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INEQUALITIES FOR s-TH MEANS FUNCTION OF ORDER k

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ABSTRACT. We establish some new inequalities for s-th functions and means of order k by using Popoviciu's, Bellman's, Menon's and Mitrinović, Bullen and Vasić's inequalities. The new inequalities in special cases yield some related inequalities published recently, which provide also new estimates on inequalities of these type.

1. Introduction

Let a be a real n-tuple, s $(\neq 0)$ be a real, k be a positive integer, and r $(1 \leqslant r \leqslant n)$ be an integer. The s-th functions of order k, $t_n^{[k,s]}(a)$, is defined by (see [**14**, p. 166])

$$\sum_{k=0}^{+\infty} t_n^{[k,s]}(a)x^k = \sum_{k=0}^{+\infty} \binom{k}{ns} \omega_n^{[k,s]}(a)x^k = \prod_{i=1}^{n}(1+a_i x)^s,$$

where $s > 0$, while the s-th mean of order k which is connected to this function is defined by

$$W_n^{[k,s]}(a) = \big(\omega_n^{[k,s]}(a)\big)^{1/k} = \left(\frac{t_n^{[k,s]}(a)}{\binom{k}{ns}}\right)^{1/k},$$

where $s > 0$.

Inequalities for s-th means and functions of order k are interesting and valuable inequalities. For now, these inequalities have attracted extensive attention and research (see [**1**, **3**–**11**, **13**, **15**–**18**, **21**, **22**]. The first aim of this paper is to give a new inequality for the s-th functions of order k.

$$(1.1)\quad \big(\omega_n^{[k,s]}(a) - \omega_n^{[k,s]}(b)\big)^2 \geqslant \big(\omega_n^{[k-1,s]}(a) - \omega_n^{[k-1,s]}(b)\big)\big(\omega_n^{[k+1,s]}(a) - \omega_n^{[k+1,s]}(b)\big),$$

with equality if and only if $a_1 = \cdots = a_n$, where a, b are non-negative n-tuples such that $a > b$ and b_r are equal, and $s > 0$, k $(1 \leqslant k < s)$ is an integer, when s

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is not an integer, or $1 \leqslant k < ns$ if s is an integer. The following inequality was established by Mitrinović, Bullen and Vasić [**13**] (see also [**14**, p. 166]).

$$\left(\omega_n^{[k,s]}(a)\right)^2 \geqslant \omega_n^{[k-1,s]}(a)\omega_n^{[k+1,s]}(a), \tag{1.2}$$

with equality if and only if $a_1 = \cdots = a_n$.

Obviously, (1.1) is a generalization of (1.2). (1.1) is a special case of (2.1) in Theorem 2.1.

The next inequality has provoked the great interest of mathematicians in [**13**] (also see [**2**, p. 167]). For $s > 0$, k is an integer, $k < s + 1$, if k is not an integer, and $a \geqslant 0$ and $b \geqslant 0$

$$W_n^{[k,s]}(a+b) \geqslant W_n^{[k,s]}(a) + W_n^{[k,s]}(b), \tag{1.3}$$

with equality if and only if a and b are proportional or $k = 1$. The another aim of this paper is to give a new inequality for the s-th mean of order k by using the Bellman's inequality.

$$\begin{aligned}\Big[\big(W_n^{[k,s]}(a+b)\big)^k - \big(W_n^{[k,s]}(c+d)\big)^k\Big]^{1/k} \geqslant \Big(\big(W_n^{[k,s]}(a)\big)^k - \big(W_n^{[k,s]}(c)\big)^k\Big)^{1/k} \\ + \Big(\big(W_n^{[k,s]}(b)\big)^k - \big(W_n^{[k,s]}(d)\big)^k\Big)^{1/k},\end{aligned} \tag{1.4}$$

with equality if and only if a and b are proportional or $k = 1$. Here $a \geqslant 0$, $b \geqslant 0$, and $c \geqslant 0$ and $d \geqslant 0$ such as $a > c$ and $b > d$, and c and d be proportional, and $s > 0$, k is an integer, $k < s + 1$ if s is not an integer.

