $b=2m$ *for any* $\beta,m\in\mathbb{N}$.

Proof. Let us assume that $\alpha K_{a,a}\cup\beta K_{b,b}$ is Seidel integral with $\mu_1^*=4a-1$. Using that $\mu_1^*+1=2\theta a$ we obtain $\theta=2$. Using the right-hand side of relation (2.6), we find that $r\alpha=3$. Since $r>1$ it follows that $\alpha=1$ or $\alpha=2$.

Case 1. ($\alpha=1$). In this situation $s=3$ and $t=1$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a=3k+b$. Using (2.7) we obtain

$$\beta=\frac{2\big((3k+b)-k\big)}{b}.$$

Consider the case when $2\mid b$. Setting $b=2m$ it follows that $m\mid(2k+2m)$. Consider the case when m is odd. Setting $m\to 2m-1$ we obtain that $k=\ell(2m-1)$ and $\beta=2(\ell+1)$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (1^0). Consider the case when m is even. Setting $m\to 2m$ we obtain that $k=\ell m$ and $\beta=\ell+2$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (2^0).

Next, consider the case when $2\nmid b$. Setting $b=2m-1$ it follows that $(2m-1)\mid(2k+(2m-1))$. Setting $k=\ell(2m-1)$ we obtain $\beta=2(2\ell+1)$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (3^0).

Case 2. ($\alpha = 2$). In this situation $s = 3$ and $t = 2$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = 3k + b$. Using (2.7) we obtain $\beta = \frac{2k+b}{b}$. Consider the case when b is odd. Setting $b \to 2m - 1$ we obtain that $k = \ell(2m - 1)$ and $\beta = 2\ell + 1$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (4^0). Consider the case when b is even. Setting $b \to 2m$ we obtain that $k = \ell m$ and $\beta = \ell + 1$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (5^0). □

Theorem 2.10. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 3a - 1$ then it belongs to one of the following classes of Seidel integral graphs:* (1^0) $K_{a,a} \cup (\beta + 1)K_{b,b}$ *where* $a = 2(5\beta + 2)m$ *and* $b = 9m$ *or* (2^0) $K_{a,a} \cup 3\beta K_{b,b}$ *where* $a = 2(5\beta - 1)m$ *and* $b = 3m$ *or* (3^0) $K_{a,a} \cup 3(3\beta - 1)K_{b,b}$ *where* $a = 2(5\beta - 2)m$ *and* $b = m$ *or* (4^0) $2K_{a,a} \cup \beta K_{b,b}$ *where* $a = 2(5\beta - 1)m$ *and* $b = 3m$ *or* (5^0) $2K_{a,a} \cup (3\beta - 1)K_{b,b}$ *where* $a = 2(5\beta - 2)m$ *and* $b = m$ *for any* $\beta, m \in \mathbb{N}$.

Proof. Let us assume that $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 3a - 1$. Using that $\mu_1^* + 1 = 2\theta a$ we obtain $2\theta = 3$. Using the right-hand side of relation (2.6), we find that $r\alpha = \frac{5}{2}$. Since $r > 1$ it follows that $\alpha = 1$ or $\alpha = 2$.

Case 1. ($\alpha = 1$). In this situation $s = 5$ and $t = 2$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = 5k + b$. Using (2.7) we obtain

$$\beta = \frac{3\big((5k + b) - 2k\big)}{2b}.$$

Consider the case when $3 \mid b$. Setting $b = 3m$ we obtain that $\beta = \frac{3(k+m)}{2m}$. Consider the case when $m \mid 3$. Setting $m \to 3m$ we obtain that $\beta = \frac{k+3m}{2m}$. Then $2m \mid (k + 3m)$ which provides that $k = (2\ell - 1)m$ and $\beta = \ell + 1$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (1^0). Consider the case when $m \nmid 3$. Then $2m \mid (k + m)$ which provides that $k = (2\ell - 1)m$ and $\beta = 3\ell$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (2^0).

Next, consider the case when $3 \nmid b$. Then $2b \mid (3k + b)$ which provides that $k = (2\ell - 1)b$ and $\beta = 3(3\ell - 1)$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (3^0).

