PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 112(126) (2022), 111–121 DOI: https://doi.org/10.2298/PIM2226111A

AN INTEGRAL FUNCTIONAL EQUATION ON ABELIAN SEMIGROUPS

Ahmed Akkaoui and Mohamed El Fatini

Abstract. We investigate a generalization of many functional equations. Namely, we consider the following functional equation

$$\int_S f(x+y+t)\,d\mu(t)+\int_S f(x+\varphi(y)+t)\,d\nu(t)=f(x)+h(y),\quad x,y\in S,$$

where $(S,+)$ is an abelian semigroup, φ is a surjective endomorphism of S, E is a linear space over the field $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ and μ,ν are linear combinations of Dirac measures. Under appropriate conditions on μ and ν and based on Stetkær's result [**9**], we find and characterize solutions of the previous functional equation.

1. Notations and preliminary results

In this work, we consider $(S,+)$ an abelian semigroup, i.e., a nonempty set equipped with an associative operation, φ an endomorphism of S, E a linear space over the field $\mathbb{K}\in\{\mathbb{R},\mathbb{C}\}$ and μ,ν linear combinations of Dirac measures. We say that the semigroup S is a topological semigroup, if S is equipped with a topology such that the product map $(x,y)\mapsto x+y$ from $S\times S$ to S is continuous, when $S\times S$ is given the topological product.

A function $A\colon S\to E$ is said to be additive if $A(x+y)=A(x)+A(y)$ for all $x,y\in S$. A map $Q(\cdot,\cdot)\colon S\times S\to E$ is said to be bi-additive if $Q(\cdot,x)\colon S\to E$ and $Q(x,\cdot)\colon S\to E$ are additive for each fixed $x\in S$, and is said to be symmetric if $Q(x,y)=Q(y,x)$ for all $x,y\in S$. The endomorphism σ of S is said to be involutive if $\sigma^2=id$. We denote by δ_z the Dirac measure concentrated at z. The number $\nu(S)$ and the measure ν^- (when S is a group) are respectively defined by

$$\nu(S):=\int_S d\nu(t)\quad\text{and}\quad\int_S g(t)\,d\nu^-(t):=\int_S g(-t)\,d\nu(t)$$

for all function $g\colon S\to E$.

2020 *Mathematics Subject Classification:* Primary 39B32; 39B52.

Key words and phrases: quadratic functional equation, additive map, endomorphism, measure, semigroup.

Communicated by Stevan Pilipović.

We recall that the second difference $C^2f\colon S\times S\times S \to E$ of a function $f\colon S\to E$ is defined by

$$C^2f(x,y,z) := f(x+y+z) - f(x+y) - f(y+z) - f(x+z) + f(x) + f(y) + f(z).$$

The functional equation $C^2f(x,y,z) = 0$ is called Whitehead's functional equation. It is equivalent to

$$f(x+y+z)+f(x)+f(y)+f(z) = f(x+y)+f(y+z)+f(x+z), \quad x,y,z\in S. \tag{1.1}$$

Our main result is based on following Stetkær's result which can be derived in [**9**, Theorems 6 and 7] as follows:

THEOREM 1.1. *Let $f\colon S\to E$ be a function.*

(1) *If f is a solution of Whitehead's functional equation* (1.1)*, then there exists an additive map $A\colon S\to E$ and a symmetric, bi-additive map $Q\colon S\times S\to E$ such that*

$$f(x) = Q(x,x) + A(x), \quad \textit{for all} \quad x\in S. \tag{1.2}$$

(2) *Assume that S is a topological semigroup and E is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$. If $f\colon S\to E$ is a continuous solution of Whitehead's functional equation* (1.1)*, then the components A and Q in decomposition* (1.2) *are continuous.*

2. Introduction

The following functional equation

$$\int_S f(x+y+t)\,d\mu(t) + \int_S f(x+\sigma(y)+t)\,d\nu(t) = f(x)+f(y), \quad x,y\in S, \tag{2.1}$$

was solved by Akkaoui et al. in [**1**], where $f\colon S\to E$ is the unknown function and σ is an involution of S.

