PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 112(126) (2022), 151–169 DOI: https://doi.org/10.2298/PIM2226151B

POLYNOMIAL OF A MEROMORPHIC FUNCTION AND ITS *k*-th DERIVATIVE SHARING A SET

Abhijit Banerjee and Molla Basir Ahamed

Abstract. With the help of weighted sharing of sets, we find out the class of meromorphic functions f, when $P(f)$ and $[P(f)]^{(k)}$ share a set $\mathcal{S}_m$ of small functions. Our results improve and extend the results of Zhang and Yang [Ann. Acad. Sci. Fenn. Math. **34** (2009), 249–260] and Xu et al. [Rev. Mat. Teor. Apl. **23**(1) (2016), 291–308]. A number of examples are exhibited to validate certain claims of the main results.

1. Introduction

Let f be a nonconstant meromorphic function in the whole complex plane $\mathbb{C}$. We shall use the following standard notations of the value distribution theory such as $m(r,f)$, the *proximity function*, $N(r,f)$, the *counting function* and $T(r,f)$, the *characteristic function* of f, etc. (see [**20**]). We denote $S(r,f)$ by any quantity satisfying $S(r,f) = O(T(r,f))$, as $r \to +\infty$ possibly outside of a set of finite measure. A meromorphic function $a \equiv a(z)$ is said to be a small function with respect to f if $T(r,a) = S(r,f)$. Let $\mathcal{S}(f)$ be the set of all small functions of f in the complex plane $\mathbb{C}$.

Let f be a nonconstant meromorphic function and $a \equiv a(z) \in \mathcal{S}(f) \cup \{\infty\}$ and $\mathcal{S} \subset \mathcal{S}(f) \cup \{\infty\}$. We define

$$E(\mathcal{S},f) := \bigcup_{a\in\mathcal{S}} \{z : f(z) - a(z) = 0, \quad \text{counting multiplicity}\},$$
$$\bar{E}(\mathcal{S},f) := \bigcup_{a\in\mathcal{S}} \{z : f(z) - a(z) = 0, \quad \text{ignoring multiplicity}\},$$

If $E(\mathcal{S},f) = E(\mathcal{S},g)$, we say that f and g share the set $\mathcal{S}$ CM; if $\bar{E}(\mathcal{S},f) = \bar{E}(\mathcal{S},g)$, we say that f and g share the set $\mathcal{S}$ IM. Especially, when $a(z)$ is constant and $\mathcal{S} = \{a\}$, we say that f and g share the value a CM if $E(\mathcal{S},f) = E(\mathcal{S},g)$; and we say that f and g share the value a IM if $\bar{E}(\mathcal{S},f) = \bar{E}(\mathcal{S},g)$. For more details

---

2010 *Mathematics Subject Classification:* Primary 30D35.

Key words and phrases: meromorphic function, weighted sharing, set sharing, uniqueness problem, derivative, small function, polynomials.

Communicated by Miloš Arsenović.

regarding values sharing by two meromorphic functions, we refer the reader to the paper [**11**] and references therein.

In 1996, Brück [**7**] initiated the research of finding a relation between an entire function f and its derivative f' counterpart sharing a value. Brück [**7**] have proposed the following conjecture which is famously known as *Brück conjecture.*

Conjecture 1.1. *Let f be a nonconstant entire function. Suppose that $\rho_1(f)$ is not a positive integer or infinite. If f and f' share one finite value a CM, then $\frac{f'-a}{f-a} = c$, for some nonzero constant c, where $\rho_1(f)$ is the first iterated order of f which is defined by*

$$\rho_1(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

Since then a widely studied subtopic of uniqueness theory have been developed as to find the relationship between a meromorphic function f and its derivative $f^{(k)}$ sharing some value or small functions (see e.g. [**9**,**15**,**21**,**23**]). A number of honest attempts have been made by many researchers such as Gundersen and Yang [**10**], Chen and Shon [**8**] and Al-Kahaladi [**1**] to solve the conjecture. In 2008, Yang and Zhang [**17**] studied Brück conjecture for a slightly different class of function to give the specific form of the function as the following.

Theorem 1.1. [**17**] *Let f be a nonconstant entire function, $n \geqslant 7$ be an integer. Denote $\mathcal{F} = f^n$. If $\mathcal{F}$ and $\mathcal{F}'$ share 1 CM, then $\mathcal{F} \equiv \mathcal{F}'$ and f assumes the form $f(z) = ce^{z/n}$, where c is a nonzero constant.*

Theorem 1.2. [**17**]*Let f be a nonconstant meromorphic function and $n \geqslant 12$ be an integer. Denote $\mathcal{F} = f^n$. If $\mathcal{F}$ and $\mathcal{F}'$ share 1 CM, then $\mathcal{F} \equiv \mathcal{F}'$, and f assumes the form $f(z) = ce^{\frac{z}{n}}$, where c is a nonzero constant.*

In 2009, Zhang and Yang [**22**] improved further the above two theorems at a large extent and proved the following results.

Theorem 1.3. [**22**] *Let f be a nonconstant entire function, n, k be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n \geqslant k+2$, then $f^n \equiv (f^n)^{(k)}$ and f assumes the form $f(z) = ce^{\lambda z/n}$, where c is a nonzero constant and $\lambda^k = 1$.*

Theorem 1.4. [**22**] *Let f be nonconstant meromorphic function, n, k be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 CM and $n > k+1+\sqrt{k+1}$, then the conclusion of Theorem 1.3 holds.*

Theorem 1.5. [**22**]*Let f be a nonconstant entire function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv 0, \infty$. If $f^n - a$ and $(f^n)^{(k)} - a$ share the value 0 IM and $n > 2k+3$, then the conclusion of Theorem 1.3 holds.*

Theorem 1.6. [**22**]*Let f be a nonconstant meromorphic function, n, k be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \not\equiv$*

$0, \infty$. *If* $f^n - a$ *and* $(f^n)^{(k)} - a$ *share the value* 0 *IM and*

$$n > 2k + 3 + \sqrt{(k+3)(2k+3)},$$

then the conclusion of Theorem 1.3 *holds.*

Though the standard definitions and notations of the value distribution theory are available in [**3**, **19**], we explain the following definitions and notations which are used in the paper.

DEFINITION 1.1. [**3**, **19**] When f and g share 1 IM, we denote by $\bar{N}_L(r, 1; f)$ the counting function of the 1-points of g. Similarly, we have $\bar{N}_L(r, 1; g)$. Let z_0 be a zero of $f - 1$ of multiplicity p and a zero of $g - 1$ of multiplicity q, we also denote by $N_{11}(r, 1; f)$ the counting function of those 1-points of f where $p = q = 1$; $\bar{N}_E^{(2}(r, 1; f)$ denotes the counting function of those 1−points of f where $p = q \geqslant 2$, each point in these counting functions is counted only once. In the same way, one can define $N_{11}(r, 1; g)$, $\bar{N}_E^{(2}(r, 1; g)$.

DEFINITION 1.2. [**5**] For $a \in \mathbb{C} \cup \{\infty\}$ and p a positive integer, let f be a nonconstant meromorphic function, we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f, denote by $N(r, a; f \mid\leqslant p)$ $(N(r, a; f \mid\geqslant p))$ the counting functions of those a-points of f whose multiplicities are not greater (less) than p where each a-point is counted according to its multiplicities. $\bar{N}(r, a; f \mid\leqslant p)$ $(\bar{N}(r, a; f \mid\geqslant p))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

DEFINITION 1.3. [**5**] For $a \in \mathbb{C} \cup \{\infty\}$ and a positive integer p we denote by

$$N_p(r, a; f) = \bar{N}(r, a; f) + \bar{N}(r, a; f \mid\geqslant 2) + \cdots + \bar{N}(r, a; f \mid\geqslant p).$$

Clearly, $N_1(r, a; f) = \bar{N}(r, a; f)$.

Next, we recall the following definition of weighted sharing of values which generally measures how closed a shared value is to being sharing IM or CM.

DEFINITION 1.4. [**12**, **13**] Let q be a nonnegative integer or infinity. For $c \in \mathbb{C} \cup \{\infty\}$, we denote by $E_f(a, q)$ the set of all $a-$points of f where an a-point of multiplicity m is counted m times if $m \leqslant q$ and $q + 1$ times if $m > q$. If $E_f(a, q) = E_g(a, q)$, we say that f, g share the value a with weight q.

We write f, g share (a, q) to mean that f, g share the value a with weight q. Clearly if f, g share (a, q), then f, g share (a, p) for all integer p $(0 \leqslant p < q)$. Also, we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ and (a, ∞) respectively.

Let $\mathcal{S}$ be a subset of $\mathcal{S}(f) \cup \{\infty\}$ and $E_f(\mathcal{S}, q)$ is defined (see [**12**]) by

$$E_f(\mathcal{S}, q) = \bigcup_{a \in \mathcal{S}} E_f(a, q).$$

We say that f and g share the set S with weight q if $E_f(\mathcal{S}, q) = E_g(\mathcal{S}, q)$. Recently, Xu et al. [**16**] have raised the following question for further investigations in this direction:

QUESTION 1.1. [**16**] *Can the nature of sharing* 1 *or* $a(z)$ *CM be further relaxed in Theorems* 1.1 *and* 1.3*?*

Define the set $\mathcal{S}_m$ by $\mathcal{S}_m = \{a(z), a(z)\zeta, a(z)\zeta^2, \ldots, a(z)\zeta^{m-1}\}$, where $a(z)$ is a small function of f and $\zeta = \cos(2\pi/m) + i\sin(2\pi/m)$ and m is a positive integer. It is easy to see that $\mathcal{S}_m$ is a set of small functions. Therefore, it is natural to ask the following question.