Case 2. ($\alpha = 2$). In this situation $s = 5$ and $t = 4$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = 5k + b$. Using (2.7) we obtain $\beta = \frac{3k+b}{2b}$. Consider the case when $3 \mid b$. Setting $b = 3m$ we obtain that $\beta = \frac{k+m}{2m}$. Then $2m \mid (k + m)$ which provides that $k = (2\ell - 1)m$ and $\beta = \ell$. Replacing ℓ with β we obtain the corresponding class of Seidel integral graphs displayed in (4^0).

Next, consider the case when $3 \nmid b$. Then $2b \mid (3k + b)$ which provides that $k = (2\ell - 1)b$ and $\beta = 3\ell - 1$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in (5^0). □

Theorem 2.11. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 2a - 1$, then it belongs to the class of Seidel integral graphs* $K_{a,a} \cup (\beta + 1)K_{b,b}$ *where* $a = (2\beta + 1)m$ *and* $b = m$ *for any* $\beta, m \in \mathbb{N}$.

Proof. Let us assume that $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 2a - 1$. Using that $\mu_1^* + 1 = 2\theta a$ we obtain $\theta = 1$. Using the right-hand side of relation (2.6), we find that $r\alpha = 2$. Consequently, since $r > 1$ we find that $\alpha = 1$. In this situation $s = 2$ and $t = 1$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = 2k + b$. Using (2.7) we obtain

$$\beta = \frac{(2k + b) - k}{b}.$$

Then $b \mid (k + b)$ which provides that $k = \ell b$ and $\beta = \ell + 1$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in Theorem 2.11. □

Theorem 2.12. *If $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = a - 1$, then it belongs to the class of Seidel integral graphs $K_{a,a} \cup \beta K_{b,b}$ where $a = 2(3\beta - 1)m$ and $b = m$ for any $\beta, m \in \mathbb{N}$.*

Proof. Let us assume that $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = a - 1$. Using that $\mu_1^* + 1 = 2\theta a$ we obtain $\theta = \frac{1}{2}$. Using the right-hand side of relation (2.6), we find that $r\alpha = \frac{3}{2}$. Consequently, since $r > 1$ we find that $\alpha = 1$. In this situation $s = 3$ and $t = 2$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = 3k + b$. Using (2.7) we obtain

$$\beta = \frac{(3k + b) - 2k}{2b}.$$

Then $2b \mid (k + b)$ which provides that $k = (2\ell - 1)b$ and $\beta = \ell$. Replacing ℓ with β and replacing b with m we obtain the corresponding class of Seidel integral graphs displayed in Theorem 2.12. □

Theorem 2.13. *There exists no Seidel integral graph from the class $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = 3a$ for any α, β, a, b and $a > b$.*

Proof. Let us assume that $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = 3a$. Using that $\mu_1^* + 1 = 2\theta a$ we obtain $\theta = \frac{3a+1}{2a}$. Using the right-hand side of relation (2.6), we find that $r\alpha = \frac{5a+1}{2a}$. Since $r > 1$ it follows that $\alpha = 1$ or $\alpha = 2$.

Case 1. $(\alpha = 1)$. In this situation we have $\frac{s}{t} = \frac{5a+1}{2a}$. We note that $(5a+1, 2a) = 1$ or $(5a + 1, 2a) = 2$. Consider the case when $(5a + 1, 2a) = 1$. Then a is an even number. Let $a = 2\varepsilon$ where $\varepsilon \in \mathbb{N}$. Since $(s, t) = 1$ we find that $s = 10\varepsilon + 1$ and $t = 4\varepsilon$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (10\varepsilon + 1)k + b$. So we obtain $2\varepsilon = (10\varepsilon + 1)k + b$, a contradiction.

Next, consider the case when $(5a + 1, 2a) = 2$. Then a is an odd number. Let $a = 2\varepsilon + 1$ where $\varepsilon \in \mathbb{N}$. Since $\frac{s}{t} = \frac{5\varepsilon+3}{2\varepsilon+1}$ and $(s, t) = 1$, $(5\varepsilon + 3, 2\varepsilon + 1) = 1$, we find that $s = 5\varepsilon + 3$ and $t = 2\varepsilon + 1$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (5\varepsilon + 3)k + b$. So we obtain $2\varepsilon + 1 = (5\varepsilon + 3)k + b$, a contradiction.