Obviously, for $c = d = 0$, (1.4) reduces to (1.3). (1.4) is a special case of (2.5) in Theorem 2.2.

Suppose that a is a nonnegative n-tuple, $\theta > 0$, $\lambda_{i,j} (1 \leqslant i \leqslant n, j = 1, 2, \dots)$ is a sequence of positive numbers and define $\nu_{i,j}$ $(1 \leqslant i \leqslant n, j = 1, 2, \dots)$ with

$$\lambda_{i,r} = \frac{1}{r!}\prod_{j=1}^{r}\nu_{i,j}.$$

Define, as it was done by Whiteley [**21**] and Bullen [**4**], the function $a \to G_n^{[k]}(a)$ of order k by

$$\sum_{k=0}^{+\infty} G_n^{[k]}(a)x^k = \theta\prod_{i=0}^{n}\left(1 + \sum_{r=1}^{+\infty}\lambda_{i,r}(a_i x)^r\right). \tag{1.5}$$

Note that the function $t_n^{[k,s]}$ is a particular case of $G_n^{[k]}$. It is enough to take $\nu_{i,j} = s - j + 1$ $(s > 0)$. What's interesting is that an inequality about $G_n^{[k]}(a)$ was established by Menon [**11**] (also see [**14**, p.168]).

$$\left(G_n^{[k]}(a+b)\right)^{1/k} \geqslant \left(G_n^{[k]}(a)\right)^{1/k} + \left(G_n^{[k]}(b)\right)^{1/k}, \quad k \geqslant 1, \tag{1.6}$$

with equality if and only if a and b are proportional or $k = 1$, where a and b are nonnegative n-tuple and if $\lambda_{i,r}$ $(r = 1, 2, \dots)$ is strictly log-concave for every i $(1 \leqslant i \leqslant n)$. The final aim of this paper is to give a new interesting inequality for the function $G_n^{[k]}(a)$.

$$\left(G_n^{[k]}(a+b) - G_n^{[k]}(c+d)\right)^{1/k} \geqslant \left(G_n^{[k]}(a) - G_n^{[k]}(c)\right)^{1/k} + \left(G_n^{[k]}(b) - G_n^{[k]}(d)\right)^{1/k}, \tag{1.7}$$

with equality if and only if a and b are proportional or $k=1$. Here a, b and k are as in (1.5), and $\lambda_{i,r}$ $(r=1,2,\dots)$ is strictly log-concave for every i $(1 \leqslant 1 \leqslant n)$, and c and d are nonnegative n-tuples such as c and d are proportional, and $a > c$ and $b > d$.

When $c = d = 0$, (1.7) becomes (1.6). (1.7) is a special case of (2.9) in Theorem 2.3.

Moreover, recent studies related to this content can be found in references [**19**, **20**].

2. Main Results

We need the following Lemmas to prove our main results.

Lemma 2.1. [**12**, p. 58] *Let $p>0, q>0$, $\frac{1}{p}+\frac{1}{q}=1$, and $a=\{a_1,\dots,a_n\}$ and $b=\{b_1,\dots,b_n\}$ be two series of positive real numbers and such that $a_1^p-\sum_{i=2}^n a_i^p > 0$ and $b_1^q-\sum_{i=2}^n b_i^q>0$. Then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}\left(b_1^q-\sum_{i=2}^n b_i^q\right)^{1/q} \leqslant a_1b_1-\sum_{i=2}^n a_ib_i,$$

with equality if and only if $a=\mu b$, where μ is a constant.

Here, we call this inequality Popoviciu's inequality.