QUESTION 1.2. [**16**] *What will happen when* 1 *or* $a(z)$ *are replaced by the set* $\mathcal{S}_m$ *in the Theorems* 1.1–1.4*?*

In order to answer this question, Xu et al. [**16**] have proved the following two results which improved Theorems 1.3 and 1.4.

THEOREM 1.7. [**16**] *Let* f *be a nonconstant entire function,* n*,* k*,* m *be positive integers and* $a(z)$ *be a small function of* f *such that* $a(z) \not\equiv 0, \infty$*. If* $E_{f^n}(\mathcal{S}_m, q) = E_{(f^n)^{(k)}}(\mathcal{S}_m, q)$ *and* $n > \max\{k+1, k+\frac{\eta}{qm}\}$*, where* $\eta = k+q+2$*, then* $f^n \equiv t(f^n)^{(k)}$ *with* $t^m = 1$ *and* f *assumes the form* $f(z) = ce^{\lambda z/n}$*, where* c *is a nonzero constant and* $\lambda^{km} = 1$*.*

THEOREM 1.8. [**16**] *Let* f *be a nonconstant meromorphic function,* n*,* k*,* m *be positive integers and* $a(z)$ *be a small function of* f *such that* $a(z) \not\equiv 0, \infty$*. If* $E_{f^n}(\mathcal{S}_m, q) = E_{(f^n)^{(k)}}(\mathcal{S}_m, q)$ *and*

$$n > \max\left\{k+1, \frac{q(m+1)k+2\eta}{2qm} + \frac{\sqrt{4\eta(\eta+qk)+(m-1)^2q^2k^2}}{2qm}\right\},$$

where $\eta = k+q+2$*, then* $f^n \equiv t(f^n)^{(k)}$ *with* $t^m = 1$ *and* f *assumes the form* $f(z) = ce^{\lambda z/n}$*, where* c *is a nonzero constant and* $\lambda^{km} = 1$*.*

Considering all the developments of *Brück conjecture* and research thereafter, it is quite natural to expect certain extensions of Theorems 1.1–1.8 up to a relation between $P(f)$ and $[P(f)]^{(k)}$ sharing a set of small functions, where $P(f)$ is a polynomial defined in Lemma 4.2.

The above discussions motivate us to raise the following question.

QUESTION 1.3. *Can the lower bounds of* n *in Theorems* 1.7 *and* 1.8 *be further reduced?*

NOTE 1.1. It is worth noticing that the Theorems *1.7* and *1.8* are in fact valid for the weight $q \geqslant 1$.

A natural question thus arises as the following.

QUESTION 1.4. *Can we obtain the same conclusions when the nature of sharing in Theorems* 1.7 *and* 1.8 *is replaced by IM* ($q = 0$) *sharing?*

2. Main results

In this paper, taking the possible answers of all the above mentioned questions into background, our aim is to prove results such that Theorems 1.1–1.8 can be accommodated under a single theorem. Henceforth, we adopt the following notations from [**6**].

Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$, where a_i $(i = 0, 1, \ldots, n)$ are all complex numbers with $a_n \neq 0$ and $n \in \mathbb{N}$. Let

$$P(f) = a_n (f - d_{l_1})^{l_1} (f - d_{l_2})^{l_2} \ldots (f - d_{l_r})^{l_r},$$

where $a_n \neq 0$ and d_{l_j} $(j = 1, 2 \ldots, r)$ are distinct finite complex numbers and $l_1, l_2, \ldots, l_r$, r, n and k are all positive integers with $\sum_{j=1}^{r} l_j = n$. Let $l = \max\{l_1, l_2, \ldots, l_r\} > k$. In view of the factorization of $P(f)$, we set a nonzero polynomial $Q(f_1)$ by

$$Q(f_1) = a_n \prod_{j=1,\ l_j \neq l}^{r} (f_1 + d_l - d_{l_j})^{l_i} = b_p f_1^p + \cdots + b_1 f_1 + b_0,$$

where $a_p = b_n$, $f_1 = f - d_l$ and $p = n - l$. Then it is easy to see that $P(f) = f_1^l Q(f_1)$. We define χ_p by

$$\chi_p := \begin{cases} 0, & \text{if } p = 0, \\ 1, & \text{if } p \geqslant 1. \end{cases}$$

We also define δ_{p-i}^m and γ_{p-i}^m, respectively, by

$$\delta_{p-i}^m := (p + k - i)m + k + 3 \quad \text{and} \quad \gamma_{p-i}^m := (p + k - i)m + 1 + \frac{1}{q},$$

where $m, q, k \in \mathbb{N}$ and $p, i \in \mathbb{N} \cup \{0\}$.

For meromorphic functions, we prove the following result which are the main results of this paper.

Theorem 2.1. *Let f be a nonconstant meromorphic function, k, $l(> k)$, n, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{P(f)}(\mathcal{S}_m, q) = E_{[P(f)]^{(k)}}(\mathcal{S}_m, q)$ and*

(i) *$q \geqslant 2$ and $n > \frac{\gamma_{p-i}^m + \gamma_0^1 + \sqrt{(\gamma_{p-i}^m - \gamma_0^1)^2 + 4C}}{2m}$, or*

(ii) *$q = 0$ and $n > \frac{\delta_{p-i}^m + \delta_0^1 + \sqrt{(\delta_{p-i}^m - \delta_0^1)^2 + 4D}}{2m}$,*

where $C = \left(k+1+\chi_p+\frac{1}{q}\right)\left(1+\frac{1}{q}\right)$ and $D = (2k+\chi_p+3)(k+3)$, then $f_1^{l+i} \equiv t\, [f_1^{l+i}]^{(k)}$ for some $i \in \{0, 1, \ldots, p\}$ with $t^m = 1$. Furthermore, f assumes the form

$$f(z) = c e^{\frac{\lambda}{l+i} z} + d_l,$$

where $c(\neq 0), d_l \in \mathbb{C}$ and $\lambda^{mk} = 1$.

For entire functions, we prove the following result.

Theorem 2.2. *Let f be a nonconstant entire function, k, $l(> k)$, n, m be positive integers and $a(z)$ be a small function of f such that $a(z) \not\equiv 0, \infty$. If $E_{P(f)}(\mathcal{S}_m, q) = E_{[P(f)]^{(k)}}(\mathcal{S}_m, q)$ and*

(i) *$q \geqslant 2$ and $n > \frac{(p+k-i)mq+q+1}{mq}$, or* (ii) *$q = 0$ and $n > \frac{(p+k-i)m+k+3}{m}$,*

then the conclusions of Theorem 2.1 hold.

In Theorems 2.1 and 2.2, if we consider $P(f) = f^n$, then it is easy to see that $\chi_p = 0$, hence we obtain some corollaries of our main results. It is worth noticing that the lower bound of n is reduced as compared to Theorems 1.7 and 1.8.

Corollary 2.1. *Let f be a nonconstant meromorphic function and k, n, m be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \neq 0, \infty$. If $E_{f^n}(\mathcal{S}_m, q) = E_{(f^n)^{(k)}}(\mathcal{S}_m, q)$ and if*

(i) $q \geqslant 2$ *and* $n > \frac{q(m+1)k+2\tau}{2qm} + \frac{\sqrt{4\tau(kq+\tau)+(m-1)^2q^2k^2}}{2qm}$, *or if*

(ii) $q = 0$ *and* $n > \frac{(m+3)k+6}{2m} + \frac{\sqrt{4(k+3)(2k+3)+(m-1)^2k^2}}{2m}$, *where* $\tau = q+1$

then $f^n \equiv t(f^n)^{(k)}$, where $t^m = 1$. Furthermore, f assumes the form $f(z) = ce^{\lambda z/n}$, where c is a nonzero constant and $\lambda^{mk} = 1$.

Remark 2.1. From Corollary 2.1, we see that $\tau = q+1 < q+k+2 = \eta$. Thus the conclusion of Theorem 1.8 can be obtained under reduced lower bound of n.

Corollary 2.2. *Let f be a nonconstant entire function and k, n, m be positive integers and $a(z)$ be a small meromorphic function of f such that $a(z) \neq 0, \infty$. If $E_{f^n}(\mathcal{S}_m, q) = E_{(f^n)^{(k)}}(\mathcal{S}_m, q)$ and if*

(i) $q \geqslant 2$ *and* $n > k + \frac{q+1}{qm}$, *or if* (ii) $q = 0$ *and* $n > k + \frac{k+3}{m}$,

then the conclusions of Corollary 2.1 hold.

Remark 2.2. From Corollary 2.2, it is easy to see that Theorem 1.7 can be obtained under reduced lower bound of n.