Case 2. $(\alpha = 2)$. In this situation we have $\frac{s}{t} = \frac{5a+1}{4a}$. Consider the case when a is an even number. Let $a = 2\varepsilon$ where $\varepsilon \in \mathbb{N}$. Since $\frac{s}{t} = \frac{10\varepsilon+1}{8\varepsilon}$ and $(s, t) = 1$, $(10\varepsilon + 1, 8\varepsilon) = 1$, we find that $s = 10\varepsilon + 1$ and $t = 8\varepsilon$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (10\varepsilon + 1)k + b$. So we obtain $2\varepsilon = (10\varepsilon + 1)k + b$, a contradiction.

Next, consider the case when a is an odd number. Setting $a = 2\varepsilon + 1$ we obtain $\frac{s}{t} = \frac{5\varepsilon+3}{2(2\varepsilon+1)}$, where $\varepsilon \in \mathbb{N}$. We note that $(5\varepsilon+3, 2(2\varepsilon+1)) = 1$ or $(5\varepsilon+3, 2(2\varepsilon+1)) = 2$. Consider the case when $(5\varepsilon + 3, 2(2\varepsilon + 1)) = 1$. Then ε is an even number. Let $\varepsilon = 2\varepsilon^\bullet$ where $\varepsilon^\bullet \in \mathbb{N}$. Since $\frac{s}{t} = \frac{10\varepsilon^\bullet+3}{2(4\varepsilon^\bullet+1)}$ and $(s,t) = 1$, $(10\varepsilon^\bullet + 3, 2(4\varepsilon^\bullet + 1)) = 1$, we find that $s = 10\varepsilon^\bullet + 3$ and $t = 2(4\varepsilon^\bullet + 1)$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (10\varepsilon^\bullet + 3)k + b$. So we obtain $4\varepsilon^\bullet + 1 = (10\varepsilon^\bullet + 3)k + b$, a contradiction.

Next, consider the case when $(5\varepsilon+3, 2(2\varepsilon+1)) = 2$. Then ε is an odd number. Let $\varepsilon = 2\varepsilon^\bullet - 1$ where $\varepsilon^\bullet \in \mathbb{N}$. Since $\frac{s}{t} = \frac{5\varepsilon^\bullet-1}{4\varepsilon^\bullet-1}$ and $(s,t) = 1$, $(5\varepsilon^\bullet - 1, 4\varepsilon^\bullet - 1) = 1$, we find that $s = 5\varepsilon^\bullet - 1$ and $t = 4\varepsilon^\bullet - 1$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (5\varepsilon^\bullet - 1)k + b$. So we obtain $2(2\varepsilon^\bullet - 1) + 1 = (5\varepsilon^\bullet - 1)k + b$, a contradiction. $\square$

Theorem 2.14. *There exists no Seidel integral graph from the class* $\alpha K_{a,a} \cup \beta K_{b,b}$ *with* $\mu_1^* = a$ *for any* α, β, a, b *and* $a > b$.

Proof. Let us assume that $\alpha K_{a,a} \cup \beta K_{b,b}$ is Seidel integral with $\mu_1^* = a$. Using that $\mu_1^* + 1 = 2\theta a$ we obtain $\theta = \frac{a+1}{2a}$. Using the right-hand side of relation (2.6), we find that $r\alpha = \frac{3a+1}{2a}$. Consequently, since $r > 1$ we find that $\alpha = 1$. We note that $(3a+1, 2a) = 1$ or $(3a+1, 2a) = 2$. We shall consider the following two cases:

Case 1. (a is even). Let $a = 2\varepsilon$ where $\varepsilon \in \mathbb{N}$. Since $\frac{s}{t} = \frac{6\varepsilon+1}{4\varepsilon}$ and $(s,t) = 1$, $(6\varepsilon + 1, 4\varepsilon) = 1$, we find that $s = 6\varepsilon + 1$ and $t = 4\varepsilon$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (6\varepsilon + 1)k + b$. So we obtain $2\varepsilon = (6\varepsilon + 1)k + b$, a contradiction.

