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INFINITE TRIPLE SERIES BY INTEGRALS

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ABSTRACT. A large class of infinite triple series are explicitly evaluated by computing definite double integrals. Several remarkable formulae are presented in terms of π and $\ln 2$.

1. Introduction and Outline

For $\lambda \in \mathbb{Z}$ and $\mu, \nu \in \mathbb{N}_0$, define the triple series

$$S_\lambda(\mu,\nu) = \sum_{n=0}^{\infty} \binom{-\lambda}{n} \Big[\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{n+i+\mu} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{n+j+\nu} \Big], \tag{1.1}$$

where the two sums with respect to i and j can also be expressed by Hurwitz–Lerch transcendent (also called Hurwitz–Lerch zeta function).

The aim of this article is to establish closed formulae for this series. Firstly, it is trivial to see that there holds the following symmetry:

$$S_\lambda(\mu,\nu) = S_\lambda(\nu,\mu). \tag{1.2}$$

Then for $m \in \mathbb{N}_0$ and $n \in \mathbb{Z}$, recall the binomial identity

$$\binom{-\lambda}{n} = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \binom{k-\lambda}{n+m},$$

which can be explained by the finite differences (i.e., the mth difference of a polynomial of degree $m+n$). When $m \leqslant \min\{\mu,\nu\}$, we can manipulate the series $S_\lambda(\mu,\nu)$ as follows:

$$\begin{aligned} S_\lambda(\mu,\nu) &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{n=0}^{\infty} \binom{k-\lambda}{n+m} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{n+i+\mu} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{n+j+\nu} \\ &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} \sum_{n=0}^{\infty} \binom{k-\lambda}{n} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{n+i+\mu-m} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{n+j+\nu-m}. \end{aligned}$$

This gives rise to the recurrence relation.

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Lemma 1.1. *Let m be an integer subject to $0 \leqslant m \leqslant \min\{\mu,\nu\}$. Then we have the following recurrence relation:*

$$S_\lambda(\mu,\nu) = \sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}S_{\lambda-k}(\mu-m,\nu-m).$$

In particular for $m=1$, we have the following simpler recurrence

$$S_\lambda(\mu,\nu) = S_{\lambda-1}(\mu-1,\nu-1) - S_\lambda(\mu-1,\nu-1). \tag{1.3}$$

By writing the two inner sums as definite integrals

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{n+i+\mu} &= \sum_{i=1}^{\infty}(-1)^{i-1}\int_0^1 x^{n+i+\mu-1}dx\\ &= \int_0^1 x^{n+\mu}\Big[\sum_{i=1}^{\infty}(-1)^{i-1}x^{i-1}\Big]dx\\ &= \int_0^1 \frac{x^{n+\mu}}{1+x}dx\end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{n+j+\nu} &= \sum_{j=1}^{\infty}(-1)^{j-1}\int_0^1 y^{n+j+\nu-1}dy\\ &= \int_0^1 y^{n+\nu}\Big[\sum_{j=1}^{\infty}(-1)^{j-1}y^{j-1}\Big]dy\\ &= \int_0^1 \frac{y^{n+\nu}}{1+y}dy,\end{aligned}$$

we can express $S_\lambda(\mu,\nu)$ as

$$\begin{aligned}S_\lambda(\mu,\nu) &= \sum_{n=0}^{\infty}\binom{-\lambda}{n}\int_0^1\frac{x^{n+\mu}}{1+x}dx\int_0^1\frac{y^{n+\nu}}{1+y}dy\\ &= \int_0^1\int_0^1\frac{x^\mu y^\nu\ dx\,dy}{(1+x)(1+y)}\sum_{n=0}^{\infty}\binom{-\lambda}{n}x^n y^n.\end{aligned}$$

Evaluating the inner sum by the binomial theorem

$$\sum_{n=0}^{\infty}\binom{-\lambda}{n}x^n y^n = (1+xy)^{-\lambda}$$

leads us to the following double integral representation.

Lemma 1.2. *Let λ,μ,ν be three integers with $\lambda\in\mathbb{Z}$ and $\mu,\nu\in\mathbb{N}_0$. The following integral representation holds:*

$$S_\lambda(\mu,\nu) = \int_0^1\int_0^1\frac{x^\mu y^\nu\ dx\,dy}{(1+x)(1+y)(1+xy)^\lambda}.$$

According to Lemmas 1.1 and 1.2, we have the following observations.