Corollary 2.3. *Let f be a nonconstant meromorphic function and k, n be positive integers and $a \equiv a(z)$ be a small meromorphic function of f such that $f^n - a$ and $(f^n)^{(k)} - a$ share $(0, q)$. If*

(i) $q \geqslant 2$ *and* $n > k + 1 + \frac{1}{q} + \sqrt{(1+1/q)(k+1+1/q)}$, *or if*

(ii) $q = 0$ *and* $n > 2k + 3 + \sqrt{(k+3)(2k+3)}$,

then $f^n \equiv (f^n)^{(k)}$ and f assumes the form $f(z) = ce^{\lambda z/n}$, where c is a nonzero constant and $\lambda^k = 1$.

Remark 2.3. From Corollary 2.3, we observe that if f be nonconstant meromorphic function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ is a small function such that $f^n - a$ and $(f^n)^{(k)} - a$ share $(0, \infty)$ and $n > k + 1 + \sqrt{k+1}$, then we obtain the conclusion of Theorem 1.4.

Corollary 2.4. *Let f be a nonconstant entire function and k, n be positive integers and $a \equiv a(z)$ is a small meromorphic function of f such that $f^n - a$ and $(f^n)^{(k)} - a$ share $(0, q)$. If*

(i) $q \geqslant 2$ *and* $n > k + 1 + \frac{1}{q}$, *or if* (ii) $q = 0$ *and* $n > 2k + 3$,

then the conclusions of Corollary 2.3 hold.

Remark 2.4. From Corollary 2.4 we see that when f be nonconstant entire function, n, k be positive integers and $a(z)(\not\equiv 0, \infty)$ be a small function such that $f^n - a$ and $(f^n)^{(k)} - a$ share $(0, 2)$ and $n \geqslant k + 2$, then we obtain the conclusion of Theorem 1.3.

3. Some examples

The following two examples validate that the conclusions of Corollaries 2.1 and 2.2 fail to hold for nonconstant entire or meromorphic functions respectively when the conditions over n are not satisfied.

Example 3.1. For $n \geqslant 2$, let the principal branch of f is given by $f(z) = (e^{\theta z} + 2a)^{\frac{1}{n}}$, where $a \neq 0$ is a constant and θ is a root of the equation $z^n + 1 = 0$. Let $\mathcal{S}_m = \{a\}$ and $P(f) = f^n$. Clearly, $P(f) = e^{\theta z} + 2a$ and $[P(f)]^{(k)} = -e^{\theta z}$ with $k = n$. Therefore, we see that $E_{P(f)}(\mathcal{S}_m, \infty) = E_{[P(f)]^{(k)}}(\mathcal{S}_m, \infty)$ and

$$n \leqslant \min\left\{k + \frac{q+1}{qm},\ k + \frac{k+3}{m}\right\} = \min\{n+1, 2n+3\} = n+1.$$

It is easy to see that $P(f) \not\equiv [P(f)]^{(k)}$ and $f(z) \neq ce^{\lambda/n}$ with $\lambda^k = 1$.

Example 3.2. Consider the meromorphic function

$$f(z) = \frac{z + 6e^{-\frac{1}{3}z} - \frac{3}{2}(e^{-\frac{1}{3}z})^2}{(1 - e^{-\frac{1}{3}z})^3}.$$

A simple computations shows that $P(f) = f$ and $[P(f)]'$ share $(1, \infty)$. The condition in Corollary 2.4 over n is not satisfied. Hence we see that $P(f) \not\equiv [P(f)]'$. Also note that $f(z) \neq ce^{z/n}$.

Remark 3.1. In case of when $Q(z)$ is a nonconstant polynomial, the next two examples show that for nonconstant entire or meromorphic functions, if conditions over n are violating, the conclusion of Theorem 2.1 fails to hold.

Example 3.3. Let

$$f(z) = \frac{-b \pm \sqrt{b^2 + 4a(ce^{e^z} + e^z)}}{2a},$$

where $a, b, c \in \mathbb{C} - \{0\}$. Evidently, $f[af + b] = ce^{e^z} + e^z$ and it is clear that $P(f) = f[af + b]$ and $[P(f)]'$ share (e^z, ∞). Here $n = 2$, $k = 1$, $m = 1$, $p = 1$ and $i = 0$ and condition in main Theorem 2.2 over n is not satisfied. Note that $p = 1 \neq 0$ and hence, $\chi_p = 1$. Clearly, $P(f) \not\equiv [P(f)]'$ and $f(z) \neq ce^{z/n}$,

Example 3.4. Let

$$f(z) = \frac{-b\sqrt{e^z + 1} \pm \sqrt{b^2(e^z + 1) + 4a(2e^z + z + 1)}}{2a\sqrt{e^z + 1}},$$

where $a, b, c \in \mathbb{C} - \{0\}$. Then we see that $f[af + b] = \frac{2e^z + z + 1}{e^z + 1}$. Therefore, it is clear that $P(f) = f[af + b]$ and $[P(f)]'$ share $(1, \infty)$. Note that here $n = 2$, $k = 1$, $m = 1$, $p = 1$ and $i = 0$. The condition in main Theorem 2.2 over n is not satisfied. Here $p = 1 \neq 0$, so $\chi_p = 1$. We see that $P(f) \not\equiv [P(f)]'$ and $f(z) \neq ce^{z/n}$.

The next two examples show that in order to obtain the specific form of the function from the assumption of the main results, the conditions over n are essential.

EXAMPLE 3.5. Let $P(f) = f$ where $f(z) = \sin \zeta z$ or $\cos \zeta z$, where ζ is a nonreal m-th roots of unity and $\mathcal{S}_m = \{a,\ a\zeta,\ a\zeta^2, \ldots, a\zeta^{m-1}\}$, where $a \equiv a(z)(\neq 0, \infty)$ is a small function of a meromorphic function and m is an even positive integer. Here we see that $n = 1$, $k = 2s$, $\forall\ s \in \mathbb{N}$, $d_l = 0$ and $E_{P(f)}(\mathcal{S}_m, \infty) = E_{[P(f)]^{(k)}}(\mathcal{S}_m, \infty)$ and

$$n \leqslant \min\left\{k + \frac{q+1}{qm},\ k + \frac{k+3}{m}\right\} = \min\left\{2s + \frac{1}{m},\ 2s + \frac{2s+3}{m}\right\} = 2s + \frac{1}{m}.$$

But $P(f) \equiv t\,[P(f)]^{(k)}$ with $t^m = 1$. Also we see that $f(z) \neq ce^{\lambda z/n}$ with $\lambda^{mk} = 1$.

EXAMPLE 3.6. Let $P(f) = f$ where $f(z) = \sin \zeta z$ or $\cos \zeta z$, where ζ is a nonreal m-th roots of unity and $\mathcal{S}_m = \{a,\ a\zeta,\ a\zeta^2, \ldots, a\zeta^{m-1}\}$, where $a \equiv a(z)(\neq 0, \infty)$ is a small function of a meromorphic function and m is an odd positive integer. Here we see that $n = 1$, $k = 4s - 2$, $\forall\ s \in \mathbb{N}$, $d_l = 0$ and $E_{P(f)}(\mathcal{S}_m, \infty) = E_{[P(f)]^{(k)}}(\mathcal{S}_m, \infty)$ and

$$n \leqslant \min\left\{4s - 2 + \frac{1}{m},\ 4s - 2 + \frac{4s+1}{m}\right\} = 4s - 2 + \frac{1}{m}.$$

But $P(f) \equiv t\,[P(f)]^{(k)}$ with $t^m = 1$. Also we see that $f(z) \neq ce^{\lambda z/n}$ with $\lambda^{mk} = 1$.

REMARK 3.2. The following example shows that the conclusion of Theorem 2.1 ceases to hold for $n = 1$.

EXAMPLE 3.7. Let $P(f) = f$, where $f(z) = \dfrac{\frac{1}{m}z^m + b}{1 + ce^{-z}}$ and $b, c\ (\neq 0)$ are complex constants and m be a positive integer. Let $a(z) = z^{m-1}$, then it is clear that $P(f) - a$ and $[P(f)]^{(k)} - a$ share $(0, \infty)$ with $n = 1 = k$ and

$$n \leqslant \min\left\{k+1+\frac{1}{q}+\sqrt{\left(1+\frac{1}{q}\right)\left(k+1+\frac{1}{q}\right)},\ 2k+3+\sqrt{(k+3)(2k+3)}\right\} = k+1+\sqrt{k+1}.$$

But we see that $P(f) \not\equiv [P(f)]^{(k)}$. Also note that Also we see that $f(z) \neq ce^{\lambda z/n}$ with $\lambda^{mk} = 1$.

The following example shows that the conditions (i) and (ii) used in Corollaries 2.1 and 2.2 are not necessary but sufficient.