Case 2. (a is odd). Let $a = 2\varepsilon + 1$ where $\varepsilon \in \mathbb{N}$. Since $\frac{s}{t} = \frac{3\varepsilon+2}{2\varepsilon+1}$ and $(3\varepsilon + 2, 2\varepsilon + 1) = 1$, we find that $s = 3\varepsilon + 2$ and $t = 2\varepsilon + 1$. Using $(\overline{4})$ and $(\overline{5})$ we find that $a = (3\varepsilon + 2)k + b$. So we obtain $2\varepsilon + 1 = (3\varepsilon + 2)k + b$, a contradiction. $\square$

Remark 2.4. The following result is also presented in [**4**] but its proof in this work is not exactly the same as in the paper [**4**]. In view of this fact, we give the following result with its proof.

Theorem 2.15. *If* $(\alpha, \beta, a, b, \delta)$ *is a positive integral solution of the Diophantine equation* (2.2) *then it is in the form:*

- $a = (t + \ell n)k + \ell m$ *and* $b = \ell m$;
- $\alpha = \dfrac{kt}{\tau} x_0 + \dfrac{mt}{\tau} z$;
- $\beta = \left[\dfrac{kt}{\tau} y_0 + \dfrac{a}{\tau} z\right] n$;
- $\delta = k\ell n + \left[\dfrac{kt}{\tau} y_0 + \dfrac{a}{\tau} z\right](t + \ell n)m$,

with the same conditions (ii)*–*(v) *which are given in Theorem* 2.3.

Proof. According to Theorem 2.3 it suffices to derive the expression for δ. First, from (2.1) we have (i) $\mu_1^* - \mu_2^* = 2\delta$ and (ii) $\mu_1^* + \mu_2^* = 2(\alpha a + \beta b) - 2(a + b + 1)$. Using (i), (ii) and the equality $\mu_1^* = 2(r\alpha - 1)a - 1$ (see (2.6)), by a straightforward calculation we obtain that $\delta = 2r\alpha a - (\alpha a + \beta b) - (a - b)$. Since $a - b = ks$ (see $(\overline{4})$ and $(\overline{5})$), $r\alpha a = ks + \eta ms$ and $\beta = \eta n$ (see $(\overline{8})$ and $(\overline{9})$), we arrive at $\delta = k\ell n + (t + \ell n)\eta m$, which completes the proof. $\square$

3. Appendix

In this section we present the data given in Table 1, which represent the set of all Seidel integral graphs from the class $\alpha K_{a,a} \cup \beta K_{b,b}$, whose order does not exceed 40. In this table a Seidel integral graph is described by the parameters α, β, a, b and ones presented in the class of Seidel integral graphs in Theorem 2.3. In Table 1 the symbol 'i' is the identification number of an integral graph.