- **Case $\boldsymbol{\lambda = 0}$.** In this case, the corresponding double integral becomes a product of two single integrals

$$S_0(\mu,\nu) = \int_0^1\int_0^1 \frac{x^\mu y^\nu}{(1+x)(1+y)}dx\,dy = \int_0^1 \frac{x^\mu}{1+x}dx\int_0^1\frac{y^\nu}{1+y}dy.$$

Both integrals are computable since their integrands are simple rational functions.
- **Case $\boldsymbol{\lambda < 0}$.** According to the binomial theorem, by expanding $(1+xy)^{-\lambda}$ in Lemma 1.2, we can express $S_\lambda(\mu,\nu)$ in terms of $S_0(\mu',\nu')$.
- **Case $\boldsymbol{\lambda > 0}$.** In view of Lemma 1.1, we can reduce $S_\lambda(\mu,\nu)$ to a linear combination of $S_{\lambda'}(\mu',0)$ or $S_{\lambda'}(0,\nu')$, where $\lambda'\in\mathbb{Z}$ and $\mu',\nu'\in\mathbb{N}_0$. By symmetry, the only series remains to be evaluated is $S_\lambda(\mu,0)$ for $\lambda\in\mathbb{Z}$ and $\mu\in\mathbb{N}_0$.

By making use of the algebraic relation

$$x^\mu = (-1)^\mu + (1+x)\sum_{k=1}^{\mu}(-1)^{\mu-k}x^{k-1},$$

we can express

$$S_\lambda(\mu,0) = (-1)^\mu S_\lambda(0,0) + \sum_{k=1}^{\mu}(-1)^{\mu-k}T_\lambda(k-1),$$

where

$$T_\lambda(\mu) = \int_0^1\int_0^1 \frac{x^\mu\ dx\,dy}{(1+y)(1+xy)^\lambda}.$$

Therefore, the problem of evaluating $S_\lambda(\mu,\nu)$ is reduced to doing that for two series $S_\lambda(0,0)$ and $T_\lambda(\mu)$. They will separately be dealt with in the next two sections. Finally, the paper will end in Section 4, where a conclusive theorem will be presented together with several tabulated sample formulae. In order to assure the accuracy of computations, numerical tests for all the equations have been made by appropriately devised *Mathematica* commands.

Throughout the paper, the following notations will be utilized. For an indeterminate x and $n\in\mathbb{N}_0$, the rising factorials are defined by

$$(x)_0 = 1 \quad\text{and}\quad (x)_n = x(x+1)\cdots(x+n-1) \quad\text{for}\quad n\in\mathbb{N}.$$

For $n\in\mathbb{N}_0$, the skew harmonic numbers (cf. [**1**, **2**, **7**, **9**]) are defined by

$$\bar{H}_0 = 0 \quad\text{and}\quad \bar{H}_n = \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}.$$

Differently from classical harmonic numbers that can be treated by generating functions [**3**], partial fractions [**5**] and hypergeometric series [**4**, **6**], the following two known integral representations will be useful in this paper

$$\int_0^1 \frac{x^n}{1+x}dx = (-1)^n\big(\ln 2 - \bar{H}_n\big), \tag{1.4}$$

$$\int_0^1 y^n \ln(1+y)dy = (-1)^{n+1}\frac{\bar{H}_{n+1}}{n+1} + \begin{cases} 0, & n-\text{odd};\\ \dfrac{2\ln 2}{n+1}, & n-\text{even}. \end{cases} \tag{1.5}$$

2. Evaluation of $S_\lambda(0,0)$

Consider the difference

$$2S_{1+\lambda}(0,0) - S_\lambda(0,0) = \int_0^1\int_0^1 \frac{(1-xy)dx\,dy}{(1+x)(1+y)(1+xy)^{\lambda+1}}.$$

By applying the equation

$$1-xy = (1+x)+(1+y)-(1+x)(1+y)$$

and then the symmetry, we can reduce the double integral to a single one

$$\begin{aligned} 2S_{1+\lambda}(0,0) - S_\lambda(0,0) &= \int_0^1\int_0^1 \frac{dx\,dy}{(1+x)(1+xy)^{\lambda+1}}\\ &+ \int_0^1\int_0^1 \frac{dx\,dy}{(1+y)(1+xy)^{\lambda+1}} - \int_0^1\int_0^1 \frac{dx\,dy}{(1+xy)^{\lambda+1}}\\ &= 2\int_0^1\int_0^1 \frac{dx\,dy}{(1+x)(1+xy)^{\lambda+1}} - \int_0^1\int_0^1 \frac{dx\,dy}{(1+xy)^{\lambda+1}}\\ &= \int_0^1\int_0^1 \frac{(1-x)dx\,dy}{(1+x)(1+xy)^{\lambda+1}}\\ &= \int_0^1 \frac{(1-x)\big[(1+x)^\lambda - 1\big]}{\lambda x(1+x)^{\lambda+1}}dx. \end{aligned}$$

Now reformulating the integrand

$$\begin{aligned} \frac{(1-x)\big[(1+x)^\lambda - 1\big]}{\lambda x(1+x)^{\lambda+1}} &= \frac{(1-x)}{\lambda(1+x)^{\lambda+1}}\sum_{k=1}^{\lambda}(1+x)^{k-1}\\ &= \frac{1}{\lambda}\sum_{k=1}^{\lambda}\Big[2(1+x)^{k-\lambda-2} - (1+x)^{k-\lambda-1}\Big], \end{aligned}$$

we can evaluate, by separating the term with $k=\lambda$, explicitly

$$\int_0^1 \frac{(1-x)\big[(1+x)^\lambda - 1\big]}{\lambda x(1+x)^{\lambda+1}}dx = \frac{1-\ln 2}{\lambda} + \sum_{k=1}^{\lambda-1}\frac{(2^{k-\lambda}-1-k+\lambda)}{\lambda(\lambda-k)(\lambda-k+1)}.$$

In general, we have the following general statement.