EXAMPLE 3.8. Let $\mathcal{S}_6 = \left\{-1, 1, \frac{1-\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}, \frac{-1+\sqrt{3}i}{2}\right\}$ and f is given by $f(z) = e^{\lambda z/6}$, where λ is a root of the equation $z^6 + 1 = 0$ and $k = 6$. Let $P(f) = f^6$. We see that $E_{P(f)}(\mathcal{S}_6, \infty) = E_{[P(f)]^{(k)}}(\mathcal{S}_6, \infty)$ and

$$n \leqslant \min\left\{k + \frac{q+1}{qm},\ k + \frac{k+3}{m}\right\} = \min\left\{\frac{37}{6}, \frac{45}{6}\right\} = \frac{37}{6}.$$

But $P(f) \equiv t[P(f)]^{(k)}$ with $t^m = (-1)^6 = 1$. Also here f assumes the form $f(z) = ce^{\lambda z/n}$, where $c = 1$ and $\lambda^{mk} = \lambda^{36} = 1$.

EXAMPLE 3.9. Let $\mathcal{S}_m = \{a, a\zeta, a\zeta^2, a\zeta^3, a\zeta^4\}$ where ζ is a nonreal 5th roots of unity. Let $P(f) = f^n$, where $f(z) = e^{\frac{1}{n}\zeta^{\frac{1}{k}} z}$ where $k = n$. Then it is clear that $E_{P(f)}(\mathcal{S}_m, \infty) = E_{[P(f)]^{(k)}}(\mathcal{S}_m, \infty)$ with $n \leqslant \min\left\{k + \frac{1}{m},\ k + \frac{k+3}{m}\right\} = n + \frac{1}{m}$. Although we see that $P(f) \equiv t[P(f)]^{(k)}$ with $t^m = 1/\zeta^5 = 1$. Also here f assumes the form $f(z) = ce^{\lambda z/n}$, where $c = 1$ and $\lambda^{mk} = (\zeta^{1/k})^{5k} = \zeta^5 = 1$.

The following examples show that the set $\mathcal{S}_m$ in the Theorems 2.1 and 2.2 can not replaced by other set.

EXAMPLE 3.10. Let $S_m = \{0, -1, 1, -i, i\}$ and $f(z) = e^{\frac{\lambda}{7}z}$, where λ is a root of the equation $z^5 + 1 = 0$. Let $P(f) = f^7$ and $k = 5$. It is clear that $E_{P(f)}(\mathcal{S}_m, \infty) = E_{[P(f)]^{(k)}}(\mathcal{S}_m, \infty)$ and $n > \max\left\{k + \frac{1}{m},\ k + \frac{k+3}{m}\right\} = \frac{33}{5}$. But $P(f) \not\equiv t[P(f)]^{(k)}$ with $t^m = 1$, although f assumes the form $f(z) = ce^{\lambda z/n}$, with $c = 1$. We also note that $\lambda^{mk} \neq 1$.

EXAMPLE 3.11. Let $\mathcal{S}_m = \left\{\frac{a\omega}{2}, \frac{a\omega}{3}, \frac{2a\omega}{3}, \frac{a\omega}{4}, \frac{3a\omega}{4}, \frac{a\omega}{5}, \frac{4a\omega}{5}\right\}$, where a is an arbitrary nonzero complex number. Let f be such that $f^n = \mathcal{B}e^{\theta z} + a\omega$, where $n \leqslant 52$ is a positive integer and θ and ω are roots of the equations $z^{n-7} + 1 = 0$ and $z^3 - 1 = 0$ respectively and $\mathcal{B} \in \mathbb{C} \setminus \{0\}$. Let $P(f) = f^n$. Then it is clear that $E_{P(f)}(\mathcal{S}_m, \infty) = E_{[P(f)]^{(k)}}(\mathcal{S}_m, \infty)$ where $k = n - 7$ and

$$n > \max\left\{k + \frac{p+1}{pm},\ k + \frac{k+3}{m}\right\}.$$

But we see that $P(f) \not\equiv t[P(f)]^{(k)}$ with $t^m = 1$and hence f does not assume the form $f(z) = ce^{\lambda z/n}$ with $\lambda^{mk} = 1$.

4. Key lemmas

In this section, we present some necessary lemmas which will be required to prove the main results of this paper. Let $\mathcal{F}$, $\mathcal{G}$ be two nonconstant meromorphic functions. Henceforth, we shall denote $\mathcal{H}$, $\mathcal{V}$ and $\mathcal{U}$ by the following functions

$$\mathcal{H} = \left(\frac{\mathcal{F}''}{\mathcal{F}'} - \frac{2\mathcal{F}'}{\mathcal{F}-1}\right) - \left(\frac{\mathcal{G}''}{\mathcal{G}'} - \frac{2\mathcal{G}'}{\mathcal{G}-1}\right). \tag{4.1}$$

$$\mathcal{V} = \left(\frac{\mathcal{F}'}{\mathcal{F}-1} - \frac{\mathcal{F}'}{\mathcal{F}}\right) - \left(\frac{\mathcal{G}'}{\mathcal{G}-1} - \frac{\mathcal{G}'}{\mathcal{G}}\right), \tag{4.2}$$

$$\mathcal{U} = \frac{\mathcal{F}'}{\mathcal{F}-1} - \frac{\mathcal{G}'}{\mathcal{G}-1}. \tag{4.3}$$

LEMMA 4.1. [**16**] *Let f be a nonconstant meromorphic function and k, p are positive integers. Then*

$$N_p(r, 0; f^{(k)}) \leqslant T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f).$$

$$N_p(r, 0; f^{(k)}) \leqslant k\bar{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).$$

LEMMA 4.2. [**18**] *Let f be a nonconstant meromorphic function and $P(f) = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_0$, where $a_0, a_1, \ldots, a_n$ are constants with $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + S(r, f)$.*

LEMMA 4.3. [**18**] *Let $\mathcal{H}$ be given by* (4.1)*, $\mathcal{F}$ and $\mathcal{G}$ be two nonconstant meromorphic functions. If $\mathcal{H} \not\equiv 0$, then $N_{11}(r, 1; \mathcal{F}) \leqslant N(r, \mathcal{H}) + S(r, \mathcal{F}) + S(r, \mathcal{G})$.*

Lemma 4.4. *Let f and hence $f_1 = f - d_l$ be a nonconstant meromorphic function and $a \equiv a(z)$ be a small meromorphic functions of f such that $a(z) \not\equiv 0, \infty$ and let*

$$\mathcal{F}_1 := \frac{P(f)}{a(z)} = \frac{f_1^l Q(f_1)}{a(z)}, \quad \mathcal{G}_1 := \frac{[P(f)]^{(k)}}{a(z)} = \frac{[f_1^l Q(f_1)]^{(k)}}{a(z)}.$$

Let $\mathcal{V}$ be given by (4.2) and $\mathcal{F} = \mathcal{F}_1^m$ and $\mathcal{G} = \mathcal{G}_1^m$. If n, m, l and k are positive integers such that $n > k+1$ and $\mathcal{V} \equiv 0$, then $f_1^{l+i} \equiv t[f_1^{l+i}]^{(k)}$ for some $i \in \{0, 1, \ldots, p\}$ and $t^m = 1$. Furthermore, f assumes the form $f(z) = ce^{\frac{\lambda}{l+i}z} + d_l$, where c is a nonzero constant and $\lambda^{mk} = 1$.

Proof. Let $\mathcal{V} \equiv 0$. Then it is easy to see that

$$1 - \frac{1}{\mathcal{F}_1^m} \equiv \mathcal{A} - \frac{\mathcal{A}}{\mathcal{G}_1^m}, \tag{4.4}$$

where $\mathcal{A}$ is a nonzero constant. We now consider the following possible cases.

Case 1. Let $\mathcal{A} \neq 1$.

Subcase 1.1. If $N(r, \infty; f) = S(r, f)$, then from (4.4), we obtain

$$\bar{N}\Big(r, \frac{1}{1-\mathcal{A}}; \mathcal{F}_1^m\Big) = \bar{N}(r, \infty; \mathcal{G}_1^m) = S(r, f).$$

By the Second Fundamental Theorem and in view of the definitions of $\mathcal{F}_1, \mathcal{G}_1$, we obtain

$$T(r, \mathcal{F}_1^m) \leqslant \bar{N}(r, \infty; \mathcal{F}_1^m) + \bar{N}(r, 0; \mathcal{F}_1^m) + \bar{N}\Big(r, \frac{1}{1-\mathcal{A}}; \mathcal{F}_1^m\Big) + S(r, f),$$

which implies that

$$mn\, T(r, f) \leqslant \bar{N}(r, 0; f_1^l Q(f_1)) + S(r, f) < n\, T(r, f) + S(r, f),$$

which is not possible.

Subcase 1.2. Let $N(r, \infty; f) \neq S(r, f)$. Then there exists a z_0 which is not a zero or pole of $a(z)$ such that $1/f(z_0) = 0$. Therefore, a simple computation shows that $1/\mathcal{F}_1(z_0) = 1/\mathcal{G}_1(z_0) = 0$. Hence it follows from (4.4) that $\mathcal{A} = 1$, which is not possible.