Lemma 2.2. [**2**, p. 38] *Let $a=\{a_1,\dots,a_n\}$ and $b=\{b_1,\dots,b_n\}$ be two series of positive real numbers and $p>1$ such that $a_1^p-\sum_{i=2}^n a_i^p>0$ and $b_1^p-\sum_{i=2}^n b_i^p>0$, then*

$$\left(a_1^p-\sum_{i=2}^n a_i^p\right)^{1/p}+\left(b_1^p-\sum_{i=2}^n b_i^p\right)^{1/p} \leqslant \left((a_1+b_1)^p-\sum_{i=2}^n (a_i+b_i)^p\right)^{1/p},$$

with equality if and only if $a=\upsilon b$ where υ is a constant.

Here, we call this inequality Bellman's inequality. Our main results are given in the following theorems.

Theorem 2.1. *Let $m\in\mathbb{N}^+$, $p>0, q>0$, $\frac{1}{p}+\frac{1}{q}=1$. If a, b be non-negative n-tuples such that $\omega_n^{[k-1,s]}(a)>m\omega_n^{[k-1,s]}(b)$, $\omega_n^{[k+1,s]}(a)>m\omega_n^{[k+1,s]}(b)$ and $b_1=\dots=b_n$. If $s>0$, k $(1\leqslant k<s)$ is an integer, when s is not an integer, or $1\leqslant k<ns$ if s is an integer, then*

$$\text{(2.1)}\qquad \mathcal{W}_n(a;k,s,p,q)-m\mathcal{W}_n(b;k,s,p,q) \geqslant \left(\omega_n^{[k-1,s]}(a)-m\omega_n^{[k-1,s]}(b)\right)^{1/p} \times\left(\omega_n^{[k+1,s]}(a)-m\omega_n^{[k+1,s]}(b)\right)^{1/q},$$

with equality if and only if $a_1=\dots=a_n$, and where

$$\mathcal{W}_n(x;k,s,p,q)=\left(\omega_n^{[k,s]}(x)\right)^2\left(\omega_n^{[k-1,s]}(x)\right)^{(1-p)/p}\left(\omega_n^{[k+1,s]}(x)\right)^{(1-q)/q}.$$

Proof. Let's prove this theorem by mathematical induction for m. First, we prove that (2.1) holds for $m=1$. From (1.2), we obtain

$$\text{(2.2)}\qquad \mathcal{W}_n(a;k,s,p,q)\geqslant\left(\omega_n^{[k-1,s]}(a)\right)^{1/p}\left(\omega_n^{[k+1,s]}(a)\right)^{1/q},$$

with equality if and only if $a_1 = \cdots = a_n$, and

$$\mathcal{W}_n(b;k,s,p,q) = \big(\omega_n^{[k-1,s]}(b)\big)^{1/p}\big(\omega_n^{[k+1,s]}(b)\big)^{1/q}, \tag{2.3}$$

From (2.2), (2.3) and in view of the Popoviciu's inequality, we have

$$\begin{aligned}\mathcal{W}_n(a;k,s,p,q) - \mathcal{W}_n(b;k,s,p,q) &\geqslant \big(\omega_n^{[k-1,s]}(a)\big)^{1/p}\big(\omega_n^{[k+1,s]}(a)\big)^{1/q}\\ &\quad - \big(\omega_n^{[k-1,s]}(b)\big)^{1/p}\big(\omega_n^{[k+1,s]}(b)\big)^{1/q}\\ &\geqslant \big(\omega_n^{[k-1,s]}(a) - \omega_n^{[k-1,s]}(b)\big)^{1/p}\\ &\quad \times \big(\omega_n^{[k+1,s]}(a) - \omega_n^{[k+1,s]}(b)\big)^{1/q}.\end{aligned}$$

From the equality conditions of (1.2) and Popoviciu's inequality, it follows that the equality in (2.1) holds if and only if $a_1 = \cdots = a_n$.

This shows (2.1) right for $m = 1$.