Our aim is to generalize (2.1), by solving the following functional equation

$$\int_S f(x+y+t)\,d\mu(t) + \int_S f(x+\varphi(y)+t)\,d\nu(t) = f(x)+h(y), \quad x,y\in S, \tag{2.2}$$

where φ is a surjective endomorphism of S. Equation (2.2) generalizes several equations which are studied in the literature such as Jensen's, Drygas' or the quadratic equations on abelian monoids. In [**4**], with $\varphi = -\,\mathrm{id}$, the functional equation (2.2) was studied in the case where $\mu = \nu^-$ and ν is a regular compactly supported the complex-valued Borel measure on a locally compact abelian Hausdorff group G such that $\nu(G) = \frac{1}{2}$ and $f\colon G\to\mathbb{C}$ is continuous. If $(S,+)$ is a monoid with a neutral element 0 and $\mu = \nu = \frac{1}{2}\delta_0$, equation (2.2) becomes respectively in the cases where $h = f$ and $h = 0$ the following generalized Jensen and quadratic functional equations:

$$f(x+y) + f(x+\varphi(y)) = 2f(x), \quad x,y\in S,$$
$$f(x+y) + f(x+\varphi(y)) = 2f(x) + 2f(y), \quad x,y\in S,$$

which are solved on an abelian semigroup (see [**7**]) in the case where φ is involutive and in the case where φ is an arbitrary endomorphism of S (see [**6**]).

Using our main result Theorem 3.1, we solve in the last section the following special cases of functional equation (2.2):

$$\int_S f(x+\sigma(y)+t)\,d\mu(t)+\int_S f(x+\tau(y)+t)\,d\nu(t)=f(x)+h(y),\quad x,y\in S, \tag{2.3}$$

$$\alpha f(x+y+a)+\beta f(x+\varphi(y)+b)=f(x)+h(y),\quad x,y\in S, \tag{2.4}$$

$$\alpha f(x+\sigma(y)+a)+\beta f(x+\tau(y)+b)=f(x)+h(y),\quad x,y\in S, \tag{2.5}$$

where σ,τ are two involutions of S, a,b are two arbitrary elements of S, and α,β are two arbitrary elements of $\mathbb{K}$. Equation (2.3) is an important generalization of (2.1) because we use here two involutions instead of one. If $(S,+)$ is a monoid with a neutral element 0, $a=b=0$, $\alpha=\beta=\frac{1}{2}$ and $h=f$, then equation (2.5) becomes

$$f(x+\sigma(y))+f(x+\tau(y))=2f(x)+2f(y),\quad x,y\in S,$$

which is solved on abelian semigroup in [**5**]. In the case where φ is involutive and $\alpha=\beta=\frac{1}{2}$, equation (2.4) is solved on monoid (see [**2**]) and on a locally compact abelian Hausdorff group [**3**] (only the complex-valued continuous solutions).

Our main contribution is to consider φ to be surjective in (2.2) and (2.4) (not necessarily involutive). Also in equations (2.3) and (2.5) σ and τ are two involutions, and also equations (2.4) and (2.5) are studied with respect to an abelian semigroup (are not necessarily on a group or a monoid).

3. Main results

To establish our Theorem 3.1, we present these two important lemmas.

Lemma 3.1. *Let $f,F:S\to E$ be two functions such that for all $x,y,z\in S$*

$$F(x+y+z)=f(x+y)+f(x+z)+f(y+z)-f(x)-f(y)-f(z), \tag{3.1}$$

then there exists an element $c\in E$ such that the function $f-c$ satisfies Whitehead's functional equation (1.1)*, i.e., $C^2(f-c)=0$.*

Proof. Let $f,F\colon S\to E$ be two functions satisfying (3.1) and let $x,y,z,r\in S$. Making the substitutions $(x+y,r,z)$ and $(x+r,y,z)$ in (3.1), we obtain

$$\begin{aligned}F(x+y+r+z)&=f(x+y+r)+f(x+y+z)+f(r+z)\\&\quad-f(x+y)-f(r)-f(z),\end{aligned} \tag{3.2}$$

$$\begin{aligned}F(x+r+y+z)&=f(x+r+y)+f(x+r+z)+f(y+z)\\&\quad-f(x+r)-f(y)-f(z).\end{aligned} \tag{3.3}$$

Subtracting (3.3) from (3.2), we get

$$f(x+y+z)+f(r+z)-f(x+y)-f(r)=f(x+z+r)+f(y+z)-f(x+r)-f(y).$$

Then for all $x,y,z,r\in S$, we have

$$f(x+y+z)-f(x+y)-f(y+z)+f(y)=f(x+r+z)-f(x+r)-f(r+z)+f(r).$$

We can deduce that the expression $f(x+y+z)-f(x+y)-f(y+z)+f(y)$ depends only on x and z. Let

$$g(x,z):=f(x+y+z)-f(x+y)-f(y+z)+f(y),\quad x,z\in S, \tag{3.4}$$

for $y\in S$.