Case 2. Let $\mathcal{A} = 1$. Thus, from (4.4) we obtain, $\mathcal{F}_1^m = \mathcal{G}_1^m$, i.e.,

$$P(f) \equiv t[P(f)]^{(k)} \quad \text{i.e.,} \quad f_1^l Q(f_1) \equiv t[f_1^l Q(f_1)]^{(k)},$$

where $t^m = 1$. By the similar argument being used in [**6**, Page 160], we see that f_1 assumes the form $f_1(z) = ce^{\frac{\lambda}{l+i}z}$ for some $i \in \{0, 1, \ldots, p\}$ and $\lambda^{mk} = 1$. Hence f assumes the form $f(z) = ce^{\frac{\lambda}{l+i}z} + d_l$, where c is a nonzero constant and $\lambda^{mk} = 1$. $\square$

Lemma 4.5. *Let $\mathcal{V}$ be given by (4.2) and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 4.4 and n, m be positive integers. If $\mathcal{V} \not\equiv 0$, then*

$$(mn-1)\bar{N}(r, \infty; f) \leqslant N(r, \infty; \mathcal{V}) + S(r, f).$$

Proof. From (4.2) and in view of the definitions of $\mathcal{F}, \mathcal{G}$, it is easy to see that if z_0 is a pole of f with the multiplicity q such that $a(z_0) \neq 0$ and $a(z_0) \neq \infty$, then z_0 is a zero of $\mathcal{F}'/(\mathcal{F}-1) - \mathcal{F}'/\mathcal{F}$ with the multiplicity $mnq-1$ and a zero of $\mathcal{G}'/(\mathcal{G}-1) - \mathcal{G}'/\mathcal{G}$ with the multiplicity $m(nq+k)-1$. Therefore, it is easy to see that z_0 is zero of $\mathcal{V}$ with multiplicity $p \geqslant \min\{mn-1, m(n+k)-1\} = mn-1$, Also, we note that $m(r,\mathcal{V}) = S(r,f)$. Therefore,

$$\begin{aligned}(mn-1)\bar{N}(r,\infty;f) &\leqslant N(r,0;\mathcal{V}) + S(r,f)\\ &\leqslant T(r,\mathcal{V}) + S(r,f) \leqslant N(r,\infty;\mathcal{V}) + S(r,f). \qquad \square\end{aligned}$$

Lemma 4.6. *Let $\mathcal{U}$ be given by (4.3) and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 4.4. If n, m are psotive integers such that $n > k$ and $\mathcal{U} \equiv 0$, then $f_1^{l+i} \equiv t[f_1^{l+i}]^{(k)}$ for some $i \in \{0,1,\ldots,p\}$ and $t^m = 1$ and f assumes the form $f(z) = ce^{\frac{\lambda}{l+i}z} + d_l$, where c is a nonzero constant and $\lambda^{mk} = 1$.*

Proof. Since $\mathcal{U} = 0$, we obtain

$$\mathcal{F} \equiv \mathcal{B}\mathcal{G} + 1 - \mathcal{B}, \tag{4.5}$$

where $\mathcal{B}$ is a nonzero constant. By the definitions of $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$, we obtain $N(r,\infty;f) = S(r,f)$. We discuss the following cases.

Case 1. Let $\mathcal{B} = 1$. Then it is easy to see that $\mathcal{F} \equiv \mathcal{G}$. Therefore, we have $\mathcal{F}_1^m \equiv \mathcal{G}_1^m$. Next proceeding exactly the same way as done in Case 2 of Lemma 4.4, we obtain $f_1^{l+i} \equiv t[f_1^{l+i}]^{(k)}$ for some $i \in \{0,1,\ldots,p\}$ and $t^m = 1$ and f assumes the form $f(z) = ce^{\frac{\lambda}{l+i}z} + d_l$, where c is a nonzero constant and $\lambda^{mk} = 1$.

Case 2. Let $\mathcal{B} \neq 1$.

Subcase 2.1. If $N(r,0;P(f)) \neq S(r,f)$, then there exists a point z_0 for which $P(f(z_0)) = 0$ but $a(z_0) \neq 0$. Since $l > k$, then it is clear that $F(z_0) = 0 = G(z_0)$. Now from (4.5), we obtain $B = 1$, which is clearly absurd.

Subcase 2.2. If $N(r,0;P(f)) = S(r,f)$, then from (4.5) and using Lemma 4.1, we obtain

$$\bar{N}(r,1-\mathcal{B};\mathcal{F}) = \bar{N}(r,0;\mathcal{G}) \leqslant N_{k+1}(r,0;P(f)) + k\,\bar{N}(r,\infty;f) \leqslant S(r,f).$$

By the Second Fundamental Theorem, in view of $N(r,0;P(f)) = N(r,\infty;f) = S(r,f)$, a simple computation shows that

$$\begin{aligned}mn\,T(r,f) &\leqslant T(r,\mathcal{F}) + S(r,f)\\ &\leqslant \bar{N}(r,\infty;\mathcal{F}) + \bar{N}(r,0;\mathcal{F}) + \bar{N}(r,1-\mathcal{B};\mathcal{F}) + S(r,f)\\ &\leqslant \bar{N}(r,0;P(f)) + \bar{N}(r,\infty;f) + \bar{N}(r,0;\mathcal{G}) + S(r,f) \leqslant S(r,f),\end{aligned}$$

which is not possible. $\square$

Lemma 4.7. *Let $\mathcal{U}$ be given by (4.3) and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 4.4. If n, m, l, i and k are positive integers such that $l > k$ and $\mathcal{U} \not\equiv 0$, then*

$$[(l+i-k)m-1]\bar{N}(r,0;P(f)) \leqslant N(r,\infty;\mathcal{U}) + S(r,f),$$

when b_i is the last nonvanishing coefficient in $Q(f_1)$ for $0 \leqslant i \leqslant p$.

Proof. Let z_0 be a zero of f_1 with multiplicity $q(\geqslant 1)$ such that $a(z_0) \neq 0, \infty$. Then z_0 is a zero of $\mathcal{F}'/(\mathcal{F}-1)$ with the multiplicity $(l+i)qm-1$ and z_0 is also a zero of $\mathcal{G}'/(\mathcal{G}-1)$ of multiplicity $((l+i)q-k)m-1$ for some $i \in \{0, 1, \ldots, p\}$. Therefore, z_0 is a zero of $\mathcal{U}$ of multiplicity at least $(l+i-k)m-1$. Since $m(r, \mathcal{U}) = S(r, f)$, we obtain

$$\begin{aligned}[(l+i-k)m-1]\bar{N}(r, 0; P(f)) &\leqslant N(r, 0; \mathcal{U}) + S(r, f)\\ &\leqslant T(r, \mathcal{U}) + S(r, f)\\ &\leqslant N(r, \infty; \mathcal{U}) + S(r, f). \qquad \square\end{aligned}$$

Lemma 4.8. *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}_1$, $\mathcal{G}_1$ are as in Lemma 4.4 and $\mathcal{V}$ as in (4.2). Now if $l > k$ and $E_{\mathcal{F}}(1, q) = E_{\mathcal{G}}(1; q)$ and $\mathcal{V} \not\equiv 0$, then the following hold:*

(i) *if $q \geqslant 2$, then*

$$\Big(mn - 1 - k - \frac{1}{q}\Big)\bar{N}(r, \infty; f) \leqslant \Big(k + 1 + \chi_{_p} + \frac{1}{q}\Big)\bar{N}(r, 0; P(f)) + S(r, f).$$

(ii) *if $q = 0$, then*

$$(mn - 2k - 3)\bar{N}(r, \infty; f) \leqslant (2k + \chi_{_p} + 3)\bar{N}(r, 0; P(f)) + S(r, f).$$

Proof. Let $q \geqslant 2$ and $\mathcal{V}$ be defined by

$$\mathcal{V} := \frac{\mathcal{F}'}{\mathcal{F}(\mathcal{F}-1)} - \frac{\mathcal{G}'}{\mathcal{G}(\mathcal{G}-1)}.$$

Since $E_q(1; \mathcal{F}) = E_q(1; \mathcal{G})$, hence it is easy to see that

$$N(r, \infty; \mathcal{V}) \leqslant \chi_{_p}\bar{N}(r, 0; \mathcal{F}) + \bar{N}(r, 0; \mathcal{G}) + \bar{N}_{(q+1}(r, 1; \mathcal{F}) + S(r, f),$$

where

$$\begin{aligned}\bar{N}_{(q+1}(r, 1; \mathcal{F}) \leqslant \frac{1}{q}N\Big(r, \frac{\mathcal{F}}{\mathcal{F}'}\Big) &\leqslant \frac{1}{q}N\Big(r, \frac{\mathcal{F}'}{\mathcal{F}}\Big) + S(r, f)\\ &\leqslant \frac{1}{q}\bar{N}(r, \infty; \mathcal{F}) + \frac{1}{q}\bar{N}(r, 0; \mathcal{F}) + S(r, f)\\ &\leqslant \frac{1}{q}\bar{N}(r, \infty; f) + \frac{1}{q}\bar{N}(r, 0; P(f)) + S(r, f).\end{aligned}$$