Suppose that (2.1) holds when $m = r - 1$, we have

$$\begin{aligned}&\mathcal{W}_n(a;k,s,p,q) - (r-1)\mathcal{W}_n(b;k,s,p,q)\\ &\quad\geqslant \big(\omega_n^{[k-1,s]}(a) - (r-1)\omega_n^{[k-1,s]}(b)\big)^{1/p}\big(\omega_n^{[k+1,s]}(a) - (r-1)\omega_n^{[k+1,s]}(b)\big)^{1/q},\end{aligned} \tag{2.4}$$

with equality if and only if $a_1 = \cdots = a_n$.

From (2.3), (2.4) and by using the Popoviciu's inequality again, we obtain

$$\begin{aligned}&\mathcal{W}_n(a;k,s,p,q) - r\mathcal{W}_n(b;k,s,p,q)\\ &\quad\geqslant \big(\omega_n^{[k-1,s]}(a) - (r-1)\omega_n^{[k-1,s]}(b)\big)^{1/p}\big(\omega_n^{[k+1,s]}(a) - (r-1)\omega_n^{[k+1,s]}(a)\big)^{1/q}\\ &\qquad - \big(\omega_n^{[k-1,s]}(b)\big)^{1/p}\big(\omega_n^{[k+1,s]}(b)\big)^{1/q}\\ &\quad\geqslant \big(\omega_n^{[k-1,s]}(a) - r\omega_n^{[k-1,s]}(b)\big)^{1/p}\big(\omega_n^{[k+1,s]}(a) - r\omega_n^{[k+1,s]}(a)\big)^{1/q},\end{aligned}$$

with equality if and only if $a_1 = \cdots = a_n$.

This shows that (2.1) is correct if $m = r-1$, then $m = r$ is also correct. Hence (2.1) is right for any $m \in \mathbb{N}^+$. □

Corollary 2.1. *If a, b, k and s are as in Theorem 2.1, then*

$$\big(\omega_n^{[k,s]}(a) - \omega_n^{[k,s]}(b)\big)^2 \geqslant \big(\omega_n^{[k-1,s]}(a) - \omega_n^{[k-1,s]}(b)\big)\big(\omega_n^{[k+1,s]}(a) - \omega_n^{[k+1,s]}(b)\big),$$

with equality if and only if $a_1 = \cdots = a_n$.

Proof. This follows immediately from the proof of Theorem 2.1. □

Theorem 2.2. *Let $m \in \mathbb{N}^+$, $a \geqslant 0$, $b \geqslant 0$, and $c \geqslant 0$ and $d \geqslant 0$ such as $W_n^{[k,s]}(a) > m^{1/k}W_n^{[k,s]}(c)$ and $W_n^{[k,s]}(b) > m^{1/k}W_n^{[k,s]}(d)$, and c and d be proportional. If $s > 0$, k is an integer, $k < s+1$ if s is not an integer, then*

$$\begin{aligned}&\Big[\big(W_n^{[k,s]}(a+b)\big)^k - m\big(W_n^{[k,s]}(c+d)\big)^k\Big]^{1/k}\\ &\quad\geqslant \Big(\big(W_n^{[k,s]}(a)\big)^k - m\big(W_n^{[k,s]}(c)\big)^k\Big)^{1/k}\\ &\qquad + \Big(\big(W_n^{[k,s]}(b)\big)^k - m\big(W_n^{[k,s]}(d)\big)^k\Big)^{1/k},\end{aligned} \tag{2.5}$$

with equality if and only if a and b are proportional or $k=1$.

