PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 112(126) (2022), 13–22 DOI: https://doi.org/10.2298/PIM2226013C

ON THE STUDY OF THE WAVE EQUATION SET ON A SINGULAR CYLINDRICAL DOMAIN

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Abstract. We give new regularity results of solutions for the linear wave equation set in a nonsmooth cylindrical domain. Different types of conditions are imposed on the boundary of the singular domain. Our study is performed in some particular anisotropic Hölder spaces.

1. Introduction and position of the problem

Set

$$\Omega := \{(x,y) \in \mathbb{R}^2 : 0 < x < a,\ \varphi_2(x) < y < \varphi_1(x)\}, \tag{1.1}$$

where $a > 0$ is a finite real number and φ_1, φ_2 are continuous real-valued functions defined on $[0,a]$ satisfying the following conditions:

(1) φ_1, φ_2 are of class C^2 on $[0,a]$,
(2) $\varphi := \varphi_1 - \varphi_2 > 0$ on $]0,a]$,
(3) for $i \in \{1,2\} : \varphi_i(0) = \varphi_i'(0) = 0$,
(4) φ_1, φ_2 are strictly monotone functions.

We suppose that the boundary $\partial\Omega$ of the cusp domain (1.1) is given by $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_1 := \{(x,\varphi_1(x)) : 0 \leqslant x \leqslant a\}, \quad \Gamma_2 := \{(x,\varphi_2(x)) : 0 \leqslant x \leqslant a\},$$
$$\Gamma_3 := \{(0,y) : \varphi_2(0) \leqslant y \leqslant \varphi_1(0)\} \cup \{(a,y) : \varphi_2(a) \leqslant y \leqslant \varphi_1(a)\}.$$

In the cylinder $\Pi = [0,1] \times \Omega$, we consider the Cauchy problem for the linear wave equation

$$\Box_\lambda u = h, \quad \lambda > 0, \tag{1.2}$$

equipped with the initial conditions

$$\mathcal{L}u|_{\{0\}\times\Omega} = 0, \quad \mathcal{L}u|_{\{1\}\times\Omega} = 0. \tag{1.3}$$

2020 *Mathematics Subject Classification:* Primary 34G10; 34K10; Secondary 12H2O; 35J25; 44A45.

Key words and phrases: operational differential equation, Hölder space, cuspidal point, wave equation.

Communicated by Stevan Pilipović.

Here, $\square_\lambda u := \partial_t^2 u - \partial_x^2 u - \partial_y^2 u - \lambda u$ and $\mathcal{L}u := \partial_t^2 u + \partial_t u + u$. In addition, we impose the following boundary conditions to the problem (1.2):

$$\partial_y u - u|_{[0,1]\times\Gamma_2} = 0, \qquad \partial u|_{[0,1]\times\Gamma_2} = 0, \tag{1.4}$$

$$\partial_x u - u|_{[0,1]\times\Gamma_3} = 0, \quad u|_{[0,1]\times(\Gamma_1\cup\Gamma_2)} = 0. \tag{1.5}$$

Questions concerning the solvability of hyperbolic problems posed on nonsmooth cylindrical domains have been studied by several authors. We cite particularly [**13, 14**], where the L^p-theory of such problems has been discussed for cylindrical domains containing a cusp bases. The methods of investigation are derived from the well known *a priori* estimates techniques and the potential theory.

In this paper, we analyze the solvability of problem (1.2)–(1.5) in the case that the right-hand side h belongs to the anisotropic Hölder space $C^\theta([0,1]; L^p(\Omega))$ with $0 < \theta < 1$ and $1 < p < \infty$, endowed with the norm

$$\|f\|_{C^\theta([0,1];L^p(\Omega))} := \sup_{t,t'\in[0,1],t\neq t'} \frac{\|f(t) - f(t')\|_{L^p(\Omega)}}{|t - t'|^\theta}.$$

Boundary conditions (1.3)–(1.5) involve the second derivative with respect to the time variable and Robin type conditions with respect to the space variables. Moreover, in our study, we consider the anisotropic character of the functional framework. The main novelty of this paper is a presentation of a new alternative abstract method for the study of (1.2)–(1.5). The main idea of this method is to transform our concrete problem to an abstract differential equation in an appropriately chosen Banach space. The use of such an argument is very effective and provide some interesting results concerning the maximal regularity of solutions for Problem (1.2)–(1.5) near the singular part of the boundary of the cylindrical domain Π. For more details about this abstract point of view, we refer the reader to [**2**] and [**4–7**], where some elliptic and parabolic problems on particular cusp domains have been successfully studied.