In view of Lemmas 4.1 and 4.5, we obtain

$$\begin{aligned}(mn-1)\bar{N}(r, \infty; f) &\leqslant \Big(\frac{1}{q} + \chi_{_p}\Big)\bar{N}(r, 0; P(f)) + \frac{1}{q}\bar{N}(r, \infty; f) + \bar{N}(r, 0; \mathcal{G}) + S(r, f)\\ &\leqslant \Big(\frac{1}{q} + \chi_{_p}\Big)\bar{N}(r, 0; P(f)) + \frac{1}{q}\bar{N}(r, \infty; f) + N_{k+1}(r, 0; P(f)) + k\,\bar{N}(r, \infty; f) + S(r, f),\end{aligned}$$

which implies that

$$\Big(mn - 1 - k - \frac{1}{q}\Big)\bar{N}(r, \infty; f) \leqslant \Big(k + 1 + \chi_{_p} + \frac{1}{q}\Big)\bar{N}(r, 0; P(f)) + S(r, f).$$

Suppose that $q = 0$. A simple computation shows that

$$N(r, \infty; \mathcal{V}) \leqslant \chi_{_p}\bar{N}(r, 0; \mathcal{F}) + \bar{N}(r, 0; \mathcal{G}) + \bar{N}_L(r, 1; \mathcal{F}) + \bar{N}_L(r, 1; \mathcal{G}) + S(r, f),$$

where

$$\begin{aligned}\bar{N}_L(r,1;\mathcal{F}) &\leqslant N\Big(r,\frac{\mathcal{F}}{\mathcal{F}'}\Big) \leqslant N\Big(r,\frac{\mathcal{F}'}{\mathcal{F}}\Big)+S(r,f)\\ &\leqslant \bar{N}(r,\infty;\mathcal{F})+\bar{N}(r,0;\mathcal{F})+S(r,f)\\ &\leqslant \bar{N}(r,\infty;f)+\bar{N}(r,0;P(f))+S(r,f).\end{aligned}$$

Similarly, applying Lemma 4.1 and using a similar argument as above, we obtain

$$\begin{aligned}\bar{N}_L(r,1;\mathcal{G}) &\leqslant \bar{N}(r,\infty;\mathcal{G})+\bar{N}(r,0;\mathcal{G})+S(r,f)\\ &\leqslant (k+1)\bar{N}(r,\infty;f)+(k+1)\bar{N}(r,0;P(f))+S(r,f).\end{aligned}$$

By Lemmas 4.1 and 4.5, we easily obtain

$$(mn-1)\bar{N}(r,\infty;f) \leqslant \{2k+\chi_p+3\}\bar{N}(r,0;P(f))+2(k+1)\bar{N}(r,\infty;f)+S(r,f),$$

which implies that

$$(mn-2k-3)\bar{N}(r,\infty;f) \leqslant (2k+\chi_p+3)\bar{N}(r,0;P(f))+S(r,f). \qquad \square$$

LEMMA 4.9. *Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{F}_1$, $\mathcal{G}_1$ are as in Lemma 4.4 and $\mathcal{U}$ as in* (4.3). *If $l>k$ and $E_{\mathcal{F}}(1,q)=E_{\mathcal{G}}(1;q)$ and $\mathcal{U}\not\equiv 0$, then the following hold:*

(i) *if $q\geqslant 2$, then*

$$\Big((l+i-k)m-1-\frac{1}{q}\Big)\bar{N}(r,0;P(f)) \leqslant \Big(1+\frac{1}{q}\Big)\bar{N}(r,\infty;f)+S(r,f).$$

(ii) *if $q=0$, then*

$$((l+i-k)m-k-3)\bar{N}(r,0;P(f)) \leqslant (k+3)\bar{N}(r,\infty;f)+S(r,f),$$

for some $i\in\{0,1,\dots,p\}$.

PROOF. Let $q\geqslant 2$. A simple computation now shows that

$$\begin{aligned}N(r,\infty;\mathcal{U}) &\leqslant \bar{N}(r,\infty;\mathcal{F})+\bar{N}_{(q+1}(r,1;\mathcal{F})+S(r,f)\\ &\leqslant \bar{N}(r,\infty;f)+\Big(\frac{1}{q}\bar{N}(r,0;P(f))+\frac{1}{q}\bar{N}(r,\infty;f)\Big)+S(r,f)\\ &\leqslant \frac{1}{q}\bar{N}(r,0;P(f))+\Big(1+\frac{1}{q}\Big)\bar{N}(r,\infty;f)+S(r,f).\end{aligned}$$

Then applying Lemma 4.7, we obtain

$$((l+i-k)m-1)\bar{N}(r,0;P(f)) \leqslant \frac{1}{q}\bar{N}(r,0;P(f))+\Big(1+\frac{1}{q}\Big)\bar{N}(r,\infty;f)+S(r,f).$$

which turns out that

$$\Big((l+i-k)m-1-\frac{1}{q}\Big)\bar{N}(r,0;P(f)) \leqslant \Big(1+\frac{1}{q}\Big)\bar{N}(r,\infty;f)+S(r,f).$$

Let $q = 0$. Applying Lemmas 4.1, 4.7 and following the similar argument as used in the proof of Lemma 4.8, we obtain

$$\begin{aligned} N(r,\infty;\mathcal{U}) &\leqslant \bar{N}(r,\infty;\mathcal{F}) + \bar{N}_L(r,1;\mathcal{F}) + \bar{N}_L(r,1;\mathcal{G}) + S(r,f) \\ &\leqslant \bar{N}(r,\infty;f) + (\bar{N}(r,0;P(f)) + \bar{N}(r,\infty;f)) + \big((k+1)\bar{N}(r,\infty;f) \\ &\qquad + (k+1)\bar{N}(r,0;P(f))\big) + S(r,f), \end{aligned}$$

which implies that

$$((l+i-k)m-k-3)\bar{N}(r,0;P(f)) \leqslant (k+3)\bar{N}(r,\infty;f) + S(r,f).$$

This completes the proof. □

Lemma 4.10. [**19**] *If $\mathcal{F}$ and $\mathcal{G}$ be two nonconstant meromorphic functions such that they share $(1,0)$ and $\mathcal{H} \not\equiv 0$, then $N_E^{1)}(r,1,\mathcal{F}) \leqslant N(r,\mathcal{H}) + S(r,\mathcal{F}) + S(r,\mathcal{G})$.*

Lemma 4.11. [**4**] *Let $\mathcal{F}$ and $\mathcal{G}$ be two nonconstant meromorphic functions sharing $(1,m)$ where $0 \leqslant m < \infty$. Then*

$$\begin{aligned} &\bar{N}(r,1;\mathcal{F}) + \bar{N}(r,1;\mathcal{G}) - N_E^{1)}(r,1,\mathcal{F}) \\ &\qquad + \Big(m - \frac{1}{2}\Big)\bar{N}_*(r,1;\mathcal{F},\mathcal{G}) \leqslant \frac{1}{2}[N(r,1;\mathcal{F}) + N(r,1;\mathcal{G})]. \end{aligned}$$

Lemma 4.12. [**14**] *Let $\mathcal{F}$ and $\mathcal{G}$ be two nonconstant meromorphic function sharing $(1,0)$, $(\infty,0)$ and $\mathcal{H} \not\equiv 0$. Then*

$$\begin{aligned} N(r,\mathcal{H}) &\leqslant \bar{N}(r,0;\mathcal{F}| \geqslant 2) + \bar{N}(r,0;\mathcal{G}| \geqslant 2) + \bar{N}_*(r,\infty;\mathcal{F},\mathcal{G}) \\ &\quad + \bar{N}_*(r,1;\mathcal{F},\mathcal{G}) + \bar{N}_0(r,0;\mathcal{F}') + \bar{N}_0(r,0;\mathcal{G}'), \end{aligned}$$

where $\bar{N}_0(r,0;\mathcal{F}')$ is the reduced counting function of the zeros of $\mathcal{F}'$ which are not the zeros of $\mathcal{F}(\mathcal{F}-1)$ and similarly, $\bar{N}_0(r,0;\mathcal{G}')$ is defined.

Lemma 4.13. *Let $\mathcal{F}$ and $\mathcal{G}$ be two nonconstant meromorphic functions such that $E_{\mathcal{F}}(1,q) = E_{\mathcal{G}}(1,q)$ and $\mathcal{H} \not\equiv 0$ and $q = 0$, then*

$$\begin{aligned} T(r,\mathcal{F}) + T(r,\mathcal{G}) &\leqslant 2N_2(r,0;\mathcal{F}) + 2N_2(r,0;\mathcal{G}) + 6\bar{N}(r,\infty;\mathcal{F}) \\ &\qquad + 3\bar{N}_L(r,1;\mathcal{F}) + 3\bar{N}_L(r,1;\mathcal{G}) + S(r,\mathcal{F}). \end{aligned}$$

Proof. Since $S(r,\mathcal{F}) = S(r,\mathcal{G})$, the Lemma can be proved by using Lemmas 4.10, 4.11 and 4.12. Hence we omit the details. □

Lemma 4.14. [**2**] *If $\mathcal{F}$ and $\mathcal{G}$ are two nonconstant meromorphic functions sharing $(1,2)$ and (∞,k), where $0 \leqslant k \leqslant \infty$, then one of the following two cases holds:*

(i) $T(r,\mathcal{F}) + T(r,\mathcal{G}) \leqslant 2[N_2(r,0;\mathcal{F}) + N_2(r,0;\mathcal{G}) + \bar{N}(r,\infty;\mathcal{F}) + \bar{N}(r,\infty;\mathcal{G}) + \bar{N}_*(r,\infty;\mathcal{F},\mathcal{G}) + S(r,\mathcal{F}) + S(r,\mathcal{G})]$,
(ii) $\mathcal{F} \equiv \mathcal{G}$,
(iii) $\mathcal{F}\mathcal{G} \equiv 1$.