PROOF. First, we prove that (2.5) holds for $m=1$. From (1.2) and (1.3), we obtain

$$\left(\frac{t_n^{[k,s]}(a+b)}{\binom{k}{ns}}\right)^{1/k} \geqslant \left(\frac{t_n^{[k,s]}(a)}{\binom{k}{ns}}\right)^{1/k} + \left(\frac{t_n^{[k,s]}(b)}{\binom{k}{ns}}\right)^{1/k}, \tag{2.6}$$

with equality if and only if a and b are proportional or $k=1$, and

$$\left(\frac{t_n^{[k,s]}(c+d)}{\binom{k}{ns}}\right)^{1/k} = \left(\frac{t_n^{[k,s]}(c)}{\binom{k}{ns}}\right)^{1/k} + \left(\frac{t_n^{[k,s]}(d)}{\binom{k}{ns}}\right)^{1/k}, \tag{2.7}$$

From (2.6), (2.7) and in view of the Bellman's inequality, we have

$$\begin{aligned}
&\left[\left(W_n^{[k,s]}(a+b)\right)^k - \left(W_n^{[k,s]}(c+d)\right)^k\right]^{1/k} \\
&\geqslant \left[\left(\left(\frac{t_n^{[k,s]}(a)}{\binom{k}{ns}}\right)^{1/k} + \left(\frac{t_n^{[k,s]}(b)}{\binom{k}{ns}}\right)^{1/k}\right)^k - \left(\left(\frac{t_n^{[k,s]}(c)}{\binom{k}{ns}}\right)^{1/k} + \left(\frac{t_n^{[k,s]}(d)}{\binom{ns}{k}}\right)^{1/k}\right)^k\right]^{1/k} \\
&\qquad \geqslant \left(\left(W_n^{[k,s]}(a)\right)^k - \left(W_n^{[k,s]}(c)\right)^k\right)^{1/k} + \left(\left(W_n^{[k,s]}(b)\right)^k - \left(W_n^{[k,s]}(d)\right)^k\right)^{1/k}.
\end{aligned}$$

From the equality conditions of (1.3) and Bellman's inequality, it follows that the equality in (2.4) holds if and only if a and b are proportional or $r=1$.

This shows (2.5) right for $m=1$.

Suppose that (2.5) holds when $m=r-1$, we have

$$\begin{aligned}
&\left[\left(W_n^{[k,s]}(a+b)\right)^k - (r-1)\left(W_n^{[k,s]}(c+d)\right)^k\right]^{1/k} \\
&\qquad \geqslant \left(\left(W_n^{[k,s]}(a)\right)^k - (r-1)\left(W_n^{[k,s]}(c)\right)^k\right)^{1/k} \\
&\qquad\qquad + \left(\left(W_n^{[k,s]}(b)\right)^k - (r-1)\left(W_n^{[k,s]}(d)\right)^k\right)^{1/k},
\end{aligned} \tag{2.8}$$

with equality if and only if a and b are proportional or $k=1$.

From (2.7), (2.8) and by using the Bellman's inequality, we obtain

$$\begin{aligned}
&\left[\left(W_n^{[k,s]}(a+b)\right)^k - r\left(W_n^{[k,s]}(c+d)\right)^k\right]^{1/k} \\
&\qquad \geqslant \Bigg\{\Big[\left(\left(W_n^{[k,s]}(a)\right)^k - (r-1)\left(W_n^{[k,s]}(c)\right)^k\right)^{1/k} \\
&\qquad + \left(\left(W_n^{[k,s]}(b)\right)^k - (r-1)\left(W_n^{[k,s]}(d)\right)^k\right)^{1/k}\Big]^k \\
&\qquad - \left[\left(\frac{t_n^{[k,s]}(c)}{\binom{k}{ns}}\right)^{1/k} + \left(\frac{t_n^{[k,s]}(d)}{\binom{k}{ns}}\right)^{1/k}\right]^k\Bigg\}^{1/k} \\
&\qquad \geqslant \left(\left(W_n^{[k,s]}(a)\right)^k - r\left(W_n^{[k,s]}(c)\right)^k\right)^{1/k} \\
&\qquad\qquad + \left(\left(W_n^{[k,s]}(b)\right)^k - r\left(W_n^{[k,s]}(d)\right)^k\right)^{1/k},
\end{aligned}$$

with equality if and only if a and b are proportional or $k=1$.