The paper is organized as follows. In the next section, we show that our problem can be transformed by a suitable changes of variables into a particular abstract second order differential equation. Section 3 is devoted to the complete study of the abstract version of the transformed problem. In Section 4, we come back to the initial problem in the cusp domain and prove our main result. That is:

THEOREM 1.1. *Let $h \in C^\theta([0,1]; L^p(\Omega))$ with $0 < \theta < 1$ and $1 < p < \infty$. Then, problem* (1.2)–(1.5) *has a unique solution u such that*

$$\varphi^{-2}u \in C^2([0,1]; L^p(\Omega)) \quad \text{and} \quad \partial_t^2 u, \partial_x^2 u, \partial_y^2 \in C^\theta([0,1]; L^p(\Omega)).$$

2. Change of variables and the abstract setting of the problem

Consider the following change of variables

$$T\colon \Pi \to \Sigma, \quad (t,x,y) \mapsto (t,\xi,\eta) := \left(t, -\int_a^x \frac{d\sigma}{\varphi(\sigma)}, \frac{y - \varphi_2(x)}{\varphi(x)}\right),$$

with Σ being the semi-infinite domain given by $\Sigma := [0,1] \times Q$, where

$$Q :=]0,+\infty] \times]0,1[\,. \tag{2.1}$$

Now, define the following change of functions

$$v(t,\xi,\eta) := (v\circ T)(t,x,y) = u(t,x,y),$$
$$g(t,\xi,\eta) := (g\circ T)(t,x,y) = h(t,x,y).$$

We have

$$\partial_x^2 u + \partial_y^2 u = \partial_\xi^2 v + \partial_\eta^2 v + \eta(\varphi_2' + \eta\varphi')^2\partial_\eta^2 v - 2(\varphi_2' + \eta\varphi')\partial_{\eta\xi}^2 v$$
$$+ \varphi'\partial_\xi v - (2\varphi'\varphi_2' - \varphi\varphi_2'' - \eta(\varphi\varphi'' - 2(\varphi')^2))\partial_\eta v.$$

To avoid the use of weighted function spaces, we introduce the following change of function $\varrho w = v$, where $\varrho = \varphi^{2/q}$ and $q = p/(p-1)$. Then we have

$$\partial_\xi^2 v + \partial_\eta^2 v = \partial_\xi^2 w + \partial_\eta^2 w + \frac{2}{q}\Big(\frac{2}{q}(\varphi')^2 + \varphi\varphi''\Big)w + (\varphi_2' + \eta\varphi')^2\partial_\eta^2 w$$
$$- \frac{4}{q}\varphi'\partial_\xi w - 2(\varphi_2' + \eta\varphi')\Big(\partial_{\eta\xi}^2 w - \frac{2}{q}\varphi'\partial_\eta w\Big) + \varphi'\Big(\partial_\xi w - \frac{2}{q}\varphi' w\Big)$$
$$+ (2\varphi'\varphi_2' - \varphi\varphi_2'' - \eta(\varphi\varphi'' - 2(\varphi')^2))\partial_\eta w.$$

Consequently, problem (1.2) becomes

$$\varrho\partial_t^2 w - \partial_\xi^2 w - \partial_\eta^2 w - \lambda w - \mathcal{P}w = f, \quad \text{in } \Sigma, \tag{2.2}$$

where $f = \varphi^{2/q} g$ and $\mathcal{P}$ is the second order differential operator with C^∞-bounded coefficients in Σ, given by

$$\mathcal{P}w := \frac{2}{q}\Big(\frac{2}{q}(\varphi')^2 + \varphi\varphi''\Big)w + (\varphi_2' + \eta\varphi')^2\partial_\eta^2 w$$
$$- \frac{4}{q}\varphi'\partial_\xi w - 2(\varphi_2' + \eta\varphi')\Big(\partial_{\eta\xi}^2 w - \frac{2}{q}\varphi'\partial_\eta w\Big) + \varphi'\Big(\partial_\xi w - \frac{2}{q}\varphi' w\Big)$$
$$+ (2\varphi'\varphi_2' - \varphi\varphi_2'' - \eta(\varphi\varphi'' - 2(\varphi')^2))\partial_\eta w.$$