Interchanging y and z in (3.4) and subtracting the new equation from (3.4), we get for all $x,y,z\in S$

$$g(x,z)-f(x+z)+f(z)=g(x,y)-f(x+y)+f(y).$$

Again the expression $g(x,y)-f(x+y)+f(y)$ depends only on x. Let

$$k(x):=g(x,y)-f(x+y)+f(y),\quad x\in S, \tag{3.5}$$

for $y\in S$. Changing the role of x and y in (3.5) and subtracting the new equation from (3.5), we get for all $x,y\in S$

$$k(x)+f(x)=k(y)+f(y),$$

because g is symmetric.

Then the function $k+f$ is a constant. Let $c\in E$ such that $k+f=c$. According to (3.5), we have for all $x,y\in S$

$$g(x,y)=f(x+y)-f(x)-f(y)+c.$$

Then (3.4) gives

$$f(x+y+z)-c=f(x+y)+f(x+z)+f(y+z)-f(x)-f(y)-f(z),$$

for all $x,y,z\in S$. So the function $V:=f-c$ satisfies Whitehead's functional equation (1.1), i.e., $C^2(f-c)=0$. □

Remark 3.1. In Lemma 3.1, we can replace the space E by an arbitrary abelian group.

Lemma 3.2. *Let $f,h\colon S\to E$ be a solution of* (2.2)*. Then there exists an element $c'\in E$ such that for all $x,y,z\in S$ we have*

$$\begin{aligned}\int_S f(\varphi(x+y+z)+t)\,d(\mu+\nu)(t)&=f\circ\varphi(x+y)+f\circ\varphi(x+z)+f\circ\varphi(y+z)\\&\quad-f\circ\varphi(x)-f\circ\varphi(y)-f\circ\varphi(z)+c'.\end{aligned}$$

Proof. Let $f,h\colon S\to E$ be a solution of (2.2) and let $x,y,z\in S$. Making the substitutions $(x+y,z)$, $(x,y+z)$ and $(x+\varphi(z),y)$ in (2.2), we get respectively

$$\int_S f(x+y+z+t)\,d\mu(t)+\int_S f(x+y+\varphi(z)+t)\,d\nu(t)=f(x+y)+h(z),$$
$$\int_S f(x+y+z+t)\,d\mu(t)+\int_S f(x+\varphi(y+z)+t)\,d\nu(t)=f(x)+h(y+z),$$
$$\int_S f(x+y+\varphi(z)+t)\,d\mu(t)+\int_S f(x+\varphi(y+z)+t)\,d\nu(t)=f(x+\varphi(z))+h(y).$$

Subtracting the middle identity from the sum of the other two we obtain

$$\int_S f(x+y+\varphi(z)+t)\,d(\mu+\nu)(t) = f(x+y)+f(x+\varphi(z))-f(x)+h(y)+h(z)-h(y+z).$$

Replacing x and y by $\varphi(x)$ and $\varphi(y)$ respectively in the last equation, we get

$$\begin{aligned}\int_S f(\varphi(x+y+z)+t)\,d(\mu+\nu)(t) &= f\circ\varphi(x+y)+f\circ\varphi(x+z)-f\circ\varphi(x)\\ &\quad + h\circ\varphi(y)+h(z)-h(\varphi(y)+z).\end{aligned} \tag{3.6}$$

Now, changing the role of x and z in the last equation and subtracting the new one from it, we get for all $x,y,z\in S$

$$f\circ\varphi(x+y)-f\circ\varphi(x)+h(\varphi(y)+x)-h(x) = f\circ\varphi(z+y)-f\circ\varphi(z)+h(\varphi(y)+z)-h(z).$$