This shows that (2.5) is correct if $m=r-1$, then $m=r$ is also correct. Hence (2.5) is right for any $m \in \mathbb{N}^+$. □

THEOREM 2.3. *Let $m \in \mathbb{N}^+$, a, b, c and d be nonnegative n-tuples such as c and d are proportional, and $G_n^{[k]}(a) > mG_n^{[k]}(c)$ and $G_n^{[k]}(b) > mG_n^{[k]}(d)$. If $\lambda_{i,r}$ $(r = 1, 2, \dots)$ is strictly log-concave for every i $(1 \leqslant i \leqslant n)$, then*

$$\begin{aligned}(2.9)\qquad \big(G_n^{[k]}(a+b) - mG_n^{[k]}(c+d)\big)^{1/k} &\geqslant \big(G_n^{[k]}(a) - mG_n^{[k]}(c)\big)^{1/k} \\ &\quad + \big(G_n^{[k]}(b) - mG_n^{[k]}(d)\big)^{1/k},\end{aligned}$$

with equality if and only if a and b are proportional or $k = 1$.

PROOF. First, we prove that (2.9) holds for $m = 1$. From (1.6), it is easy to obtain

$$(2.10)\qquad \big(G_n^{[k]}(a+b)\big)^{1/k} \geqslant \big(G_n^{[k]}(a)\big)^{1/k} + \big(G_n^{[k]}(b)\big)^{1/k},$$

with equality if and only if a and b are proportional or $k = 1$, and

$$(2.11)\qquad \big(G_n^{[k]}(c+d)\big)^{1/k} = \big(G_n^{[k]}(c)\big)^{1/k} + \big(G_n^{[k]}(d)\big)^{1/k},$$

From (2.10), (2.11) and the Bellman's inequality, we have

$$\begin{aligned}&\big(G_n^{[k]}(a+b) - G_n^{[k]}(c+d)\big)^{1/k} \\ &\quad\geqslant \big\{\big[\big(G_n^{[k]}(a)\big)^{1/k} + \big(G_n^{[k]}(b)\big)^{1/k}\big]^k - \big[\big(G_n^{[k]}(c)\big)^{1/k} + \big(G_n^{[k]}(d)\big)^{1/k}\big]^k\big\}^{1/k} \\ &\quad\geqslant \big(G_n^{[k]}(a) - G_n^{[k]}(c)\big)^{1/k} + \big(G_n^{[k]}(b) - G_n^{[k]}(d)\big)^{1/k}.\end{aligned}$$

From the equality conditions of (1.6) and Bellman's inequality, it follows that the equality in (2.9) holds if and only if a and b are proportional or $k = 1$.

This shows (2.9) right for $m = 1$.

Suppose that (2.9) holds when $m = r - 1$, we have

$$\begin{aligned}(2.12)\qquad \big(G_n^{[k]}(a+b) - (r-1)G_n^{[k]}(c+d)\big)^{1/k} &\geqslant \big(G_n^{[k]}(a) - (r-1)G_n^{[k]}(c)\big)^{1/k} \\ &\quad + \big(G_n^{[k]}(b) - (r-1)G_n^{[k]}(d)\big)^{1/k},\end{aligned}$$

with equality if and only if a and b are proportional or $k = 1$.

From (2.11), (2.12) and by using the Bellman's inequality again, we obtain

$$\begin{aligned}\big(G_n^{[k]}(a+b) - rG_n^{[k]}(c+d)\big)^{1/k} &\geqslant \big\{\big[\big(G_n^{[k]}(a) - (r-1)G_n^{[k]}(c)\big)^{1/k} \\ &\quad + \big(G_n^{[k]}(b) - (r-1)G_n^{[k]}(d)\big)^{1/k}\big]^k \\ &\quad - \big(\big(G_n^{[k]}(c)\big)^{1/k} + \big(G_n^{[k]}(d)\big)^{1/k}\big)^k\big\}^{1/k} \\ &\geqslant \big(G_n^{[k]}(a) - rG_n^{[k]}(c)\big)^{1/k} + \big(G_n^{[k]}(b) - rG_n^{[k]}(d)\big)^{1/k},\end{aligned}$$

with equality if and only if a and b are proportional or $k = 1$.