It is easy to see that conditions (1.4) imply that $\partial_\xi w - w|_{\xi=0} = 0$, $w|_{\xi=+\infty} = 0$, as well as that conditions (1.5) lead to $\partial_\eta w - w|_{\eta=0} = 0$, $w|_{\eta=1} = 0$. Taking into account the conditions imposed on the functions of parametrization, the problem (2.2) can be viewed as a certain perturbation of the following one:

$$\varrho\partial_t^2 w - \partial_\xi^2 w - \partial_\eta^2 w - \lambda w = f, \quad \text{in } \Sigma, \tag{2.3}$$

accompanied with the following conditions:

$$\mathcal{L}w|_{t=0} = 0, \quad \mathcal{L}w|_{t=1} = 0, \tag{2.4}$$

and

$$\partial_\xi w - w|_{\xi=0} = 0, \quad w|_{\xi=+\infty} = 0, \quad \partial_\eta w - w|_{\eta=0} = 0, \quad w|_{\eta=1} = 0. \tag{2.5}$$

In what follows, we will focus our attention to the study of the principal problem (2.3)–(2.5). First of all, we will present some information about the regularity of a new hand term f:

Lemma 2.1. *Let $0 < \theta < 1$ and $1 < p < \infty$. Then*

$$h \in C^\theta([0,1]; L^p(\Omega)) \Leftrightarrow f \in C^\theta([0,1]; L^p(Q)).$$

Proof. The result is easily obtained by reiterating the same techniques used in [**7**, Proposition 3.1] and [**10**, Section 2.2]. □

Now, let us write the abstract version of (2.3)–(2.5). Define the following vector-valued functions:

$$w : [0,1] \to E;\ t \longrightarrow w(t); \quad w(t)(\eta, \upsilon) = w(t, \xi, \eta),$$
$$f : [0,1] \to E;\ \xi \longrightarrow f(t); \quad f(t)(\xi, \eta) = f(t, \xi, \eta),$$

where $E := L^p(Q)$. Problem (2.3)–(2.5) can be reduced to the abstract differential equation

$$\varrho w''(t) - Aw(t) - \lambda w(t) = f(t), \quad 0 \leqslant t \leqslant 1,$$

where the operator

$$Aw := \partial_\xi^2 w + \partial_\eta^2 w \tag{2.6}$$

acts with its natural domain $D(A)$ consisting of all functions $w \in W^{2,p}(Q)$ for which $\partial_\xi w - w|_{\xi=0} = 0$, $w_{\xi=+\infty} = 0$, $\partial_\eta w - w|_{\eta=0} = 0$ and $w|_{\eta=1} = 0$. To make the notation less cluttered, we study the following problem

$$w''(t) - Aw(t) - \lambda w(t) = f(t), \quad 0 \leqslant t \leqslant 1, \tag{2.7}$$

accompanied with the following boundary conditions:

$$\mathcal{L}w|_{t=0} = 0, \quad \mathcal{L}w|_{t=1} = 0. \tag{2.8}$$

Remark 2.1. In the remaining part of this work, the letter C will denote a generic positive constant not necessarily the same at each occurrence.

3. On the study of the abstract problem

In the semi-infinite strip Q defined by (2.1), we consider the following spectral problem

$$\partial_\xi^2 \upsilon + \partial_\eta^2 \upsilon - \lambda \upsilon = k, \tag{3.1}$$

$$\partial_\eta \upsilon - \upsilon|_{\eta=0} = 0, \quad \upsilon|_{\eta=1} = 0, \tag{3.2}$$

$$\partial_\xi \upsilon - \upsilon|_{\xi=0} = 0, \quad \upsilon|_{\xi=+\infty} = 0, \tag{3.3}$$

where $k \in L^p(Q)$ and λ is a positive spectral parameter. It is important here to note that the spectral analysis of the linear operator defined by (2.6) is based essentially on the study of problem (3.1)–(3.3). To this end, we use the sum's operator theory developed in [**8**].