PROPOSITION 2.1. *For $\lambda \in \mathbb{Z}$, we have the following formulae:*

$$\lambda \leqslant 0: \quad S_\lambda(0,0) = \int_0^1 \int_0^1 \frac{(1+xy)^{-\lambda}}{(1+x)(1+y)} dx\, dy = \sum_{k=0}^{-\lambda} \binom{-\lambda}{k} \big(\ln 2 - \bar{H}_k\big)^2;$$

$$\lambda = 1: \quad S_1(0,0) = \frac{\pi^2}{24}, \qquad \text{(Furdui and Bradie [8])}$$

$$\lambda \geqslant 1: \quad S_{1+\lambda}(0,0) = \frac{1}{2} S_\lambda(0,0) + \frac{1-\ln 2}{2\lambda} + \sum_{k=1}^{\lambda-1} \frac{(2^{k-\lambda} - 1 - k + \lambda)}{2\lambda(\lambda-k)(\lambda-k+1)}.$$

REMARK. By iterating λ times the recurrence in Proposition 2.1, we can derive, for $\lambda \geqslant 1$, the following explicit formula

$$S_\lambda(0,0) = \frac{\pi^2}{12 \times 2^\lambda} + \sum_{j=1}^{\lambda-1} \frac{1-\ln 2}{j \times 2^{\lambda-j}} + \sum_{j=1}^{\lambda-1} \sum_{k=1}^{j-1} \frac{2^{j-\lambda}(2^{k-j} - 1 - k + j)}{j(j-k)(j-k+1)}.$$

PROOF. For the $\lambda > 0$ case, the series $S_\lambda(0,0)$ can be evaluated explicitly by the recurrence relation as long as the initial value $S_1(0,0)$ is determined. First we reduce it to a single integral by

$$\begin{aligned} S_1(0,0) &= \int_0^1 \int_0^1 \frac{dx\, dy}{(1+x)(1+y)(1+xy)} \\ &= \int_0^1 \int_0^1 \left[\frac{1}{(1-x^2)(1+y)} - \frac{x}{(1-x^2)(1+xy)} \right] dx\, dy \\ &= \int_0^1 \left[\frac{\ln 2}{1-x^2} - \frac{\ln(1+x)}{1-x^2} \right] dx = \int_0^1 \frac{\ln \frac{2}{1+x}\, dx}{1-x^2}. \end{aligned}$$

Then making the change of variables by $x \to \frac{1-y}{1+y}$, we can evaluate

$$S_1(0,0) = \int_0^1 \frac{\ln(1+y)\ dy}{2y} = \frac{\pi^2}{24}.$$

We remark that evaluating the above triple series $S_1(0,0)$ was proposed as a Monthly problem [**8**] by Furdui and resolved by Bradie subsequently. This has been the primary inspiration for the author to work on this subject.

The $\lambda \leqslant 0$ case can be verified as follows:

$$\begin{aligned} S_\lambda(0,0) &= \int_0^1 \int_0^1 \frac{(1+xy)^{-\lambda}}{(1+x)(1+y)} dx\, dy \quad \left[\text{where } (1+xy)^{-\lambda} = \sum_{k=0}^{-\lambda} \binom{-\lambda}{k} x^k y^k \right] \\ &= \sum_{k=0}^{-\lambda} \binom{-\lambda}{k} \int_0^1 \frac{x^k}{1+x} dx \int_0^1 \frac{y^k}{1+y} dy = \sum_{k=0}^{-\lambda} \binom{-\lambda}{k} \big(\ln 2 - \bar{H}_k\big)^2, \end{aligned}$$

where the last passage is justified by the integral formula (1.4). □

According to Proposition 2.1, we deduce in succession the following explicit formulae:

$$S_1(0,0)=\frac{\pi^2}{24}, \qquad S_{-1}(0,0)=1+2\ln^2 2-2\ln 2,$$
$$S_2(0,0)=\frac{1}{2}+\frac{\pi^2}{48}-\frac{\ln 2}{2}, \qquad S_{-2}(0,0)=\frac{9}{4}+4\ln^2 2-5\ln 2,$$
$$S_3(0,0)=\frac{9}{16}+\frac{\pi^2}{96}-\frac{\ln 2}{2}, \qquad S_{-3}(0,0)=\frac{40}{9}+8\ln^2 2-\frac{32\ln 2}{3},$$
$$S_4(0,0)=\frac{151}{288}+\frac{\pi^2}{192}-\frac{5\ln 2}{12}, \qquad S_{-4}(0,0)=\frac{1241}{144}+16\ln^2 2-\frac{131\ln 2}{6},$$
$$S_5(0,0)=\frac{1075}{2304}+\frac{\pi^2}{384}-\frac{\ln 2}{3}; \qquad S_{-5}(0,0)=\frac{30167}{1800}+32\ln^2 2-\frac{661\ln 2}{15}.$$