LEMMA 4.15. *Let $\mathcal{F}$ and $\mathcal{G}$ be two nonconstant meromorphic functions such that $E_{\mathcal{F}}(1,q) = E_{\mathcal{G}}(1,q)$ and $\mathcal{H} \not\equiv 0$ and $q \geqslant 2$, then*

$$T(r,\mathcal{F}) + T(r,\mathcal{G}) \leqslant 2N_2(r,0;\mathcal{F}) + 2N_2(r,0;\mathcal{G}) + 6\bar{N}(r,\infty;\mathcal{F}) + S(r,\mathcal{F}).$$

PROOF. Since $\mathcal{F}$ and $\mathcal{G}$ share $(1,q)$ where $q \geqslant 2$, hence it is easy to see that $\mathcal{F}$ and $\mathcal{G}$ share $(1,2)$. Now, the Lemma can be easily obtained using Lemma 4.14. □

LEMMA 4.16. *Let $\mathcal{H}$ be given by (4.1) and $\mathcal{F}, \mathcal{G}, \mathcal{F}_1$ and $\mathcal{G}_1$ be given by Lemma 4.4. If n, m, l and k are positive integers such that $l > k$ and*

$$\bar{N}(r,\infty;f) = N(r,0;P(f)) = S(r,f)$$

and $\mathcal{H} \equiv 0$, then $P(f) \equiv t\ [P(f)]^{(k)}$ i.e., $f_1^l Q(f_1) \equiv t\ [f_1^l Q(f_1)]^{(k)}$, where $t^m = 1$ and f assumes the form $f(z) = ce^{\frac{\lambda}{l+i}z} + d_i$, for some $i \in \{0,1,\ldots,p\}$ and c is a nonzero constant and $\lambda^{mk} = 1$.

PROOF. Since $\mathcal{H} \equiv 0$, by integration, we obtain

$$\frac{1}{\mathcal{F}-1} \equiv \frac{\mathcal{C}}{\mathcal{G}-1} + \mathcal{D}, \tag{4.6}$$

where $\mathcal{C}(\neq 0)$ and $\mathcal{D}$ are constants. It follows from (4.6) that $\mathcal{G} \equiv \frac{(\mathcal{D}-\mathcal{C})\mathcal{F}+(\mathcal{C}-\mathcal{D}-1)}{\mathcal{D}\mathcal{F}-(\mathcal{D}+1)}$, which can be written as

$$\mathcal{G}_1^m \equiv \frac{(\mathcal{D}-\mathcal{C})\mathcal{F}_1^m + (\mathcal{C}-\mathcal{D}-1)}{\mathcal{D}\mathcal{F}_1^m - (\mathcal{D}+1)}. \tag{4.7}$$

We now discuss the following possible cases.

CASE 1. Let $\mathcal{D} \neq 0, -1$. Therefore, it follows from (4.7) that

$$\bar{N}\Big(r, \frac{\mathcal{D}+1}{\mathcal{D}}; \mathcal{F}_1^m\Big) = \bar{N}(r,\infty;\mathcal{G}_1^m).$$

By the Second Fundamental Theorem and in view of $S(r,\mathcal{F}) = S(r,f)$, a simple computation shows that

$$\begin{aligned} mn\,T(r,f) &= T(r,\mathcal{F}_1^m) + S(r,f) \\ &\leqslant \bar{N}(r,\infty;\mathcal{F}_1^m) + \bar{N}(r,0;\mathcal{F}_1^m) + \bar{N}\Big(r, \frac{\mathcal{D}+1}{\mathcal{D}}; \mathcal{F}_1^m\Big) + S(r,f) \\ &\leqslant \bar{N}(r,\infty;f) + \bar{N}(r,0;P(f)) + \bar{N}(r,\infty;\mathcal{G}_1^m) + S(r,f) \leqslant S(r,f), \end{aligned}$$

which is not possible.

CASE 2. Suppose $\mathcal{D} = 0$. Then from (4.7), it is easy to see that

$$\bar{N}\Big(r, \frac{\mathcal{C}-1}{\mathcal{C}}; \mathcal{F}_1^m\Big) = \bar{N}(r,0;\mathcal{G}_1^m).$$

Subcase 2.1. Let $\mathcal{C} \neq 1$. By the Second Fundamental Theorem and using Lemma 4.1, we obtain

$$\begin{aligned} mn\,T(r,f) &= T(r,\mathcal{F}_1^m) + S(r,f) \\ &\leqslant \bar{N}(r,\infty;\mathcal{F}_1^m) + \bar{N}(r,0;\mathcal{F}_1^m) + \bar{N}\Big(r,\frac{\mathcal{C}-1}{\mathcal{C}};\mathcal{F}_1^m\Big) + S(r,f) \\ &\leqslant \bar{N}(r,\infty;f) + \bar{N}(r,0;P(f)) + \bar{N}(r,0;\mathcal{G}_1^m) + S(r,f) \\ &\leqslant (k+1)N(r,0;P(f)) + (k+1)\bar{N}(r,\infty;f) + S(r,f) \leqslant S(r,f), \end{aligned}$$

which is not possible.

Subcase 2.2. Let $\mathcal{C} = 1$. Then it is easy to see that $\mathcal{F}_1^m \equiv \mathcal{G}_1^m$ and this can be written as $P(f) \equiv t\,[P(f)]^{(k)}$ .e., $f_1^l Q(f_1) \equiv t\,[f_1^l Q(f_1)]^{(k)}$. By the same argument used in Case 2 of Lemma 4.4, it is easy to see that f assumes the form

$$f(z) = ce^{\frac{\lambda}{l+i}z} + d_l,$$

where $c(\neq 0), d_l \in \mathbb{C}$ and $\lambda^{mk} = 1$.

Case 3. Let $\mathcal{D} = -1$, then from (4.7), we obtain

$$\mathcal{G}_1^m \equiv \frac{(\mathcal{C}+1)\mathcal{F}_1^m - \mathcal{C}}{\mathcal{F}_1^m}. \tag{4.8}$$

Following Case 2, it is easy to show that $\mathcal{C} = -1$. Therefore, from (4.8) we obtain $\mathcal{F}_1^m\mathcal{G}_1^m \equiv 1$ which turns out that $P(f)[P(f)]^{(k)} \equiv t\ a^2$, where t is a constant satisfying $t^m = 1$.

Since $\bar{N}(r,\infty;f) = S(r,f) = N(r,0;P(f))$, a simple computation shows that

$$\begin{aligned} 2n\,T(r,f) = 2T\Big(r,\frac{P(f)}{a}\Big) &= T\Big(r,\frac{ta^2}{(P(f))^2}\Big) + O(1) \\ &\leqslant T\Big(r,\frac{[P(f)]^{(k)}}{P(f)}\Big) + O(1) \\ &\leqslant m\Big(r,\frac{[P(f)]^{(k)}}{P(f)}\Big) + N\Big(r,\frac{[P(f)]^{(k)}}{P(f)}\Big) + O(1) \\ &\leqslant N(r,\infty;[P(f)]^{(k)}) + N(r,0;P(f))) + O(1) \\ &\leqslant \bar{N}(r,\infty;f) + N(r,0;P(f)) + O(1) \\ &\leqslant S(r,f), \end{aligned}$$

which is not possible. □

5. Proof of the main results

Proof of Theorem 2.1. Let $\mathcal{F}_1 = \frac{P(f)}{a(z)} = \frac{f_1^l Q(f_1)}{a(z)}$ and $\mathcal{G}_1 = \frac{[P(f)]^{(k)}}{a(z)} = \frac{[f_1^l Q(f_1)]^{(k)}}{a(z)}$ and $\mathcal{F} = \mathcal{F}_1^m$, $\mathcal{G} = \mathcal{G}_1^m$, where f and hence $f_1 = f - d_l$ is a nonconstant meromorphic function. We discuss here the following cases.

Case 1. If $\mathcal{UV} \equiv 0$, then by using Lemmas 4.4 and 4.6, we obtain the conclusions of Theorem 2.1.

Case 2. Let $\mathcal{U}\mathcal{V} \not\equiv 0$, then from the assumption of Theorem 2.1, we see that $E_{\mathcal{F}}(1,q) = E_{\mathcal{G}}(1,q)$.