3.1. On the sum's operator theory. Let E a complex Banach space and M, N be two closed linear operators with domains $D(A)$, $D(N)$. Let S be the operator defined by

$$Su := Mu + Nu - \lambda u, \quad u \in D(S) := D(M) \cap D(N), \tag{3.4}$$

where M and N verify the assumptions

$$\text{(H.1)}\begin{cases}\text{i)} & \rho(M)\sum_M = \{\mu : |\mu| \geqslant r,\ |\operatorname{Arg}(\mu)| < \pi - \epsilon_M\},\\ & \forall\mu \in \sum_M \quad \|(M-\mu I)^{-1}\|_{L(E)} \leqslant C/|\mu|,\\ \text{ii)} & \rho(N) \supset \sum_N = \{\mu : |\mu| \geqslant r \mid \operatorname{Arg}(\mu)| < \pi - \epsilon_N\},\\ & \forall\mu \in \sum_N \quad \|(N-\mu I)^{-1}\|_{L(E)} \leqslant C/|\mu|,\\ \text{iii)} & \epsilon_M + \epsilon_N < \pi,\\ \text{iv)} & \overline{D(M)+D(N)} = E,\end{cases}$$

and

$$\text{(H.2)}\begin{cases}\forall\mu_1 \in \rho(M),\ \forall\mu_2 \in \rho(N):\\ (M-\mu_1 I)^{-1}(N-\mu_2 I)^{-1} - (N-\mu_2 I)^{-1}(M-\mu_1 I)^{-1}\\ = [(M-\mu_1 I)^{-1}(N-\mu_2 I)^{-1}] = 0,\end{cases}$$

with $\rho(M)$ and $\rho(N)$ being the resolvent sets of M and N and $L(E)$ being the space of all linear continuous operators from E into E. The main result proved in [**8**] is given by the following

Theorem 3.1. *Assume* (H.1)–(H.2). *Then the operator S defined by* (3.4) *is invertible and one has*

$$S^{-1}: u \to -\frac{1}{2i\pi}\int_\Gamma (M+z)^{-1}(N-\lambda-z)^{-1}u\,dz, \tag{3.5}$$

where Γ is a suitable sectorial curve lying in $\rho(M)\cap\rho(-N)$.

3.2. On the study of Laplace operator on unbounded strip. With a little abuse of notation, the abstract version of (3.1)) is formulated as follows

$$Hv + Bv - \lambda v = k, \quad v \in D(B)\cap D(H), \tag{3.6}$$

where

$$(Hv)(\eta) := v''(\eta), \quad D(H) := \{v \in W^{2,p}(0,1) : v'(0) - v(0) = 0, v(1) = 0\}, \tag{3.7}$$

(3.8)

$$(Bv)(\xi) := v''(\xi), \quad D(B) := \{v \in W^{2,p}(\mathbb{R}^+) : v'(0) - v(0) = 0, v(+\infty) = 0\}.$$

First of all, let us observe the following:

Lemma 3.1. *Let H be the linear operator defined by* (3.7). *Then there exists $C > 0$ such that*

$$(0,\infty) \subset \rho(H) \quad \textit{and} \quad \|(H-\mu I)^{-1}\|_{L(E)} \leqslant C/\mu, \quad \mu > 0. \tag{3.9}$$

