PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 112(126) (2022), 143–150 DOI: https://doi.org/10.2298/PIM2226143P

ASYMPTOTIC DOUBLE SUBSERIES

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Abstract. In 1964 Šalát presented a notion of asymptotic density of single dimensional subseries. Using this notion he presented a series of theorems similar to the following. If d_n be a decreasing sequence such that $\liminf_n nd_n > 0$ and let the subseries $(x) = \sum_{k=1}^{\infty} \epsilon_k(x)d_k$ of the series $\sum_{k=1}^{\infty} d_k$ be convergent, then $p(x) = \lim_n \frac{p(n,x)}{n} = 0$. Following Šalát's patten, we present a notion of double subseries and a natural variation of Šalát's theorem.

1. Definitions, Notations, and Preliminary Results

The main goal of this paper is introduce the notion of double subseries. To that end we begin with the following notions of convergence in the Pringshiem sense and subsequence of a double sequence.

Definition 1.1 (Pringsheim, 1900). A double sequence $x = \{x_{k,l}\}$ has a *Pringsheim limit* L (denoted by P-$\lim x = L$) provided that, given an $\epsilon > 0$ there exists an $N \in \mathbf{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > N$. Such an $\{x\}$ is described more briefly as "P-convergent".

In the later stage different classes of double sequences were introduced and studied in [**1**, **7**–**11**].

Definition 1.2. A double sequence is *totally null* provided that its Pringsheim, rows, and columns limits are all zero.

Following this notion of convergent in the Pringsheim sense the author introduce the following notion of subsequence of a double sequence.

Definition 1.3 (Patterson, 2000). A double sequence $\{y\}$ is a *double subsequence* of $\{x\}$ provided that there exist increasing index sequences $\{n_j\}$ and $\{k_j\}$

2020 *Mathematics Subject Classification:* Primary 40B05, Secondary 40C05.
Key words and phrases: double sequences, double subseries, P-convergent.
Communicated by Gradimir Milovanović.

such that, if $\{x_j\} = \{x_{n_j,k_j}\}$, then $\{y\}$ is formed by

$$\begin{matrix} x_1 & x_2 & x_5 & x_{10} \\ x_4 & x_3 & x_6 & - \\ x_9 & x_8 & x_7 & - \\ - & - & - & - \end{matrix}.$$

Using this notion the author present a series of natural characterization of double sequences in [**4**]. In a similar manner the goal includes the introduce of new notion of double subseries and the presentation of natural variation and implications of the following theorem. If d_n be a decreasing sequence such that $\liminf_n nd_n > 0$ and let the subseries $(x) = \sum_{k=1}^{\infty} \epsilon_k(x)d_k$ of the series $\sum_{k=1}^{\infty} d_k$ be convergent then $\lim_n \frac{1}{n} \sum_{k=1}^{n} \epsilon_k(x) = 0$. It shall be observed that in 2012 Ganguly and Dafadar presented a definition of double subseries similar to the one presented here and two theorems that are similar to the main theorems below without proof in [**3**]. To that end let us consider the following double series

$$S = \sum_{m,n=1,1}^{\infty,\infty} a_{m,n} = \begin{bmatrix} a_{1,1} & + & a_{1,2} & + & a_{1,3} & +\cdots \\ a_{2,1} & + & a_{2,2} & + & a_{2,3} & +\cdots \\ \vdots & + & \vdots & + & \vdots & +\cdots \\ \vdots & + & \vdots & + & \vdots & +\cdots \end{bmatrix} \tag{1.1}$$

and let

$$\begin{matrix} d_{1,1} & < & d_{1,2} & < & d_{1,3} & < & \cdots & < & d_{1,n} & < & \cdots \\ d_{2,1} & < & d_{2,2} & < & d_{2,3} & < & \cdots & < & d_{2,n} & < & \cdots \\ d_{3,1} & < & d_{3,2} & < & d_{3,3} & < & \cdots & < & d_{3,n} & < & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \cdots \end{matrix}$$

be an increasing sequence of natural numbers in each row and define $k_1^j = d_{j,1}$, $k_2^j = d_{j,2}$, $k_3^j = d_{j,3}, \cdots$ and $l_1^i = d_{1,i}$, $l_2^i = d_{2,i}$, $l_3^i = d_{3,i}, \cdots$, where $i, j = 1, 2, 3, 4, \ldots$. The double series