Hence we can deduce that the expression

$$f\circ\varphi(z+y)-f\circ\varphi(z)+h(\varphi(y)+z)-h(z)$$

depends only on y. Let

$$H(y) := f\circ\varphi(z+y)-f\circ\varphi(z)+h(\varphi(y)+z)-h(z) \quad y\in S,$$

for $z\in S$. Then for all $y,z\in S$ we have

$$h(z)-h(\varphi(y)+z) = f\circ\varphi(z+y)-f\circ\varphi(z)-H(y).$$

Identity (3.6) becomes

$$\begin{aligned}\int_S f(\varphi(x+y+z)+t)\,d(\mu+\nu)(t) &= f\circ\varphi(x+y)+f\circ\varphi(x+z)+f\circ\varphi(y+z)\\ &\quad - f\circ\varphi(x)-f\circ\varphi(z)+h\circ\varphi(y)-H(y).\end{aligned} \tag{3.7}$$

Interchanging the role of y and z in (3.7) and subtracting the new equation from it, we get for all $x,y\in S$

$$f\circ\varphi(z)+h\circ\varphi(z)-H(z) = f\circ\varphi(y)+h\circ\varphi(y)-H(y).$$

Then the function $f\circ\varphi+h\circ\varphi-H$ is constant, say c'. Hence identity (3.7) becomes

$$\begin{aligned}\int_S f(\varphi(x+y+z)+t)\,d(\mu+\nu)(t) &= f\circ\varphi(x+y)+f\circ\varphi(x+z)+f\circ\varphi(y+z)\\ &\quad - f\circ\varphi(x)-f\circ\varphi(y)-f\circ\varphi(z)+c',\end{aligned}$$

which yields the result of Lemma 3.2. □

Proposition 3.1. *If $f,h\colon S\to E$ is a solution of* (2.2)*, then there exists an element $c\in E$ such that $f\circ\varphi-c$ satisfies Whitehead's functional equation* (1.1)*, i.e., $C^2(f\circ\varphi-c)=0$.*

Proof. Let $f,h\colon S\to E$ be a solution of (2.2). According to Lemma 3.2, there exists an element $c'\in E$ such that

$$\begin{aligned}\int_S f(\varphi(x+y+z)+t)\,d(\mu+\nu)(t) &= f\circ\varphi(x+y)+f\circ\varphi(x+z)+f\circ\varphi(y+z)\\ &\quad - f\circ\varphi(x)-f\circ\varphi(y)-f\circ\varphi(z)+c'.\end{aligned}$$

Putting $F(x) := \int_S f(\varphi(x)+t)\,d(\mu+\nu)(t) - c'$, then

$$F(x+y+z) = f\circ\varphi(x+y)+f\circ\varphi(x+z)+f\circ\varphi(y+z)-f\circ\varphi(x)-f\circ\varphi(y)-f\circ\varphi(z). \tag{3.8}$$

Applying Lemma 3.1 to identity (3.8), then there exists an element $c \in E$ such that $C^2(f\circ\varphi - c) = 0$. □

The following main theorem solves functional equation (2.2) on an arbitrary abelian semigroup.

THEOREM 3.1. *Suppose that φ is surjective and let $(\alpha,\beta) := (\mu(S),\nu(S)) \in \mathbb{K}^2$. Then the pair $f, h\colon S \to E$ is a solution of (2.2) if and only if we have the following possibilities:*

(1) *If $\alpha+\beta \neq 1$, then $f = c$, $h = (\alpha+\beta-1)c$, where c is a constant belonging to E.*
(2) *If $\alpha+\beta = 1$, then*
   (a) *If $\alpha = 1$ and $\beta = 0$, then $f = A + c$, $h = A + \int_S A(t)\,d(\mu+\nu)(t)$, where $A\colon S \to X$ is an additive map and c is a constant belonging to E.*
   (b) *If $\alpha = 0$ and $\beta = 1$, then $f = A+c$, $h = A\circ\varphi + \int_S A(t)\,d(\mu+\nu)(t)$, where $A\colon S \to E$ is an additive map and c is a constant belonging to E.*
   (c) *If $\alpha \neq 0$ and $\beta \neq 0$, then*

$$f(x) = Q(x,x) + A(x) + c, \tag{3.9}$$

$$\begin{aligned} h(x) = {}& \frac{\alpha}{\beta}Q(x,x) + \alpha A(x) + \beta A(\varphi(x)) + 2\int_S Q(x,t)\,d\Big(\mu - \frac{\alpha}{\beta}\nu\Big)(t) \\ &+ \int_S \{Q(t,t) + A(t)\} d(\mu+\nu)(t), \end{aligned}$$

where $A\colon S \to E$ is an additive map, $Q\colon S\times S \to E$ is a symmetric, bi-additive map such that for all $x, y \in S$

(i) $Q(x,\varphi(y)) = -\frac{\alpha}{\beta}Q(x,y)$, (ii) $\int_S Q(x,t)\,d(\mu+\nu)(t) = 0$,

and c is a constant belonging to E.