3. Evaluation of $T_\lambda(\mu)$

According to the partial fraction decomposition

$$\frac{1}{(1+y)(1+xy)^\lambda}=\frac{1}{(1-x)^\lambda(1+y)}-\sum_{k=1}^{\lambda}\frac{x}{(1-x)^{1+\lambda-k}(1+xy)^k},$$

we can integrate with respect to y

$$\int_0^1\frac{1}{(1+y)(1+xy)^\lambda}dy=\frac{\ln\frac{2}{1+x}}{(1-x)^\lambda}+\sum_{k=2}^{\lambda}\frac{1-(1+x)^{k-1}}{(k-1)(1+x)^{k-1}(1-x)^{1+\lambda-k}}.$$

This reduces $T_\lambda(\mu)$ to a single integral

$$T_\lambda(\mu)=\int_0^1\mathcal{R}(\lambda,\mu;x)dx,$$

where the integrand is given by

$$\mathcal{R}(\lambda,\mu;x)=\frac{x^\mu\ln\frac{2}{1+x}}{(1-x)^\lambda}+\sum_{k=2}^{\lambda}\frac{x^\mu\left[1-(1+x)^{k-1}\right]}{(k-1)(1+x)^{k-1}(1-x)^{1+\lambda-k}}.$$

Observing that

$$\mathcal{R}(\lambda,\mu;x)-\mathcal{R}(\lambda,\mu+1;x)=\mathcal{R}(\lambda-1,\mu;x)+\frac{x^\mu\left[1-(1+x)^{\lambda-1}\right]}{(\lambda-1)(1+x)^{\lambda-1}}$$

and

$$\begin{aligned}\int_0^1 \frac{x^\mu\big[1-(1+x)^{\lambda-1}\big]}{(1+x)^{\lambda-1}}dx &= -\int_0^1 x^\mu dx + \int_0^1 \frac{x^\mu}{(1+x)^{\lambda-1}}dx\\ &= \frac{-1}{\mu+1} + \sum_{k=0}^{\mu}\int_0^1 (-1)^{\mu-k}\binom{\mu}{k}(1+x)^{1+k-\lambda}dx\\ &= \frac{-1}{\mu+1} + \sum_{k=0}^{\mu}(-1)^{\mu-k}\binom{\mu}{k}\begin{cases}\ln 2, & k=\lambda-2;\\ \dfrac{2^{2+k-\lambda}-1}{2+k-\lambda}, & k\neq\lambda-2.\end{cases}\end{aligned}$$

We find the explicit formulae as in the following proposition.

Proposition 3.1. *For $\lambda\in\mathbb{Z}$ and $\mu\in\mathbb{N}_0$, we have the following formulae:*

$$\begin{aligned}\lambda\leqslant 0: T_\lambda(\mu) &= \int_0^1\int_0^1 \frac{x^\mu(1+xy)^{-\lambda}}{1+y}dx\,dy = \frac{\mu!(-\lambda)!}{(1-\lambda+\mu)!}\big(\ln 2-\bar{H}_{-\lambda}\big)\\ &\quad+\sum_{k=1}^{-\lambda}(-1)^{-\lambda-k}\binom{-\lambda}{k}\frac{\mu!(-\lambda-k)!}{(1-\lambda+\mu-k)!}\sum_{i=1}^{k}\binom{k-1}{i-1}\frac{1}{1-\lambda-i};\end{aligned}$$

$$\lambda=1: T_1(0)=\frac{\pi^2}{12}-\frac{\ln^2 2}{2},$$

$$T_1(\mu+1)-T_1(\mu)=(-1)^\mu\frac{\ln 2}{\mu+1}-(-1)^\mu\frac{\bar{H}_{\mu+1}}{\mu+1};$$

$$\lambda>1: T_\lambda(0)=\sum_{k=1}^{\lambda-1}\frac{1-2^{-k}}{k(\lambda-1)},$$

$$\begin{aligned}T_\lambda(\mu+1) &= T_\lambda(\mu)-T_{\lambda-1}(\mu)\\ &\quad+\frac{1}{(\lambda-1)(\mu+1)}-\sum_{k=0}^{\mu}\frac{(-1)^{\mu-k}}{\lambda-1}\binom{\mu}{k}\begin{cases}\ln 2, & k=\lambda-2;\\ \dfrac{2^{2+k-\lambda}-1}{2+k-\lambda}, & k\neq\lambda-2.\end{cases}\end{aligned}$$