Proof. As in [**12**], a direct computation shows that

$$(H-\mu I)^{-1}v = \int_0^{+\infty} G_{1,\sqrt{\mu}}(\eta,s)v(s)\,ds,$$

where $v \in L^p(0,1)$ and

$$G_{1,\sqrt{\mu}}(\xi,s)=\begin{cases}\dfrac{\sinh\sqrt{\mu}(1-\eta)[\sinh\sqrt{\mu}s+\sqrt{\mu}\cosh\sqrt{\mu}s]}{\sqrt{\mu}[\sinh\sqrt{\mu}+\sqrt{\mu}\cosh\sqrt{\mu}]}, & 0\leqslant s\leqslant \eta,\\ \dfrac{\sinh\sqrt{\mu}(1-s)[\sinh\sqrt{\mu}\eta+\sqrt{\mu}\cosh\sqrt{\mu}\eta]}{\sqrt{\mu}[\sinh\sqrt{\mu}+\sqrt{\mu}\cosh\sqrt{\mu}]}, & \eta\leqslant s\leqslant 1.\end{cases}$$

Here, $\sqrt{\mu}$ is the analytic determination defined by $\Re\sqrt{\mu}>0$. Observe that

$$\left|\sinh\sqrt{\mu}+\sqrt{\mu}\cosh\sqrt{\mu}\right|=\left|\frac{e^{\Re\sqrt{\mu}}}{2}\left(a_{\sqrt{\mu}}+ib_{\sqrt{\mu}}\right)+\frac{e^{-\Re\sqrt{\mu}}}{2}\left(c_{\sqrt{\mu}}+id_{\sqrt{\mu}}\right)\right|,$$

where

$$\begin{aligned}a_{\sqrt{\mu}}&=1+\Re\sqrt{\mu}\cos\Im\sqrt{\mu}-\Im\sqrt{\mu}\sin\Im\sqrt{\mu},\\ b_{\sqrt{\mu}}&=1+\Re\sqrt{\mu}\sin\Im\sqrt{\mu}+\Im\sqrt{\mu}\cos\Im\sqrt{\mu},\\ c_{\sqrt{\mu}}&=(\Re\sqrt{\mu}-1)\cos\Im\sqrt{\mu}-\Im\sqrt{\mu}\sin\Im\sqrt{\mu},\\ d_{\sqrt{\mu}}&=(1-\Re\sqrt{\mu})\sin\Im\sqrt{\mu}+\Im\sqrt{\mu}\cos\Im\sqrt{\mu}.\end{aligned}$$

Then, we obtain

$$\begin{aligned}&|\sinh\sqrt{\mu}+\sqrt{\mu}\cosh\sqrt{\mu}|\\ &\geqslant\frac{e^{\Re\sqrt{\mu}}}{2}[(1+\Re\sqrt{\mu})^2+(\Im\sqrt{\mu})^2]^{1/2}-\frac{e^{-\Re\sqrt{\mu}}}{2}[(1-\Re\sqrt{\mu})^2+(\Im\sqrt{\mu})^2]^{1/2}\\ &\qquad\geqslant\sinh\Re\sqrt{\mu}[1+(\Re\sqrt{\mu})^2+2\Re\sqrt{\mu}]^{1/2}\end{aligned}$$

and

$$|\sinh\sqrt{\mu}+\sqrt{\mu}\cosh\sqrt{\mu}|\geqslant\sinh\Re\sqrt{\mu}[1+(\Re\sqrt{\mu})]$$

To obtain the desired result, it suffices to see that

$$\begin{aligned}\frac{\sinh(1-\eta)\Re\sqrt{\mu}}{\sinh\Re\sqrt{\mu}[1+(\Re\sqrt{\mu})]}&=\frac{e^{\Re\sqrt{\mu}(1-\eta)}-e^{-\Re\sqrt{\mu}(1-\eta)}}{(e^{\Re\sqrt{\mu}}-e^{-\Re\sqrt{\mu}})(1+(\Re\sqrt{\mu}))}\\ &=\frac{e^{-\Re\sqrt{\mu}\eta}[1-e^{-\Re\sqrt{\mu}(2-\eta)}]}{(1-e^{-2\Re\sqrt{\mu}})(1+(\Re\sqrt{\mu}))}\leqslant\frac{Ce^{-\Re\sqrt{\mu}\eta}}{(1+(\Re\sqrt{\mu}))},\end{aligned}$$

from which we deduce that

$$\int_0^1\left|\frac{\sinh(1-\eta)\Re\sqrt{\mu}}{\sinh\Re\sqrt{\mu}[1+(\Re\sqrt{\mu})]}\right|^p d\eta\leqslant\int_0^1\left|e^{-\Re\sqrt{\mu}\eta}\right|^p d\eta\leqslant\frac{1}{p\Re\sqrt{\mu}}.\qquad\square$$

Arguing as before, we may conclude that the following result holds true.