Moreover, if S is a topological semigroup, E is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$, and $f\colon S \to E$ is continuous, then the components A and Q in decomposition (3.9) are continuous and if φ is continuous, then h is continuous.

PROOF. It is easy to check that the functions $f, h : S \to E$ presented in cases (1) and (2) are solutions of (2.2). Conversely, let $f, h\colon S \to E$ be a solution of (2.2). According to Proposition 3.1, there exists an element $c \in E$ such that $f\circ\varphi - c$ satisfies Whitehead's functional equation (1.1), i.e., $C^2(f\circ\varphi - c) = 0$. Since φ is surjective, we get $C^2(f - c) = 0$.

According to Theorem 1.1, there exists an additive map $A\colon S \to E$ and a symmetric, bi-additive map $Q\colon S\times S \to E$ such that $f(x) = Q(x,x) + A(x) + c$. Substituting f in (2.2) and using the fact that $\mu(S) = \alpha$ and $\nu(S) = \beta$, we get after calculation

$$\begin{aligned}(3.10)\qquad h(y) = {} & (\alpha+\beta-1)[Q(x,x)+A(x)+c] + \alpha Q(y,y) + \beta Q(\varphi(y),\varphi(y)) \\ & + \alpha A(y) + \beta A\circ\varphi(y) + 2\alpha Q(x,y) \\ & + 2\beta Q(x,\varphi(y)) + 2\int_S Q(x,t)\,d(\mu+\nu)(t) \\ & + 2\int_S Q(y,t)\,d\mu(t) + 2\int_S Q(\varphi(y),t)\,d\nu(t) + \theta,\end{aligned}$$

where $\theta := \int_S \{Q(t,t)+A(t)\} d(\mu+\nu)(t)$.

For a fixed $y \in S$, the right-hand side of (3.10) is a function of $x \in S$. Then using the observation that was used in the proof of [**1**, Proposition 3.10], we see that

$$(3.11)\qquad 0 = (\alpha+\beta-1)Q(x,x),$$

$$(3.12)\qquad 0 = (\alpha+\beta-1)A(x) + 2\alpha Q(x,y) + 2\beta Q(x,\varphi(y)) + 2\int_S Q(x,t)\,d(\mu+\nu)(t),$$

$$\begin{aligned}(3.13)\qquad h(y) = {} & (\alpha+\beta-1)c + \theta + \alpha Q(y,y) + \beta Q(\varphi(y),\varphi(y)) \\ & + \alpha A(y) + \beta A(\varphi(y)) + 2\int_S Q(y,t)\,d\mu(t) + 2\int_S Q(\varphi(y),t)\,d\nu(t).\end{aligned}$$

By applying the same observation to (3.12) for the variable y by fixing x, we get

$$(3.14)\qquad (\alpha+\beta-1)A(x) + 2\int_S Q(x,t)\,d(\mu+\nu)(t) = 0,$$

$$(3.15)\qquad \alpha Q(x,y) + \beta Q(x,\varphi(y)) = 0.$$

(1) If $\alpha+\beta-1 \neq 0$, then we derive from (3.11), that $Q(x,x)=0$, which implies $Q=0$ (because $2Q(x,y) = Q(x+y,x+y) - Q(x,x) - Q(y,y)$) and from (3.14), we have $A=0$. Then $f=c$ and from (3.13), $h = (\alpha+\beta-1)c$.

So we deduce the result (1) of Theorem.