Proof. The recurrence relation for $\lambda>1$ is already determined. But we have to check the case $\lambda=1$. It is not hard to compute the initial value

$$T_1(0)=\int_0^1\int_0^1\frac{dx\,dy}{(1+y)(1+xy)}=\int_0^1\frac{\ln(1+y)}{y(1+y)}dy=\frac{\pi^2}{12}-\frac{\ln^2 2}{2}.$$

Next, by reducing the integral

$$\begin{aligned}T_1(\mu) &= \int_0^1\int_0^1\frac{x^\mu}{(1+y)(1+xy)}dx\,dy\\ &= \int_0^1\int_0^1\left[\frac{x^\mu}{(1-x)(1+y)}-\frac{x^{1+\mu}}{(1-x)(1+xy)}\right]dx\,dy\\ &= \int_0^1\frac{x^\mu\ln\frac{2}{1+x}}{1-x}dx\end{aligned}$$

and then invoking (1.5), we confirm the difference respect to μ

$$T_1(\mu+1)-T_1(\mu)=\int_0^1 x^\mu\ln(1+x)dx-\ln 2\int_0^1 x^\mu dx$$
$$=(-1)^\mu\frac{\ln 2}{\mu+1}-(-1)^\mu\frac{\bar{H}_{\mu+1}}{\mu+1}.$$

When $\lambda\leqslant 0$, the integral can be done as follows:

$$\begin{aligned}T_\lambda(\mu)&=\int_0^1\int_0^1\frac{x^\mu(1+xy)^{-\lambda}}{1+y}dx\,dy\quad[\text{writing } 1+xy=(1+y)-y(1-x)]\\&=\sum_{k=0}^{-\lambda}(-1)^{k-\lambda}\binom{-\lambda}{k}\int_0^1 x^\mu(1-x)^{-\lambda-k}dx\int_0^1 y^{-\lambda-k}(1+y)^{k-1}dy\\&=\frac{\mu!(-\lambda)!}{(1-\lambda+\mu)!}\big(\ln 2-\bar{H}_{-\lambda}\big)\qquad(\text{the initial term with } k=0)\\&+\sum_{k=1}^{-\lambda}(-1)^{-\lambda-k}\binom{-\lambda}{k}\frac{\mu!(-\lambda-k)!}{(1-\lambda+\mu-k)!}\sum_{i=1}^{k}\binom{k-1}{i-1}\frac{1}{1-\lambda-i}.\end{aligned}$$

For the remaining case $\lambda>1$, we have

$$T_\lambda(0)=\int_0^1\int_0^1\frac{dx\,dy}{(1+y)(1+xy)^\lambda}=\int_0^1\frac{(1+y)^{\lambda-1}-1}{(\lambda-1)y(1+y)^\lambda}dy.$$

By writing

$$(1+y)^{\lambda-1}-1=y\sum_{k=1}^{\lambda-1}(1+y)^{\lambda-k-1}$$

we can evaluate

$$T_\lambda(0)=\sum_{k=1}^{\lambda-1}\int_0^1\frac{dy}{(\lambda-1)(1+y)^{k+1}}=\sum_{k=1}^{\lambda-1}\frac{1-2^{-k}}{k(\lambda-1)}.\qquad\square$$

According to Proposition 3.1, it is possible to compute $T_\lambda(\mu)$ when $\lambda\in\mathbb{Z}$ and $\mu\in\mathbb{N}_0$ are specified by small values. For $-3\leqslant\lambda\leqslant 3$ and $0\leqslant\mu\leqslant 3$, we tabulate the values of $T_\lambda(\mu)$ below, which shows that the corresponding integrals result in rational numbers when $\lambda-\mu\geqslant 2$.