Lemma 3.2. *Let B the linear operator defined by* (3.8). *Then, there exists $C>0$ such that*

$$(0,\infty)\subset\rho(B)\quad\textit{and}\quad\|(B-\mu I)^{-1}\|_{L(E)}\leqslant C/\mu,\quad\mu>0.\tag{3.10}$$

Proof. Let $v\in L^p((0,\infty))$. We have

$$(B-\mu I)^{-1}v=\int_0^{+\infty}G_{2,\sqrt{\mu}}(\xi,s)v(s)\,ds,$$

where

$$G_{2,\sqrt{\mu}}(\eta, s) = \begin{cases} \dfrac{1}{2\sqrt{\mu}} e^{-\sqrt{\mu}s}[e^{\sqrt{\mu}\xi} + c\sqrt{\mu}e^{-\sqrt{\mu}\xi}], & 0 \leqslant s \leqslant \xi, \\ \dfrac{1}{2\sqrt{\mu}} e^{-\sqrt{\mu}\xi}[e^{\sqrt{\mu}s} + c\sqrt{\mu}e^{-\sqrt{\mu}s}], & \xi \leqslant s \leqslant +\infty, \end{cases}$$

and $c(\sqrt{\mu}) = (\sqrt{\mu}-1)/(\sqrt{\mu}+1)$. The estimate (3.10) is handled by using the same argument delivered in [**9**, p. 1916]. □

The following lemma is needed to justify the use of the commutative version of the sum's operator theory:

LEMMA 3.3. *Let B and H be the linear operators defined by* (3.7)–(3.8). *Then* $\forall \mu_1 \in \rho(B)\, \forall \mu_2 \in \rho(H)$ $(B-\mu_1 I)^{-1}(H-\mu_2 I)^{-1} - (H-\mu_2 I)^{-1}(B-\mu_1 I)^{-1} = 0.$

PROOF. Let $\upsilon \in E$. Then the required equality follows from the following computation:

$$\begin{aligned}(B-\mu_1 I)^{-1}(H-\mu_2 I)^{-1}\upsilon(\xi) &= \int_0^{+\infty} G_{1,\sqrt{\mu_1}}(\xi,s)(H-\mu_2 I)^{-1}\upsilon(\xi)(s)\,ds \\ &= \int_0^{+\infty} G_{1,\sqrt{\mu_1}}(\xi,s)\int_0^1 G_{2,\sqrt{\mu_2}}(\xi,s)[\upsilon(s)](\tau)\,d\tau\,ds \\ &= \int_0^1 G_{2,\sqrt{\mu_2}}(\xi,s)\bigg(\int_0^{+\infty} G_{1,\sqrt{\mu_1}}(\xi,s)[\upsilon(s)](\tau)\,ds\bigg)d\tau \\ &= \int_0^1 G_{2,\sqrt{\mu_2}}(\xi,s)\bigg(\int_0^{+\infty} G_{1,\sqrt{\mu_1}}(\xi,s)\upsilon(s)\,ds\bigg)(\tau)\,d\tau \\ &= (H-\mu_2 I)^{-1}(B-\mu_1 I)^{-1}\upsilon(\xi).\end{aligned}$$
□

We conclude this section with the following useful observation:

REMARK 3.1. It is necessary to note that:

(1) The use of a classical argument of analytic continuation of the resolvent allows us to say that the previous estimates hold true in the sectors

$$\sum_B = \{\mu : |\mu| \geqslant r, |\operatorname{Arg}(\mu)| < \pi - \epsilon_M\},$$

$$\sum_H = \{\mu : |\mu| \geqslant r, |\operatorname{Arg}(\mu)| < \pi - \epsilon_H\},$$

provided that $\epsilon_M + \epsilon_N < \pi$.