(2) If $\alpha+\beta-1=0$, then
   - (a) If $\alpha=1$ and $\beta=0$, then equality (3.15) implies that $Q=0$, hence $f = A+c$ and from (3.13), we get $h = A + \int_S A(t)\,d(\mu+\nu)(t)$. So we deduce (2)(a).
   - (b) If $\alpha=0$ and $\beta=1$, from equality (3.15) we have $Q(x,\varphi(y))=0$ for all $x,y\in S$, which implies that $Q=0$, because φ is surjective. Then $f=A+c$ and $h = A\circ\varphi + \int_S A(t)\,d(\mu+\nu)(t)$. So we conclude (2)(b).
   - (c) If $\alpha\neq 0$ and $\beta\neq 0$, equality (3.15) gives $Q(x,\varphi(y)) = -\frac{\alpha}{\beta}Q(x,y)$ for all $x,y\in S$ and we get from (3.13)

$$\begin{aligned}h(x) = {} & \frac{\alpha}{\beta}Q(x,x) + \alpha A(x) + \beta A(\varphi(x)) + 2\int_S Q(x,t)\,d\Big(\mu - \frac{\alpha}{\beta}\nu\Big)(t) \\ & + \int_S \{Q(t,t)+A(t)\} d(\mu+\nu)(t).\end{aligned}$$

From (3.14), we have $\int_S Q(x,t)\,d(\mu+\nu)(t) = 0$. So we obtain (c)(ii). The continuity statements follow from Theorem 1.1 (ii). □

REMARK 3.2. In Theorem 3.1, we have considered linear combinations of Dirac measures because, quite simply, the unknowns are defined on a semigroup. This result remains valid provided that the integral that defines equation (2.2) exists.

As an application of Theorem 3.1, we present the following example.

EXAMPLE 3.1. Let $S := (\mathbb{R}^2, +)$, $E := (\mathbb{C}, +)$, $a = (1,1)$, $b = (2,2)$, $\mu = \frac{4}{3}\delta_a$, $\nu = -\frac{1}{3}\delta_b$, and let φ be the endomorphism of S defined by $\varphi(x) = (4x_1, 4x_2)$ for all $x := (x_1, x_2) \in S$. Note that φ is not involutive but it is surjective.

Functional equation (2.2) becomes

$$4f(x_1+y_1+1, x_2+y_2+1) - f(x_1+4y_1+2, x_2+4y_2+2) = 3f(x_1,x_2) + 3h(y_1,y_2),$$

for all $(x_1,x_2),(y_1,y_2) \in S$.

The continuous symmetric and bi-additive map $Q\colon S \times S \to \mathbb{C}$, as known in the literature (e.g., [**8**, Lemma 2.14]), takes the following form

$$Q(x,y) := px_1y_1 + qx_2y_2 + s(x_1y_2 + x_2y_1),$$

for all $x := (x_1,x_2), y := (y_1,y_2) \in S$, where $p,q,s \in \mathbb{C}$. The function Q verifies the condition $Q(x,\varphi(y)) = -\frac{\alpha}{\beta}Q(x,y)$ for all $x,y \in S$ ($\alpha = \frac{4}{3}$, $\beta = -\frac{1}{3}$) and also verifies the condition $\int_S Q(x,t)\,d(\mu+\nu)(t) = 0$ for all $x \in S$ if and only if $p = q = -s$.

The continuous additive map $A\colon S \to \mathbb{C}$, as known in the literature (e.g., [**8**, Corollary 2.4]), takes the following form $A(x) := \omega x_1 + \eta x_2$, for all $x := (x_1,x_2) \in S$, where $\omega, \eta \in \mathbb{C}$.

According to Theorem 3.1, the continuous solutions $f,h\colon \mathbb{R}^2 \to \mathbb{C}$ of the above functional equation are

$$\begin{aligned} f(x_1,x_2) &= p(x_1-x_2)^2 + \omega x_1 + \eta x_2 + c,\\ h(x_1,x_2) &= -4p(x_1-x_2)^2 + \tfrac{2}{3}(\omega+\eta), \end{aligned}$$

where $p,\omega,\eta,c \in \mathbb{C}$.

4. Applications

As immediate consequences of Theorem 3.1, we have the following corollaries. The first one solves equation (2.3).