This shows that (2.9) is correct if $m = r - 1$, then $m = r$ is also correct. Hence (2.9) is right for any $m \in \mathbb{N}^+$. □

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References

1. V. J. Baston, *Two inequalities for the complete symmetric function*, Math. Proc. Camb. Philos. Soc. **84**(1) (1978), 1–3.
2. E. F. Bechenbach, R. Bellman, *Inequalities*, Springer-Verlag, Berlin-Göttingen, Heidelberg, 1961.
3. P. S. Bullen, *On some forms of Whiteley*, Publ. Elektroteh. Fak., Univ. Beogr., Ser. Mat. Fiz. **41** (1975), 59–64.
4. P. S. Bullen, M. Marcus, *Symmetric means and matrix inequalities*, Proc. Am. Math. Soc. **12** (1961), 286–289.
5. G. H. E. Duchamp, F. Hivert, J-C. Novelli, J-Y. Thibon, *Noncommutative symmetric functions VII: free quasi-symmetric functions revisited*, Ann. Comb. **15** (2011), 655–673.
6. K. Guan, *Schur-convexity of the complete elementary symmetric function*, J. Inequal. Appl. **2006** (2006), 67624.
7. I. R. Kayumov, *On holomorphic motions of n-symmetric functions*, Math. Notes **87**(5-6) (2010), 828–833.
8. T. I. Krasnova, *The conjunction complexity asymptotic of self-correcting circuits for monotone symmetric functions with threshold*, Moscow Univ. Math. Bull. **69**(3) (2014), 121-124.
9. M. Marcus, L. Lopes, *Inequalities for symmetric functions and Hermitian matrices*, Can. J. Math. **9** (1957), 305–312.
10. J. B. Mcleod, *On four inequalities in symmetric functions*, Proc. Edinb. Math. Soc., II. Ser. **11** (1958/59), 211–219.
11. K. V. Menon, *Inequalities for symmetric functions*, Duke Math. J. **35** (1968), 37–45.
12. D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag Berlin, Heidelberg, New York, 1970.
13. D. S. Mitrinović, P. S. Bullen, P. M. Vasić, *Sredine i sa njima provezane nejednakosti I*, Publ. Elektroteh. Fak., Univ. Beogr., Ser. Mat. Fiz. **600** (1997), 1–232.
14. D. S. Mitrinović, J. E. Pećarić, A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer academic publisher, Dordrecht-Boston-London, 1993.
15. N. Ozeki, *On the convex sequences* (V), J. Coll. Arts Sci. Chiba Univ. **B5** (1972), 1–4.
16. Z. Páles, *Inequalities for sums of multipowers*, Acta Math. Hung. **56**(1–2) (1990), 165–175.
17. H. N. Shi, *Refinement and generalization of a class of inequalities for symmetric functions*, Math. Pract. Theory **29**(4) (1999), 81-84.
18. C. Tănăsescu, *Concavity via hyperbolic forms*, Rad. JAZU **9**(450) (1990), 53–67.
19. J. Tian, M. Ha, *Properties and refinements of Aczél-type inequalities*, J. Math. Inequal. **12**(1) (2018), 175–189.
20. J. Tian, S. Wu, *New refinements of generalized Aczél's inequality and their applications*, J. Math. Inequal. **10**(1) (2016), 247–259.
21. J. N. Whiteley, *Two theorems on convolutions*, Proc. Lond. Math. Soc. **37** (1962), 450–568.
22. L. Zhu, *New refinements of Young's inequality*, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM **113**(2) (2019), 909–915.

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