(2) Thanks to [**11**, Proposition 2.1.1], we have that B and H are densely defined.

Define a closed operator A by $Aw := \partial_\xi^2 w + \partial_\eta^2 w$, where $D(A)$ is consisted of all functions $w \in W^{2,p}(Q)$ such that $\partial_\xi w - w|_{\xi=0} = 0$, $w|_{\xi=+\infty} = 0$, $\partial_\eta w - w|_{\eta=0} = 0$, and $w|_{\eta=1} = 0$.

LEMMA 3.4. *Let A be defined as above. Then there exists $C > 0$ such that*

$$(0,\infty) \subset \rho(A) \quad \text{and} \quad \|(A-\lambda I)^{-1}\|_{L(E)} \leqslant C/\lambda, \quad \lambda > 0. \tag{3.11}$$

PROOF. This result is a direct consequence of the commutative version of the sum's operator theory, showing that $(A-\lambda I)^{-1}$ is well defined. Estimate (3.11) is easily handled from the formula (3.5). □

3.3. Some regularity results for the abstract problem. In order to give a fairly complete study of abstract Cauchy problem (2.7)–(2.8), we follow the method developed in [**1**]. We build the natural representation of the solution by using the operational calculus and the Dunford integral. Towards this end, consider the following scalar problem

$$w''(t) - zw(t) = f(t), \quad 0 \leqslant t \leqslant 1, \tag{3.12}$$

equipped with the following boundary conditions:

$$w''(0) + w'(0) + w(0) = 0 \quad \text{and} \quad w''(1) + w'(1) + w(1) = 0. \tag{3.13}$$

A direct computation shows that the unique solution of (3.12)–(3.13) is given by

$$\begin{aligned} w(t) &= \frac{1}{2(1-e^{-2\sqrt{z}})\sqrt{z}} \int_0^1 e^{-\sqrt{z}(2-s+t)} f(s)\,ds \\ &\quad + \frac{1}{2(1-e^{-2\sqrt{z}})\sqrt{z}} \int_0^1 e^{-\sqrt{z}(s-t+2)} f(s)\,ds \\ &\quad - \frac{(z+\sqrt{z}+1)}{(z-\sqrt{z}+1)} \int_0^1 e^{-\sqrt{z}(s+t)} f(s)\,ds - \frac{(z-\sqrt{z}+1)}{(z+\sqrt{z}+1)} \int_0^1 e^{-\sqrt{z}(2-s-t)} f(s)\,ds \\ &\quad + \frac{1}{2\sqrt{z}} \int_0^t e^{-\sqrt{z}(t-s)} f(s)\,ds + \frac{1}{2\sqrt{z}} \int_t^1 e^{-\sqrt{z}(s-t)} f(s)\,ds. \end{aligned}$$

It is well known that (3.11) implies the existence of numbers $\delta \in \left]0, \frac{\pi}{2}\right[$ and $r > 0$ such that the resolvent set of A contains the following sector of the complex plane

$$\Pi_{\delta,r} = \{z \in \mathbb{C}^* : |\arg(z)| \leqslant \delta\} \cup \{z \in \mathbb{C} : |z| \leqslant r\}.$$

If Γ denotes the sectorial boundary curve of $\Pi_{\delta,r}$ oriented positively, then the natural representation of the solution of (2.7)–(2.8), in the abstract case, is given by

$$w(t) = \sum_{i=1}^{6} S_i(t,f), \tag{3.14}$$

where

$$\begin{aligned} S_1(t,f) &= \frac{1}{2i\pi} \int_\Gamma \int_0^1 \frac{1}{2(1-e^{-2\sqrt{z}})\sqrt{z}} e^{-\sqrt{z}(2-s+t)} (A-\lambda I)^{-1} f(s)\,ds, \\ S_2(t,f) &= \frac{1}{2i\pi} \int_\Gamma \int_0^1 \frac{1}{2(1-e^{-2\sqrt{z}})\sqrt{z}} e^{-\sqrt{z}(s-t+2)} (A-\lambda I)^{-1} f(s)\,ds, \\ S_3(t,f) &= -\frac{1}{2i\pi} \int_\Gamma \int_0^1 \frac{(z+\sqrt{z}+1)}{(z-\sqrt{z}+1)} e^{-\sqrt{z}(s+t)} (A-\lambda I)^{-1} f(s)\,ds\,dz, \end{aligned}$$