i	x_0	y_0	z	o	α	β	a	b	τ	t	k	ℓ	m	n	μ_1^*	μ_2^*
1	0	−1	1	10	1	1	4	1	2	2	1	1	1	1	3	−5
2	0	−1	1	10	1	2	3	1	1	1	1	1	1	1	5	−5
3	0	−1	1	16	1	2	6	1	3	3	1	1	1	2	7	−7
4	0	−1	1	16	1	3	5	1	1	1	2	1	1	1	9	−7
5	0	−1	1	20	1	1	9	1	3	3	2	1	1	1	5	−7
6	0	−1	1	20	1	1	8	2	4	2	2	1	2	1	7	−9
7	0	−1	1	20	1	2	6	2	2	1	2	1	2	1	11	−9
8	0	−1	1	20	1	6	4	1	1	1	1	1	1	2	15	−7
9	0	−1	1	22	1	3	8	1	4	4	1	1	1	3	11	−9
10	0	−1	1	22	1	4	7	1	1	1	3	1	1	1	13	−9
11	0	−1	2	22	2	3	4	1	2	2	1	1	1	1	15	−5
12	0	−1	2	22	2	5	3	1	1	1	1	1	1	1	17	−5
13	0	−1	1	24	1	2	10	1	2	2	3	1	1	1	9	−9
14	0	−1	1	24	1	6	6	1	2	2	1	1	1	3	17	−9
15	1	−1	0	26	1	1	9	4	3	3	1	2	2	1	11	−13
16	0	−1	1	26	1	4	5	2	1	1	1	2	1	1	19	−9
17	0	−1	1	28	1	4	10	1	5	5	1	1	1	4	15	−11
18	1	1	0	28	2	2	6	1	2	4	1	1	1	1	17	−5
19	0	−1	1	28	1	5	9	1	1	1	4	1	1	1	17	−11
20	1	1	0	28	2	2	5	2	1	2	1	1	2	1	19	−7
21	0	−1	1	30	1	1	12	3	6	2	3	1	3	1	11	−13
22	0	−1	1	30	1	2	9	3	3	1	3	1	3	1	17	−13
23	0	−1	1	32	1	2	12	2	3	3	2	2	1	1	15	−13
24	0	−1	1	32	1	3	10	2	2	1	4	1	2	1	19	−13
25	0	−1	1	34	1	1	16	1	4	4	3	1	1	1	7	−9
26	0	−1	1	34	1	2	15	1	5	5	2	1	1	2	11	−11
27	0	−1	1	34	1	5	12	1	6	6	1	1	1	5	19	−13
28	0	−1	1	34	1	6	11	1	1	1	5	1	1	1	21	−13
29	0	−1	1	34	1	3	8	3	2	2	1	3	1	1	23	−13
30	0	−1	1	34	1	4	9	2	3	3	1	2	1	2	23	−13
31	0	−1	3	34	3	5	4	1	2	2	1	1	1	1	27	−5
32	0	−1	1	34	1	10	7	1	1	1	2	1	1	2	27	−11
33	0	−1	3	34	3	8	3	1	1	1	1	1	1	1	29	−5
34	0	−1	1	34	1	12	5	1	1	1	1	1	1	3	29	−9
35	0	−1	2	36	2	6	6	1	3	3	1	1	1	2	27	−7

36	0	−1	2	36	2	8	5	1	1	1	2	1	1	1	29	−7
37	0	−1	1	38	1	3	16	1	2	2	5	1	1	1	15	−13
38	−1	−1	1	38	2	1	8	3	4	4	1	1	3	1	23	−9
39	1	3	−1	38	2	5	7	1	1	2	2	1	1	1	27	−7
40	1	3	0	38	1	3	7	4	1	1	1	2	2	1	27	−13
41	2	3	−1	38	3	4	5	1	1	3	1	1	1	1	29	−5
42	2	3	0	38	2	3	5	3	1	1	1	1	3	1	29	−9
43	0	−1	1	38	1	10	9	1	3	3	1	1	1	5	29	−13
44	0	−1	1	40	1	1	18	2	6	3	4	1	2	1	11	−13
45	0	−1	1	40	1	1	16	4	8	2	4	1	4	1	15	−17
46	−1	−1	1	40	2	1	9	2	3	6	1	1	2	1	23	−7
47	0	−1	1	40	1	6	14	1	7	7	1	1	1	6	23	−15
48	0	−1	1	40	1	2	12	4	4	1	4	1	4	1	23	−17
49	0	−1	1	40	1	7	13	1	1	1	6	1	1	1	25	−15
50	0	−1	1	40	1	6	8	2	1	1	2	2	1	1	31	−13

Table 1

Next, graphs represented in Table 1 with identification numbers $i = 8$, 11, 32, 39 are Seidel integral graphs with $\mu_1^* = 4ab - 1$. In view of this, there exist exactly 4 non-isomorphic Seidel integral graphs from the class $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = 4ab - 1$, whose order does not exceed 40.

Next, graphs represented in Table 1 with identification[2] numbers $i = 14$, 18 are Seidel integral graphs with $\mu_1^* = 3ab - 1$. In view of this, there exist exactly 2 non-isomorphic Seidel integral graphs from the class $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = 3ab - 1$, whose order does not exceed 40. In particular, there exist exactly 8, 9 and 1 non-isomorphic Seidel integral graphs with $\mu_1^* = 3ab - 1$ and order $o \leqslant 142$, which belong to the classes Theorem 2.6 (2.8), (2.9) and (2.10), respectively. In view of this, there exist exactly $8 + 9 + 1 = 18$ non-isomorphic Seidel[3] integral graphs from the class $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = 3ab - 1$, whose order does not exceed 142.