TABLE 1. **Values for $T_\lambda(\mu)$**

$\lambda \backslash \mu$	0	1	2	3
-3	$\frac{29}{24} + \frac{\ln 2}{4}$	$\frac{19}{24} + \frac{\ln 2}{20}$	$\frac{53}{90} + \frac{\ln 2}{60}$	$\frac{197}{420} + \frac{\ln 2}{140}$
-2	$\frac{5}{6} + \frac{\ln 2}{3}$	$\frac{13}{24} + \frac{\ln 2}{12}$	$\frac{2}{5} + \frac{\ln 2}{30}$	$\frac{19}{60} + \frac{\ln 2}{60}$
-1	$\frac{1}{2} + \frac{\ln 2}{2}$	$\frac{1}{3} + \frac{\ln 2}{6}$	$\frac{1}{4} + \frac{\ln 2}{12}$	$\frac{1}{5} + \frac{\ln 2}{20}$
0	$\ln 2$	$\frac{\ln 2}{2}$	$\frac{\ln 2}{3}$	$\frac{\ln 2}{4}$
1	$\frac{\pi^2}{12} - \frac{\ln^2 2}{2}$	$\frac{\pi^2}{12} - 1 + \ln 2 - \frac{\ln^2 2}{2}$	$\frac{\pi^2}{12} - \frac{3}{4} + \frac{\ln 2}{2} - \frac{\ln^2 2}{2}$	$\frac{\pi^2}{12} - \frac{37}{36} + \frac{5\ln 2}{6} - \frac{\ln^2 2}{2}$
2	$\frac{1}{2}$	$\frac{3}{2} - \frac{\pi^2}{12} - \ln 2 + \frac{\ln^2 2}{2}$	$2 - \frac{\pi^2}{6} - \ln 2 + \ln^2 2$	$\frac{43}{12} - \frac{\pi^2}{4} - \frac{5\ln 2}{2} + \frac{3\ln^2 2}{2}$
3	$\frac{7}{16}$	$\frac{3}{16}$	$\frac{\pi^2}{12} - \frac{13}{16} + \frac{\ln 2}{2} - \frac{\ln^2 2}{2}$	$\frac{\pi^2}{4} - \frac{163}{48} + \frac{5\ln 2}{2} - \frac{3\ln^2 2}{2}$

4. Evaluation of $S_\lambda(\mu,\nu)$

Finally, we are ready to evaluate $S_\lambda(\mu,\nu)$ in general. Since $S_\lambda(\mu,\nu)$ is symmetric with respect to μ and ν where $\mu,\nu \in \mathbb{N}_0$, we assume that $\mu \geqslant \nu$. By making use of Lemma 1.1, we can write

$$S_\lambda(\mu,\nu) = \sum_{k=0}^{\nu} (-1)^{\nu-k} \binom{\nu}{k} S_{\lambda-k}(\mu-\nu, 0).$$

In view of the algebraic relation

$$x^{\mu-\nu} = (-1)^{\mu-\nu} + (1+x) \sum_{j=1}^{\mu-\nu} (-1)^{\mu-\nu-j} x^{j-1},$$

we can proceed further with

$$\begin{aligned} S_{\lambda-k}(\mu-\nu,0) &= \int_0^1 \int_0^1 \frac{x^{\mu-\nu}}{(1+x)(1+y)(1+xy)^{\lambda-k}} dx\, dy \\ &= (-1)^{\mu-\nu} S_{\lambda-k}(0,0) + \sum_{j=1}^{\mu-\nu} (-1)^{\mu-\nu-j} T_{\lambda-k}(j-1). \end{aligned}$$

By substitution, we arrive at the following general theorem.

THEOREM 4.1 ($\lambda \in \mathbb{Z}$ and $\mu,\nu \in \mathbb{N}_0$). *For any triplet integers λ,μ,ν with $\lambda \in \mathbb{Z}$ and $\mu,\nu \in \mathbb{N}_0$, the corresponding triple series $S_\lambda(\mu,\nu)$ has always the value in $\mathbb{Q}\langle 1, \ln 2, \ln^2 2\rangle$ and $\mathbb{Q}\langle 1, \pi^2, \ln 2, \ln^2 2\rangle$, respectively, for $\lambda \leqslant 0$ and $\lambda > 0$, where $\mathbb{Q}\langle\Lambda\rangle$ is the $\mathbb{Q}$-linear space generated by $\Lambda \subset \mathbb{R}$. More precisely, assuming $S_\lambda(0,0)$ and $T_\lambda(\mu)$ as in Propositions 2.1 and 3.1, respectively, the following infinite series identity holds:*

$$S_\lambda(\mu,\nu) = \sum_{k=0}^{\nu} (-1)^{\mu-k} \binom{\nu}{k} S_{\lambda-k}(0,0) + \sum_{k=0}^{\nu} \binom{\nu}{k} \sum_{j=1}^{\mu-\nu} (-1)^{\mu-k-j} T_{\lambda-k}(j-1).$$

In order to facilitate reader's reference, we record the summation formulae for $S_\lambda(\mu,\nu)$ with $-4 \leqslant \lambda \leqslant 5$ and $0 \leqslant \mu,\nu \leqslant 3$ in the following two tables.