$$\sum_{m,n=1,1}^{\infty,\infty} a_{k_m^n, l_n^m} = \begin{bmatrix} a_{k_1^1,l_1^1} & + & a_{k_1^2,l_2^1} & + & a_{k_1^3,l_3^1} & + & \cdots & + & a_{k_1^n,l_n^1} & +\cdots \\ a_{k_2^1,l_1^2} & + & a_{k_2^2,l_2^2} & + & a_{k_2^3,l_3^2} & + & \cdots & + & a_{k_2^n,l_n^2} & +\cdots \\ \vdots & + & \vdots & + & \vdots & + & \cdots & + & \vdots & +\cdots \\ \vdots & + & \vdots & + & \vdots & + & \cdots & + & \vdots & +\cdots \end{bmatrix}$$

is called the double subseries of the double series (1.1). Let us now consider a single dimensional sequence of numbers of in the interval $(0, 1]$ in their dyadic expansions with infinity many digits to, thus to each element x^k of the sequence of $\{x^k\}_{k=1}^{\infty}$ in $(0, 1]$, will yields following expression

$$x^k = \sum_{l=1}^{\infty} \alpha_{k,l}(x)2^{-l}. \tag{1.2}$$

where $\alpha_{k,l}(x) = 0$ or 1 and for infinite number of l with fixed k, $\alpha_{k,l}(x) = 1$. The following double infinite series can be associated with (1.2)

$$S_x = \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l}(x) a_{k,l}. \tag{1.3}$$

The method grants us to tools to transfer the set of all double series of a given series into $(0,1]$. Let us consider the subseries (1.3) of the series (1.1), and define it in the following manner

$$p(k,l,x) = \sum_{\delta,\delta'=1,1}^{k,l} \alpha_{\delta,\delta'}(x).$$

Then numbers

$$p_1(x) = \text{P-}\liminf_{k,l\to\infty} \frac{p(k,l,x)}{kl} \quad \text{and} \quad p_2(x) = \text{P-}\limsup_{k,l\to\infty} \frac{p(k,l,x)}{kl},$$

shall be called the Pringsheim lower and upper asymptotic density of the series (1.3) in (1.1). In addition if these limits exist we shall called the following number the Pringsheim asymptotic density of the series (1.3) in (1.1):

$$p(x) = \text{P-}\lim_{k,l\to\infty} \frac{p(k,l,x)}{kl}.$$

Let us now define lower and upper; row and column asymptotic densities and their corresponding row and column densities. The respective pair of numbers defined lower and upper row asymptotic densities and lower and upper column asymptotic densities,

$$p_1^r(x) = \liminf_{l\to\infty} \frac{p(k,l,x)}{l}, \text{ for each } k, \quad p_2^r(x) = \limsup_{l\to\infty} \frac{p(k,l,x)}{l}, \text{ for each } k,$$
$$p_1^c(x) = \liminf_{k\to\infty} \frac{p(k,l,x)}{k}, \text{ for each } l, \quad p_2^c(x) = \limsup_{k\to\infty} \frac{p(k,l,x)}{k}, \text{ for each } l.$$

Corresponding to the lower and upper column and row densities we are granted the following densities

$$p^r(x) = \lim_{l\to\infty} \frac{p(k,l,x)}{l}, \text{ for each } k; \quad p^c(x) = \lim_{k\to\infty} \frac{p(k,l,x)}{l}, \text{ for each } l.$$

It should be noted that $p_1^r(x), p_2^r(x), p^r(x), p_1^c(x), p_2^c(x), p^r(x), p_1(x), p_2(x), p(x)$ are all in $[0,1]$. The following is an example where the Pringsheim, row, and column densities all exist.

Example 1.1. If $a_{k,l} = \frac{1}{kl}$; $k,l = 1,2,3,4,5,\ldots$ and $S_x = \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l}(x) a_{k,l}$ is convergent in the Pringsheim, row-wise, and column-wise sense them

$$p_1^r(x), p_2^r(x), p^r(x), p_1^c(x), p_2^c(x), p^r(x), p_1(x), p_2(x), p(x)$$

are zero.