[2] The Seidel integral graphs represented in Table 1 with identification numbers $i = 14$, 18 belong to the class Theorem 2.6 (2.9). In particular, the Seidel integral graph with identification number $i = 14$ is obtained for $s = 2$, $t = 1$, $(x_0^+, y_0^+) = (1, 3)$, $(x_0^-, y_0^-) = (2, 3)$, $z^+ = 0$, $z^- = -1$, $n = 2$ and $m = 1$. In view of these values, we find that $\alpha = 1$, $\beta = 6$, $a = 6$ and $b = 1$. The Seidel integral graph with identification number $i = 18$ is obtained for $s = 2$, $t = 2$, $(x_0^+, y_0^+) = (1, 3)$, $(x_0^-, y_0^-) = (2, 3)$, $z^+ = 0$, $z^- = -1$, $n = 2$ and $m = 1$. In view of these values, we find that $\alpha = 2$, $\beta = 2$, $a = 6$ and $b = 1$.

[3] There exists no Seidel integral graph from the classes Theorem 2.6 (2.8) and (2.10), whose order does not exceed 40. We here present a Seidel integral graph which belongs to the class Theorem 2.6 (2.8), obtained for $s = 1$, $t = 1$, $(x_0, y_0) = (1, 1)$, $z^+ = 1$, $z^- = 0$, $n = 5$ and $m = 2$. In view of these values, we find that $\alpha = 2$, $\beta = 5$, $a = 6$ and $b = 2$. Note that $2K_{6,6} \cup 5K_{2,2}$ is the Seidel integral graph of the least order $o = 44$ which belongs to the class Theorem 2.6 (2.8). We here also present a Seidel integral graph which belongs to the class Theorem 2.6 (2.10), obtained for $s = 3$, $t = 1$, $(x_0^+, y_0^+) = (1, 2)$, $(x_0^-, y_0^-) = (2, 9)$, $z^+ = 1$, $z^- = -1$, $n = 5$ and $m = 1$. In view of these values, we find that $\alpha = 1$, $\beta = 15$, $a = 11$ and $b = 4$. Note that $K_{11,11} \cup 15K_{4,4}$ is the Seidel integral graph of the least order $o = 142$ which belongs to the class Theorem 2.6 (2.10).

Next, graphs represented in Table 1 with identification numbers $i = 2, 4, 10, 16, 19, 20, 28, 42, 49, 50$ are Seidel integral graphs with $\mu_1^* = 2ab - 1$. In view of this, there exist exactly 10 non-isomorphic Seidel integral graphs from the class $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = 2ab - 1$, whose order does not exceed 40.

Next, graphs represented in Table 1 with identification numbers $i = 1, 7, 13, 24, 29, 37, 38, 40$ are Seidel integral graphs with $\mu_1^* = ab - 1$. In view of this, there exist exactly 8 non-isomorphic Seidel[4] integral graphs from the class $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = ab - 1$, whose order does not exceed 40.

Next, graphs represented in Table 1 with identification numbers $i = 8, 11, 16, 20, 32, 39, 40, 50$ are Seidel integral graphs with $\mu_1^* = 4a - 1$. In view of this, there exist exactly 8 non-isomorphic Seidel[5] integral graphs from the class $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = 4a - 1$, whose order does not exceed 40.

Next, graphs represented in Table 1 with identification numbers $i = 14, 18, 29, 38$ are Seidel integral graphs with $\mu_1^* = 3a - 1$. In view of this, there exist exactly 4 non-isomorphic Seidel[6] integral graphs from the class $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = 3a - 1$, whose order does not exceed 40.

Next, graphs represented in Table 1 with identification numbers $i = 2, 4, 7, 10, 19, 22, 24, 28, 48, 49$ are Seidel integral graphs with $\mu_1^* = 2a - 1$. In view of this, there exist exactly 10 non-isomorphic Seidel integral graphs from the class $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = 2a - 1$, whose order does not exceed 40.