TABLE 2. **Values for $S_\lambda(\mu,\nu)$ with $-4 \leqslant \lambda \leqslant 0$**

	$\mu\backslash\nu$	0	1	2	3
$\lambda=0$	0	$\ln^2 2$			
	1	$\ln 2-\ln^2 2$	$1+\ln^2 2-2\ln 2$		
	2	$\ln^2 2-\frac{\ln 2}{2}$	$\frac{3\ln 2}{2}-\frac{1}{2}-\ln^2 2$	$\frac{1}{4}+\ln^2 2-\ln 2$	
	3	$\frac{5\ln 2}{6}-\ln^2 2$	$\frac{5}{6}+\ln^2 2-\frac{11\ln 2}{6}$	$\frac{4\ln 2}{3}-\ln^2 2-\frac{5}{12}$	$\frac{25}{36}+\ln^2 2-\frac{5\ln 2}{3}$
$\lambda=-1$	0	$1+2\ln^2 2-2\ln 2$			
	1	$\frac{5\ln 2}{2}-\frac{1}{2}-2\ln^2 2$	$\frac{5}{4}+2\ln^2 2-3\ln 2$		
	2	$\frac{5}{6}+2\ln^2 2-\frac{7\ln 2}{3}$	$\frac{17\ln 2}{6}-\frac{11}{12}-2\ln^2 2$	$\frac{17}{18}+2\ln^2 2-\frac{8\ln 2}{3}$	
	3	$\frac{29\ln 2}{12}-\frac{7}{12}-2\ln^2 2$	$\frac{9}{8}+2\ln^2 2-\frac{35\ln 2}{12}$	$\frac{11\ln 2}{4}-\frac{65}{72}-2\ln^2 2$	$\frac{149}{144}+2\ln^2 2-\frac{17\ln 2}{6}$
$\lambda=-2$	0	$\frac{9}{4}+4\ln^2 2-5\ln 2$			
	1	$\frac{16\ln 2}{3}-\frac{17}{12}-4\ln^2 2$	$\frac{79}{36}+4\ln^2 2-\frac{17\ln 2}{3}$		
	2	$\frac{47}{24}+4\ln^2 2-\frac{21\ln 2}{4}$	$\frac{67\ln 2}{12}-\frac{131}{72}-4\ln^2 2$	$\frac{95}{48}+4\ln^2 2-\frac{11\ln 2}{2}$	
	3	$\frac{317\ln 2}{60}-\frac{187}{120}-4\ln^2 2$	$\frac{149}{72}+4\ln^2 2-\frac{337\ln 2}{60}$	$\frac{83\ln 2}{15}-\frac{443}{240}-4\ln^2 2$	$\frac{7159}{3600}+4\ln^2 2-\frac{167\ln 2}{30}$
$\lambda=-3$	0	$\frac{40}{9}+8\ln^2 2-\frac{32\ln 2}{3}$			
	1	$\frac{131\ln 2}{12}-\frac{233}{72}-8\ln^2 2$	$\frac{601}{144}+8\ln^2 2-\frac{67\ln 2}{6}$		
	2	$\frac{145}{36}+8\ln^2 2-\frac{163\ln 2}{15}$	$\frac{667\ln 2}{60}-\frac{2639}{720}-8\ln^2 2$	$\frac{3571}{900}+8\ln^2 2-\frac{166\ln 2}{15}$	
	3	$\frac{653\ln 2}{60}-\frac{619}{180}-8\ln^2 2$	$\frac{2899}{720}+8\ln^2 2-\frac{167\ln 2}{15}$	$\frac{133\ln 2}{12}-\frac{839}{225}-8\ln^2 2$	$\frac{14171}{3600}+8\ln^2 2-\frac{111\ln 2}{10}$
$\lambda=-4$	0	$\frac{1241}{144}-\frac{131\ln 2}{6}+16\ln^2 2$			
	1	$\frac{661\ln 2}{30}-\frac{4969}{720}-16\ln^2 2$	$\frac{29309}{3600}-\frac{667\ln 2}{30}+16\ln^2 2$		
	2	$\frac{1933}{240}+16\ln^2 2-22\ln 2$	$\frac{111\ln 2}{5}-\frac{8873}{1200}-16\ln^2 2$	$\frac{1897}{240}-\frac{133\ln 2}{6}+16\ln^2 2$	
	3	$\frac{2311\ln 2}{105}-\frac{5171}{720}-16\ln^2 2$	$\frac{200353}{25200}-\frac{2332\ln 2}{105}+16\ln^2 2$	$\frac{4657\ln 2}{210}-\frac{37817}{5040}-16\ln^2 2$	$\frac{1382131}{176400}-\frac{1553\ln 2}{70}+16\ln^2 2$

TABLE 3. **Values for $S_\lambda(\mu,\nu)$ with $1 \leqslant \lambda \leqslant 5$**