2. Main Results

THEOREM 2.1. *Let $\{a_{k,l}\}$ be a totally null double sequence and*

$$\text{P-}\lim_{k,l}\inf kla_{k,l} > 0.$$

Let the double subseries $S_x = \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l}(x)a_{k,l}$ of the double series $\sum_{k,l=1,1}^{\infty,\infty} a_{k,l}$ be P-convergent. Then

$$p(x) = \text{P-}\lim_{k,l} \frac{p(k,l,x)}{kl} > 0$$

implies that S_x is divergent.

PROOF. Let us assume $p_2(x) = \text{P-}\limsup_{k,l\to\infty} \frac{p(k,l,x)}{kl} > 0$, then there exist k and l such that $p(k,l,x) > \delta' kl$. In addition there exists $\delta > 0$ such that $kla_{k,l} \geqslant \delta$ for all $k,l = 1,2,3,4,\ldots$. In a manner to Šalát presentation in [**6**] we now consider arbitrary natural numbers K and L and let $\epsilon = \frac{1}{2}\delta'\delta$ and choose k_0 and l_0 such that $k_0 > K$ and $l_0 > L$, $p(k_0,l_0,x) > \delta' k_0 l_0$, and $KLa_{k_0,l_0} < \frac{1}{2}\delta'\delta = \epsilon$. Now it is clear for the k_0 and l_0 above we are granted the following estimation

$$\begin{aligned}
\sum_{r,s=K+1,L+1}^{k_0,l_0} \alpha_{r,s}(x)a_{r,s} &\geqslant a_{k_0,l_0} \sum_{r,s=K+1,L+1}^{k_0,l_0} a_{r,s}(x) \\
&= a_{k_0,l_0}\left[\sum_{r,s=1,1}^{k_0,l_0} a_{r,s}(x) - \sum_{r,s=1,1}^{K,L} a_{r,s}(x)\right] \\
&> a_{k_0,l_0}[\delta' k_0 l_0 - KL] \\
&> \epsilon.
\end{aligned}$$

Thus via the Cauchy-Bolzano theorem [**2**] the double series S_x is not convergent. □

The following corollaries are natural extension of the above theorem.

COROLLARY 2.1. *Let $\{a_{k,l}\}$ be a column null double sequence (i.e. each column is a null sequence) and $\liminf_{k\to\infty} ka_{k,l} > 0$ for each l. Let the double subseries $S_x = \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l}(x)a_{k,l}$ of the double series $\sum_{k,l=1,1}^{\infty,\infty} a_{k,l}$ be column convergent. Then*

$$p^c(x) = \lim_{k\to\infty} \frac{p(k,l,x)}{k} = 0, \text{ for each } l.$$

COROLLARY 2.2. *Let $\{a_{k,l}\}$ be a row null double sequence (i.e. each row is a null sequence) and $\liminf_{l\to\infty} la_{k,l} > 0$, for each k. Let the double subseries $S_x = \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l}(x)a_{k,l}$ of the double series $\sum_{k,l=1,1}^{\infty,\infty} a_{k,l}$ be row convergent. Then*

$$p^r(x) = \lim_{l\to\infty} \frac{p(k,l,x)}{l} = 0, \text{ for each } k.$$

That the conditions $d_{i,j} \geqslant d_{i,j+1}$, $d_{i,j} \geqslant d_{i+1,j}$, and $d_{i,j} \geqslant d_{i+1,j+1}$ in the above theorem are important, illustrates by the following example.

Example 2.1.