Next, graphs represented in Table 1 with identification numbers $i = 1, 6, 13, 21, 37, 45$ are Seidel integral graphs with $\mu_1^* = a - 1$. In view of this, there exist exactly 6 non-isomorphic Seidel integral graphs from the class $\alpha K_{a,a} \cup \beta K_{b,b}$ with $\mu_1^* = a - 1$, whose order does not exceed 40. This completes[7] my explanation on Table 1.

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[4]We note (i) graphs represented in Table 1 with identification numbers $i = 7, 24$ belong to the class Theorem 2.8 (2.13); (ii) graphs represented in Table 1 with identification numbers $i = 1, 13, 29, 37, 38$ belong to the class Theorem 2.8 (2.14) and (iii) graph represented in Table 1 with identification number $i = 40$ belongs to the class Theorem 2.8 (2.15).

[5]We note (i) graphs represented in Table 1 with identification numbers $i = 16, 50$ belong to the class Theorem 2.9 (1^0); (ii) graph represented in Table 1 with identification number $i = 40$ belongs to the class Theorem 2.9 (2^0); (iii) graphs represented in Table 1 with identification numbers $i = 8, 32$ belong to the class Theorem 2.9 (3^0); (iv) graphs represented in Table 1 with identification numbers $i = 11, 39$ belong to the class Theorem 2.9 (4^0) and (v) graph represented in Table 1 with identification number $i = 20$ belongs to the class Theorem 2.9 (5^0).

[6]First, there exists no Seidel integral graph from the class Theorem 2.10 (1^0), whose order does not exceed 40. We note (i) graph represented in Table 1 with identification number $i = 29$ belongs to the class Theorem 2.10 (2^0); (ii) graph represented in Table 1 with identification number $i = 14$ belongs to the class Theorem 2.10 (3^0); (iii) graph represented in Table 1 with identification number $i = 38$ belongs to the class Theorem 2.10 (4^0) and (iv) graph represented in Table 1 with identification number $i = 18$ belongs to the class Theorem 2.10 (5^0).

[7]In this work the data given in Tables 1 and 2 are obtained in two different ways: (i) they are generated by using relation (2.5) and (ii) by varying the parameters α, β, a, b in all possible ways in equation (2.2).