$$S_4(t,f) = -\frac{1}{2i\pi}\int_\Gamma\int_0^1 \frac{(z-\sqrt{z}+1)}{(z+\sqrt{z}+1)} e^{-\sqrt{z}(2-s-t)}(A-\lambda I)^{-1} f(s)\,ds\,dz,$$
$$S_5(t,f) = \frac{1}{2i\pi}\int_\Gamma\int_0^t \frac{e^{-\sqrt{z}(t-s)}}{2\sqrt{z}}(A-\lambda I)^{-1} f(s)\,ds,$$
$$S_6(t,f) = \frac{1}{2i\pi}\int_\Gamma\int_t^1 \frac{e^{-\sqrt{z}(s-t)}}{2\sqrt{z}}(A-\lambda I)^{-1} f(s)\,ds.$$

Observe that estimate (3.11) yields the convergence of the integrals occurring in 3.14. Furthermore, we have the following proposition:

PROPOSITION 3.1. *Let $f \in C^\theta([0,1];E)$, with $0<\theta<1$ and $1<p<\infty$. Then we have*

(1) *for all $t \in [0,1] : S_i(t,f) \in D(A)$, $i = 1,2,\dots,6$;*
(2) *for all $t \in [0,1] : AS_i(t,f) \in C^\theta([0,1];E)$, $i = 1,2,\dots,6$.*

PROOF. The proof is carried out by analogy with the proof of Proposition 3.1 in [**3**]. □

As an immediate consequence of this proposition, the main result concerning problem (2.7)–(2.8) reads as follows:

THEOREM 3.2. *Let $f \in C^\theta([0,1];L^p(Q))$ with $0<\theta<1$ and $1<p<\infty$. Then Problem (2.7)–(2.8) has a unique solution $w \in C([0,1];D(A)) \cap C^2([0,1];E)$ satisfying w'' and $Aw \in C^\theta([0,1];E)$.*

Applying all the preceding abstract results to the transformed problem (2.2)–(2.4)–(2.5), we obtain the following proposition:

PROPOSITION 3.2. *Let $f \in C^\theta([0,1];L^p(Q))$, with $0<\theta<1$ and $1<p<+\infty$. Then, Problem (2.2)–(2.4)–(2.5) has a unique solution $w \in C^2([0,1];L^p(Q))$. Moreover, w satisfies the maximal regularity*

$$\varrho\partial_t^2 w,\ \partial_\xi^2 w + \partial_\eta^2 w \in C^\theta([0,1];L^p(Q)).$$

4. Resolution of the original problem

Recall that $w = \varphi^{-2/q} v = \varphi^{2/p}\varphi^{-2} u$. Proposition 3.2 allows us to conclude that $\varphi^{-2}u \in C^2([0,1];L^p(\Omega))$. On the other hand, observe that

$$\varrho\partial_t^2 w = \varphi^{2/q}\partial_t^2 w = \varphi^{2/q}\varphi^{-2/q}\partial_t^2 v = \partial_t^2 u,$$
$$\partial_\eta^2 w = \varphi^{-2/q}\varphi^2\partial_y^2 u = \varphi^{2/p}\partial_y^2 u.$$

Keeping in mind that $\varrho\partial_t^2 w,\ \partial_\xi^2 w + \partial_\eta^2 w \in C^\theta([0,1];L^p(Q))$, we may deduce that $\partial_t^2 u,\ \partial_x^2 u + \partial_y^2$ belong to $C^\theta([0,1];L^p(\Omega))$. This, in turn, implies that $\partial_t^2 u,\ \partial_x^2 u,\ \partial_y^2$ are in the class $C^\theta([0,1];L^p(\Omega))$, which completes the proof of Theorem 1.1.

Acknowledgement. The second named author is partially supported by Grant 174024 of Ministry of Science and Technological Development, Republic of Serbia.

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(Received 07 07 2018)

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