Subcase 2.1. When $q \geqslant 2$, then by using Lemmas 4.8 and 4.9, we obtain

$$\begin{aligned}(5.1)\quad &\Big(mn-1-k-\frac{1}{q}\Big)\Big((l+i-k)m-1-\frac{1}{q}\Big)\bar{N}(r,\infty;f)\\ &\qquad\leqslant \Big(k+1+\chi_{p}+\frac{1}{q}\Big)\Big(1+\frac{1}{q}\Big)\bar{N}(r,\infty;f)+S(r,f),\end{aligned}$$

$$\begin{aligned}(5.2)\quad &\Big(mn-1-k-\frac{1}{q}\Big)\Big((l+i-k)m-1-\frac{1}{q}\Big)\bar{N}(r,0;P(f))\\ &\qquad\leqslant \Big(k+1+\chi_{p}+\frac{1}{q}\Big)\Big(1+\frac{1}{q}\Big)\bar{N}(r,0;P(f)))+S(r,f).\end{aligned}$$

Therefore, it follows from (5.1) and (5.2) that

$$((mn-\gamma_{0}^{1})(mn-\gamma_{p-i}^{m})-C)\bar{N}(r,\infty;f)\leqslant S(r,f), \tag{5.3}$$

$$((mn-\gamma_{0}^{1})(mn-\gamma_{p-i}^{m})-C)\bar{N}(r,0;P(f))\leqslant S(r,f), \tag{5.4}$$

where $\gamma_{p-i}^{m}=(p+k-i)m+1+\frac{1}{q}$ and $C=\big(k+1+\chi_{p}+\frac{1}{q}\big)\big(1+\frac{1}{q}\big)$.

It is easy to see that

$$\begin{aligned}(mn-\gamma_{p-i}^{m})(mn-\gamma_{0}^{1})-C &= m^{2}n^{2}-m(\gamma_{p-i}^{m}+\gamma_{0}^{1})n+(\gamma_{0}^{1}\gamma_{p-i}^{m}-C)\\ &= m^{2}\Big(n-\frac{\gamma_{p-i}^{m}+\gamma_{0}^{1}+\sqrt{(\gamma_{p-i}^{m}-\gamma_{0}^{1})^{2}+4C}}{2m}\Big)\\ &\quad\times\Big(n-\frac{\gamma_{p-i}^{m}+\gamma_{0}^{1}-\sqrt{(\gamma_{p-i}^{m}-\gamma_{0}^{1})^{2}+4C}}{2m}\Big).\end{aligned}$$

In view of the assumptions of Theorem 2.1, it follows from (5.3) and (5.4) that

$$\bar{N}(r,0;P(f))=S(r,f)=\bar{N}(r,\infty;f). \tag{5.5}$$

We consider the following two cases:

Case 2.1.1. Let $\mathcal{H}\not\equiv 0$. Using Lemmas 4.13, 4.15 and (5.5), we obtain $T(r,f)=S(r,f)$, which is a contradiction.

Case 2.1.2. Let $\mathcal{H}\equiv 0$. Then from Lemma 4.16, we obtain the conclusion of Theorem 2.1.

Subcase 2.2. When $q=0$, using Lemmas 4.8 and 4.9, a simple computation shows that

$$\begin{aligned}(5.6)\quad &(mn-2k-3)((l+i-k)m-k-3)\bar{N}(r,\infty;f)\\ &\qquad\leqslant (2k+\chi_{p}+3)(k+3)\bar{N}(r,\infty;f)+S(r,f),\end{aligned}$$

$$\begin{aligned}(5.7)\quad &(mn-2k-3)((l+i-k)m-k-3)\bar{N}(r,0;P(f))\\ &\qquad\leqslant (2k+\chi_{p}+3)(k+3)\bar{N}(r,0;P(f))+S(r,f).\end{aligned}$$

In view of equations (5.6) and (5.7) and following Subcase 2.1, rest of the proof can be carried out, hence we omit the details. □

Proof of Theorem 2.2. Since f is an entire function, we have $N(r,\infty;f)=S(r,f)$. If $\mathcal{U}\equiv 0$, then using Lemma 4.6, we obtain the conclusion of Theorem 2.2. If $\mathcal{U}\not\equiv 0$, then using Lemma 4.9 for $q\geqslant 2$, we obtain

$$(mn-\delta^m_{p-i})\bar{N}(r,0;P(f))\leqslant S(r,f).$$

Since $n>\frac{(p+k-i)mq+q+1}{mq}$, we arrive at a contradiction.

When $q=0$, applying Lemma 4.9, we obtain

$$(mn-\delta^m_{p-i})\bar{N}(r,0;P(f))\leqslant S(r,f),$$

which is a contradiction since $n>\frac{(p+k-i)m+k+3}{m}$.

Thus, $\bar{N}(r,0;P(f))=S(r,f)$. Rest of the proof follows from Cases 1 and 2 of the proof of Theorem 2.1. □

6. Some questions

In the study of sharing set problem by meromorphic functions, reducing cardinality of the set is the main trend of the research. Hence, it is our utmost interest to see the possible answer of the following question.

Question 6.1. *Can the lower bound of n be further reduced in Theorems* 2.1 *and* 2.2*?*

To prove the main results of this paper, we have used Lemma 4.13 for $q=0$ and Lemma 4.15 for $q\geqslant 2$. But we are unable to prove the main results for $q=1$. Hence we have utmost interests to see the possible answer of the following question.

Question 6.2. *Can we prove Theorems* 2.1 *and* 2.2 *for $q=1$?*

Acknowledgment. The authors thank the referees for their helpful suggestions and insightful comments to improve the exposition of the paper. The first author is supported PRG of University of Kalyani.

References

1. A. Al-Khaladi, *On meromorphic functions that share one value with their derivative*, Analysis **25** (2005), 131–140.
2. A. Banerjee, *Meromorphic functions sharing two sets*, Czech. Math. J. **57**(132) (2007), 1199–1214.
3. ______, *Weighted sharing of a small function by a meromorphic function and its derivative*, Comput. Math. Appl. **53** (2007), 1750–1761.
4. ______, *Uniqueness of meromorphic functions sharing two sets with finite weight II*, Tamkang J. Math. **41**(4) (2010), 379–392.
5. A. Banerjee, M. B. Ahamed, *Uniqueness of a polynomial and a differential monomial sharing a small function*, An. Univ. Vest Timiș., Ser. Mat.-Inform. **54**(1) (2016), 55–71.
6. A. Banerjee, S. Majumder, *On a generalization of a result of Zhang and Yang*, Mat. Vesn. **68**(3) (2016), 155–163.
7. R. Brück, *On entire functions which share one value CM with their first derivative*, Result. Math. **30** (1996), 21–24.

8. Z. X. Chen, K. H. Shon, *On conjecture of R. Brück concerning entire function sharing one value CM with its derivative*, Taiwanese J. Math. **8** (2004), 235–244.
9. G. G. Gundersen, *Meromorphic functions that share finite values with their derivative*, J. Math. Anal. Appl. **75** (1980), 441–446.
10. G. G. Gundersen, L. Z. Yang, *Entire functions that share one value with one or two of their derivatives*, J. Math. Anal. Appl. **223** (1998), 88–95.
11. W. K. Hayman, *Meromorphic function*, Clarendon Press, Oxford, 1964.
12. I. Lahiri, *Weighted sharing and uniqueness of meromorphic functions*, Nagoya Math. J. **16**(1) (2001), 193–206.
13. ______, *Weighted value sharing and uniqueness of meromorphic functions*, Complex Variables, Theory Appl. **46** (2001), 241–253.
14. I. Lahiri, A. Banerjee, *Weighted sharing of two sets*, Kyungpook Math. J. **46** (2006), 79–87.
15. E. Mues, N. Steinmetz, *Meromorphic function, die mit iher ersten und zweiten Ablenitung einen endlichen Wert teilen*, Complex Variables, Theory Appl. **6** (1986), 51–71.
16. H. Y. Xu, C. F. Yi, H. Wang, *On a conjecture of R. Brück concerning meromorphic function sharing small functions*, Rev. Mat. Teor. Apl. **23**(1) (2016), 291–308.
17. L. Z. Yang, J. L Zhang, *Non-existance of meromorphic functions of Fermat type functional equation*, Aequationes Math. **76**(2008), 140–150.
18. H. X. Yi, *Unicity theorems for meromorphic function whose n-th derivatives share the same 1-points*, Complex Variables, Theory Appl. **34** (1997), 421–436.
19. ______, *Meromorphic functions that share one or two values II*, Kodai Math. J. **22**(1999), 264–272.
20. H. X. Yi, C. C. Yang, *Uniqueness Theory of Meromorphic Functions*, Science Press/Kluwer, Beijing/ New York, 1995/2003.
21. Q. C. Zhang, *Meromorphic function that shares one entire function with its derivative*, J. Inequal. Pure Appl. Math. **6** (2005), 116.
22. J. L. Zhang, L. Z. Yang, *A power of a meromorphic function sharing a small function with its derivative*, Ann. Acad. Sci. Fenn. Math. **34** (2009), 249–260.
23. J. H. Zheng, S. P. Wang, *On unicity properties of meromorphic functions and their derivatives*, Adv. Math., Beijing **21** (1992), 334–341.

Department of Mathematics
University of Kalyani
West Bengal
India
abanerjee_kal@yahoo.co.in, abanerjeekal@gmail.com

(Received 24 02 2017)
(Revised 01 07 2021)

Department of Mathematics
Kalipada Ghosh Tarai Mahavidyalaya
West Bengal
India

Department of Mathematics
Jadavpur University
West Bengal
Kolkata
India
mbahamed.math@jadavpuruniversity.in