$$\sum_{k,l=1,1}^{\infty,\infty} a_{k,l} = \Big(\frac{1}{1\cdot 1}+\frac{1}{2\cdot 1}+\frac{1}{2^2\cdot 1}\Big)+\Big(\frac{1}{2\cdot 1}+\frac{1}{2^2\cdot 1}+\frac{1}{2^4\cdot 1}\Big)$$
$$+\Big(\frac{1}{3\cdot 1}+\frac{1}{2^5\cdot 1}+\frac{1}{2^6\cdot 1}\Big)+\Big(\frac{1}{4\cdot 1}+\frac{1}{2^7\cdot 1}+\frac{1}{2^8\cdot 1}\Big)$$
$$\vdots$$
$$+\Big(\frac{1}{n\cdot 1}+\frac{1}{2^{2n-1}\cdot 1}+\frac{1}{2^{2n}\cdot 1}\Big)$$
$$+\Big(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 2}+\frac{1}{2^2\cdot 2}\Big)+\Big(\frac{1}{2\cdot 2}+\frac{1}{2^2\cdot 2}+\frac{1}{2^4\cdot 1}\Big)$$
$$+\Big(\frac{1}{3\cdot 2}+\frac{1}{2^5\cdot 2}+\frac{1}{2^6\cdot 2}\Big)+\Big(\frac{1}{4\cdot 2}+\frac{1}{2^7\cdot 2}+\frac{1}{2^8\cdot 2}\Big)$$
$$\vdots$$
$$+\Big(\frac{1}{n\cdot 2}+\frac{1}{2^{2n-1}\cdot 2}+\frac{1}{2^{2n}\cdot 2}\Big)$$
$$\vdots$$
$$+\Big(\frac{1}{1\cdot 2^m}+\frac{1}{2\cdot 2^m}+\frac{1}{2^2\cdot 2^m}\Big)+\Big(\frac{1}{2\cdot 2^m}+\frac{1}{2^2\cdot 2^m}+\frac{1}{2^4\cdot 1}\Big)$$
$$+\Big(\frac{1}{3\cdot 2^m}+\frac{1}{2^5\cdot 2^m}+\frac{1}{2^6\cdot 2^m}\Big)+\Big(\frac{1}{4\cdot 2^m}+\frac{1}{2^7\cdot 2^m}+\frac{1}{2^8\cdot 2^m}\Big)$$
$$\vdots$$
$$+\Big(\frac{1}{n\cdot 2^m}+\frac{1}{2^{2n-1}\cdot 2^m}+\frac{1}{2^{2n}\cdot 2^m}\Big)$$
$$\vdots$$

Now clearly the double subseries $\sum_{i,j=1,1}^{\infty,\infty}\frac{1}{2^{i+j}}$ is a P-convergent sequence and its asymptotic density in the original series is $\frac{4}{9}$.

We will also establish by example that the condition P-$\liminf_{k,l} kla_{k,l}>0$ can to be replace by the weaker condition $\sum_{k,l=1,1}^{\infty,\infty} a_{k,l}=+\infty$.

Example 2.2. If we let $m,n=1,2,3,4,\ldots$

$$a_{m^{m+k_m},n^{n+l_n}}=\frac{1}{m^{m+2}n^{n+2}},$$

where k_m and l_n are integers such that $0\leqslant k_m<(m+1)^{m+1}-m^m$ and $0\leqslant l_n<(n+1)^{n+1}-n^n$.

For a double sequence with elements defined as follow $k_m=2m^m$ and $l_n=2n^n$ we granted that $P(2m^m,2n^n,x)\geqslant m^m n^n$ which implies that $p_2(x)\geqslant\frac{1}{4}$ and with $k_m=(m+1)^{m+1}$ and $l_n=(n+1)^{n+1}$ and consider

$$P((m+1)^{m+1}-1,(n+1)^{n+1}-1,x)$$

this implies that $p_1(x) = 0$. Thus Theorem 2.1 false.

THEOREM 2.2. *Let* $\sum_{k,l=1,1}^{\infty,\infty} a_{k,l} = \infty$*, let there exist* r *and* s *such that*

$$\begin{array}{ccccccccccc}
d_{\phi,\psi} & < & d_{\phi,\psi+1} & < & d_{\phi,\psi+2} & < & \cdots & < & d_{\phi,\psi+l} & < & \cdots \\
d_{\phi+1,\psi} & < & d_{\phi+1,\psi+1} & < & d_{\phi+1,\psi+2} & < & \cdots & < & d_{\phi+1,\psi+l} & < & \cdots \\
d_{\phi+2,\psi} & < & d_{\phi+2,\psi+1} & < & d_{\phi+2,\psi+2} & < & \cdots & < & d_{\phi+2,\psi+l} & < & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots. \\
d_{\phi+k,\psi} & < & d_{\phi+k,\psi+1} & < & d_{\phi+k,\psi+2} & < & \cdots & < & d_{\phi+k,\psi+l} & < & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots
\end{array}$$