010^{02}	016^{02}	020^{04}	022^{04}	024^{02}	026^{02}	028^{04}	030^{02}	032^{02}	034^{10}
036^{02}	038^{07}	040^{07}	044^{06}	046^{07}	048^{06}	050^{06}	052^{10}	054^{08}	056^{07}
058^{06}	060^{08}	062^{07}	064^{09}	066^{09}	068^{14}	070^{08}	072^{07}	074^{06}	076^{21}
078^{08}	080^{15}	082^{11}	084^{09}	086^{09}	088^{14}	090^{05}	092^{17}	094^{13}	096^{10}
098^{07}	100^{17}	102^{17}	104^{18}	106^{14}	108^{15}	110^{11}	112^{18}	114^{17}	116^{15}
118^{19}	120^{19}	122^{15}	124^{25}	126^{04}	128^{17}	130^{19}	132^{18}	134^{10}	136^{40}
138^{17}	140^{22}	142^{12}	144^{19}	146^{11}	148^{23}	150^{16}	152^{30}	154^{14}	156^{22}
158^{11}	160^{32}	162^{17}	164^{29}	166^{23}	168^{21}	170^{22}	172^{27}	174^{17}	176^{36}
178^{23}	180^{15}	182^{10}	184^{28}	186^{12}	188^{28}	190^{29}	192^{25}	194^{15}	196^{24}
198^{18}	200^{25}	202^{31}	204^{40}	206^{24}	208^{32}	210^{19}	212^{30}	214^{25}	216^{34}
218^{24}	220^{32}	222^{24}	224^{23}	226^{15}	228^{39}	230^{25}	232^{29}	234^{28}	236^{33}
238^{28}	240^{36}	242^{15}	244^{40}	246^{29}	248^{42}	250^{28}	252^{30}	254^{15}	256^{34}
258^{26}	260^{45}	262^{24}	264^{42}	266^{14}	268^{35}	270^{22}	272^{54}	274^{22}	276^{53}
278^{20}	280^{44}	282^{20}	284^{31}	286^{36}	288^{32}	290^{32}	292^{34}	294^{20}	296^{43}
298^{22}	300^{48}	302^{18}	304^{53}	306^{29}	308^{31}	310^{35}	312^{38}	314^{21}	316^{35}
318^{37}	320^{53}	322^{24}	324^{43}	326^{22}	328^{58}	330^{28}	332^{63}	334^{29}	336^{43}
338^{22}	340^{51}	342^{37}	344^{48}	346^{42}	348^{50}	350^{21}	352^{58}	354^{32}	356^{48}
358^{39}	360^{37}	362^{26}	364^{51}	366^{28}	368^{52}	370^{32}	372^{45}	374^{40}	376^{46}
378^{24}	380^{58}	382^{31}	384^{56}	386^{29}	388^{56}	390^{36}	392^{45}	394^{36}	396^{42}
398^{39}	400^{51}	402^{31}	404^{54}	406^{32}	408^{81}	410^{33}	412^{57}	414^{34}	416^{50}
418^{44}	420^{48}	422^{25}	424^{65}	426^{29}	428^{56}	430^{40}	432^{64}	434^{31}	436^{60}
438^{31}	440^{59}	442^{43}	444^{68}	446^{20}	448^{53}	450^{33}	452^{46}	454^{40}	456^{74}
458^{37}	460^{70}	462^{34}	464^{41}	466^{31}	468^{63}	470^{34}	472^{70}	474^{39}	476^{57}
478^{28}	480^{75}	482^{26}	484^{41}	486^{43}	488^{61}	490^{34}	492^{66}	494^{42}	496^{74}
498^{44}	500^{59}	502^{37}	504^{59}	506^{30}	508^{51}	510^{47}	512^{51}	514^{50}	516^{69}
518^{31}	520^{70}	522^{37}	524^{48}	526^{46}	528^{91}	530^{41}	532^{52}	534^{54}	536^{57}
538^{42}	540^{75}	542^{34}	544^{85}	546^{36}	548^{54}	550^{32}	552^{86}	554^{38}	556^{69}
558^{34}	560^{76}	562^{32}	564^{63}	566^{46}	568^{71}	570^{57}	572^{67}	574^{27}	576^{62}
578^{35}	580^{94}	582^{44}	584^{66}	586^{50}	588^{61}	590^{50}	592^{78}	594^{51}	596^{58}
598^{43}	600^{79}	602^{30}	604^{59}	606^{59}	608^{81}	610^{51}	612^{75}	614^{33}	616^{67}
618^{52}	620^{76}	622^{44}	624^{77}	626^{37}	628^{52}	630^{35}	632^{77}	634^{33}	636^{84}
638^{37}	640^{85}	642^{61}	644^{64}	646^{56}	648^{85}	650^{47}	652^{60}	654^{61}	656^{88}
658^{52}	660^{73}	662^{28}	664^{94}	666^{54}	668^{71}	670^{52}	672^{72}	674^{26}	676^{63}
678^{45}	680^{95}	682^{54}	684^{90}	686^{31}	688^{80}	690^{49}	692^{69}	694^{57}	696^{88}
698^{35}	700^{69}	702^{56}	704^{88}	706^{60}	708^{67}	710^{50}	712^{84}	714^{69}	716^{78}
718^{48}	720^{80}	722^{43}	724^{72}	726^{56}	728^{84}	730^{47}	732^{81}	734^{36}	736^{84}
738^{64}	740^{77}	742^{43}	744^{84}	746^{32}	748^{90}	750^{64}			

Table 2.

There exists exactly 14541 non-isomorphic Seidel integral graphs which belong to the class $\alpha K_{a,a} \cup \beta K_{b,b}$, whose order does not exceed 750. Table 2 contains a distribution of those graphs in respect to their orders. In Table 2 the symbol o^n denotes the number of integral graphs of the corresponding order $o = 1, 2, \ldots, 750$. In this table o^n is omitted if the corresponding number $n = 0$.

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