	$\mu\diagdown\nu$	0	1	2	3
$\lambda=1$	0	$\frac{\pi^2}{24}$			
	1	$\frac{\pi^2}{24}-\frac{\ln^2 2}{2}$	$\ln^2 2-\frac{\pi^2}{24}$		
	2	$\ln 2+\frac{\pi^2}{24}-1$	$\ln 2-\frac{\pi^2}{24}-\frac{\ln^2 2}{2}$	$1+\frac{\pi^2}{24}-2\ln 2$	
	3	$\frac{1}{4}+\frac{\pi^2}{24}-\frac{\ln^2 2}{2}-\frac{\ln 2}{2}$	$1-\frac{\pi^2}{24}+\ln^2 2-\frac{3\ln 2}{2}$	$\frac{\pi^2}{24}-\frac{\ln^2 2}{2}+\frac{\ln 2}{2}-\frac{1}{2}$	$\ln^2 2+\ln 2-\frac{\pi^2}{24}-\frac{3}{4}$
$\lambda=2$	0	$\frac{1}{2}+\frac{\pi^2}{48}-\frac{\ln 2}{2}$			
	1	$\frac{\ln 2}{2}-\frac{\pi^2}{48}$	$\frac{\pi^2}{48}+\frac{\ln 2}{2}-\frac{1}{2}$		
	2	$\frac{3}{2}-\frac{\pi^2}{16}+\frac{\ln^2 2}{2}-\frac{3\ln 2}{2}$	$\frac{\pi^2}{16}-\frac{\ln^2 2}{2}-\frac{\ln 2}{2}$	$\frac{1}{2}-\frac{\pi^2}{16}+\ln^2 2-\frac{\ln 2}{2}$	
	3	$\frac{1}{2}-\frac{5\pi^2}{48}+\frac{\ln^2 2}{2}+\frac{\ln 2}{2}$	$\frac{5\pi^2}{48}-\frac{5}{2}-\frac{\ln^2 2}{2}+\frac{5\ln 2}{2}$	$\frac{3\ln 2}{2}-\frac{5\pi^2}{48}$	$\frac{1}{2}+\frac{5\pi^2}{48}-\ln^2 2-\frac{3\ln 2}{2}$
$\lambda=3$	0	$\frac{9}{16}+\frac{\pi^2}{96}-\frac{\ln 2}{2}$			
	1	$\frac{\ln 2}{2}-\frac{\pi^2}{96}-\frac{1}{8}$	$\frac{\pi^2}{96}-\frac{1}{16}$		
	2	$\frac{5}{16}+\frac{\pi^2}{96}-\frac{\ln 2}{2}$	$\frac{1}{8}-\frac{\pi^2}{96}$	$\frac{\pi^2}{96}+\frac{\ln 2}{2}-\frac{7}{16}$	
	3	$\frac{7\pi^2}{96}-\frac{\ln^2 2}{2}-\frac{9}{8}+\ln 2$	$\frac{19}{16}-\frac{7\pi^2}{96}+\frac{\ln^2 2}{2}-\ln 2$	$\frac{7\pi^2}{96}-\frac{\ln^2 2}{2}-\frac{1}{8}-\frac{\ln 2}{2}$	$\frac{15}{16}-\frac{7\pi^2}{96}+\ln^2 2-\ln 2$
$\lambda=4$	0	$\frac{151}{288}+\frac{\pi^2}{192}-\frac{5\ln 2}{12}$			
	1	$\frac{5\ln 2}{12}-\frac{13}{96}-\frac{\pi^2}{192}$	$\frac{11}{288}+\frac{\pi^2}{192}-\frac{\ln 2}{12}$		
	2	$\frac{85}{288}+\frac{\pi^2}{192}-\frac{5\ln 2}{12}$	$\frac{1}{96}-\frac{\pi^2}{192}+\frac{\ln 2}{12}$	$\frac{\pi^2}{192}-\frac{29}{288}+\frac{\ln 2}{12}$	
	3	$\frac{5\ln 2}{12}-\frac{19}{96}-\frac{\pi^2}{192}$	$\frac{5}{288}+\frac{\pi^2}{192}-\frac{\ln 2}{12}$	$\frac{11}{96}-\frac{\pi^2}{192}-\frac{\ln 2}{12}$	$\frac{\pi^2}{192}-\frac{97}{288}+\frac{5\ln 2}{12}$
$\lambda=5$	0	$\frac{1075}{2304}+\frac{\pi^2}{384}-\frac{\ln 2}{3}$			
	1	$\frac{\ln 2}{3}-\frac{67}{576}-\frac{\pi^2}{384}$	$\frac{133}{2304}+\frac{\pi^2}{384}-\frac{\ln 2}{12}$		
	2	$\frac{587}{2304}+\frac{\pi^2}{384}-\frac{\ln 2}{3}$	$\frac{\ln 2}{12}-\frac{11}{576}-\frac{\pi^2}{384}$	$\frac{\pi^2}{384}-\frac{5}{256}$	
	3	$\frac{\ln 2}{3}-\frac{11}{64}-\frac{\pi^2}{384}$	$\frac{31}{768}+\frac{\pi^2}{384}-\frac{\ln 2}{12}$	$\frac{17}{576}-\frac{\pi^2}{384}$	$\frac{\pi^2}{384}-\frac{187}{2304}+\frac{\ln 2}{12}$

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