Then

$$S_x = \sum_{k,l=1,1}^{\infty,\infty} \alpha_{k,l}(x) a_{k,l} < +\infty$$

where $\alpha_{k,l}(x)$ *is defined above. Then*

$$p_1(x) = \text{P-}\liminf_{k,l\to\infty} \frac{p(k,l,x)}{kl} = 0.$$

PROOF. Let us assume $p_1(x) > 0$. Then there exists $\delta > 0$ such that for all $k \geqslant r \geqslant \phi$ and $l \geqslant s \geqslant \psi$ with $r \geqslant 2$ and $s \geqslant 2$, respectively, and $p(k,l,x) \geqslant kl\delta > 0$. Then by the following variation of double Able's summation formula we obtain the following

$$\begin{aligned}
\sum_{i,j=r,s}^{r+\alpha',s+\beta'} \alpha_{i,j}(x) a_{i,j} = & -p(r-1,s-1,x)a_{r,s-1} + p(r,s-1,x)(a_{r,s-1} - a_{r+1,s-1}) \\
& + p(r+1,s-1,x)(a_{r+1,s-1} - a_{r+2,s-1}) \\
& \vdots \\
& + p(r+\alpha'-1,s-1,x)(a_{r+\alpha'-1,s-1} - a_{r+\alpha',s-1}) \\
& + -p(r-1,s,x)a_{r,s} + p(r,s,x)(a_{r,s} - a_{r+1,s}) \\
& + p(r+1,s,x)(a_{r+1,s} - a_{r+2,s}) \\
& \vdots \\
& + p(r+\alpha'-1,s,x)(a_{r+\alpha'-1,s} - a_{r+\alpha',s}) \\
& + -p(r-1,s+1,x)a_{r,s+1} + p(r,s+1,x)(a_{r,s+1} - a_{r+1,s+1}) \\
& + p(r+1,s+1,x)(a_{r+1,s+1} - a_{r+2,s+1}) \\
& \vdots \\
& + p(r+\alpha'-1,s+1,x)(a_{r+\alpha'-1,s+1} - a_{r+\alpha',s+1}) \\
& \vdots \\
& + -p(r-1,s+\beta'-1,x)a_{r,s+\beta'-1}
\end{aligned}$$

$$
\begin{aligned}
&+ p(r, s+\beta'-1, x)(a_{r,s+\beta'-1} - a_{r+1,s+\beta'-1}) \\
&+ p(r+1, s+\beta', x)(a_{r+1,s+\beta'} - a_{r+2,s+\beta'-1}) \\
&\vdots \\
&+ p(r+\alpha'-1, s+\beta', x)(a_{r+\alpha'-1,s+\beta'-1} - a_{r+\alpha',s+\beta'-1}) \\
\geqslant& -\big[p(r-1, s-1, x) + p(r-1, s, x) + p(r-1, s+1, x) \\
&\quad + \cdots + p(r-1, s+\beta'-1, x)\big] + \delta \sum_{i,j=r,s}^{r+\alpha',s+\beta'} a_{i,j}
\end{aligned}
$$

it is clear the that right side sum to ∞. Thus the theorem is proven. □

Similar to Corollary 2.1 and theorem 2.2 above we are granted the following:

Corollary 2.3. *Let $\sum_{l=1}^{\infty} a_{k,l} = \infty$ for each k, let there exist r such that*

$$
\begin{array}{ccccccccccc}
d_{r,s} & < & d_{r,s+1} & < & d_{r,s+2} & < & \cdots & < & d_{r,s+n} & < & \cdots \\
d_{r+1,s} & < & d_{r+1,s+1} & < & d_{r+1,s+2} & < & \cdots & < & d_{r+1,s+n} & < & \cdots \\
d_{r+2,s} & < & d_{r+2,s+1} & < & d_{r+2,s+2} & < & \cdots & < & d_{r+2,s+n} & < & \cdots. \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \vdots & \vdots & \cdots
\end{array}
$$

Then

$$S_x^k = \sum_{l=1}^{\infty} \alpha_{k,l}(x) a_{k,l} < +\infty$$

for each k where $\alpha_{k,l}(x)$ is defined above. Then

$$p_1^r(x) = \liminf_{l\to\infty} \frac{p(k,l,x)}{l} = 0$$

for each k.

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(Received 02 02 2021)

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