COROLLARY 4.1. *Let $(\alpha,\beta) := (\mu(S),\nu(S)) \in \mathbb{K}^2$. The pair $f,h\colon S \to E$ is a solution of* (2.3) *if and only if we have the following possibilities:*

(1) *If $\alpha+\beta \neq 1$, then $f = c$, $h = (\alpha+\beta-1)c$, where c is a constant belonging to E.*
(2) *If $\alpha+\beta = 1$, then*
   (a) *If $\alpha = 1$ and $\beta = 0$, then $f = A + c$, $h = A\circ\sigma + \int_S A(t)\,d(\mu+\nu)(t)$, where $A\colon S \to E$ is an additive map and c is a constant belonging to E.*
   (b) *If $\alpha = 0$ and $\beta = 1$, then $f = A + c$, $h = A\circ\tau + \int_S A(t)\,d(\mu+\nu)(t)$, where $A\colon S \to E$ is an additive map and c is a constant belonging to E.*

(c) *If $\alpha \neq 0$ and $\beta \neq 0$, then*

$$f(x) = Q(x,x) + A(x) + c, \tag{4.1}$$

$$\begin{aligned} h(x) = {} & \frac{\alpha}{\beta} Q(\sigma(x), \sigma(x)) + \alpha A(\sigma(x)) + \beta A(\tau(x)) \\ & + 2 \int_S Q(\sigma(x), t)\, d\Big(\mu - \frac{\alpha}{\beta}\nu\Big)(t) + \int_S \{Q(t,t) + A(t)\} d(\mu + \nu)(t), \end{aligned}$$

where $A\colon S \to E$ is an additive map, $Q\colon S \times S \to E$ is a symmetric, bi-additive map such that for all $x, y \in S$ we have

(i) $Q(x, \tau(y)) = -\dfrac{\alpha}{\beta} Q(x, \sigma(y))$, (ii) $\int_S Q(x,t)\, d(\mu+\nu)(t) = 0$,

and c is a constant belonging to E.

Moreover, if S is a topological semigroup, E is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$, and $f\colon S \to E$ is continuous, then the components A and Q in decomposition (4.1) are continuous and if σ and τ are continuous, then h is continuous.

Proof. By putting $\varphi = \tau \circ \sigma$ and replacing h by $h \circ \sigma$ in Theorem 3.1, we get the desired result. □

Let α, β be two elements of $\mathbb{K}$, and let a, b be two arbitrary fixed elements of the semigroup S. We have the following corollaries.

Corollary 4.2. *Suppose that φ is surjective. The pair $f, h : S \to E$ is a solution of (2.4) if and only if we have the following possibilities:*

(1) *If $\alpha + \beta \neq 1$, then $f = c$, $h = (\alpha + \beta - 1)c$, where c is a constant belonging to E.*

(2) *If $\alpha + \beta = 1$, then*

(a) *If $\alpha = 1$ and $\beta = 0$, then $f = A + c$, $h = A + A(a)$, where $A\colon S \to X$ is an additive map and c is a constant belonging to E.*

(b) *If $\alpha = 0$ and $\beta = 1$, then $f = A + c$, $h = A \circ \varphi + A(b)$, where $A\colon S \to E$ is an additive map and c is a constant belonging to E.*

(c) *If $\alpha \neq 0$ and $\beta \neq 0$, then*

$$\begin{aligned} f(x) &= Q(x,x) + A(x) + c, \\ h(x) &= \frac{\alpha}{\beta} Q(x+a, x+a) + \alpha A(x+a) + \beta A(\varphi(x) + b), \end{aligned} \tag{4.2}$$

where $A\colon S \to E$ is an additive map, $Q\colon S \times S \to E$ is a symmetric, bi-additive map such that for all $x, y \in S$

(i) $Q(x, \varphi(y)) = -\frac{\alpha}{\beta} Q(x,y)$, (ii) $Q(x,b) = -\frac{\alpha}{\beta} Q(x,a)$,

and c is a constant belonging to E.

Moreover, if S is a topological semigroup, E is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$, and $f\colon S \to E$ is continuous, then the components A and Q in decomposition (4.2) are continuous and if φ is continuous, then h is continuous.

Proof. By putting $\mu = \alpha\delta_a$, $\nu = \beta\delta_b$ in Theorem 3.1, we get the desired result. □

Corollary 4.3. *The pair $f, h\colon S \to E$ is a solution of* (2.5) *if and only if we have the following possibilities:*

(1) *If $\alpha + \beta \neq 1$, then $f = c$, $h = (\alpha + \beta - 1)c$, where c is a constant belonging to E.*
(2) *If $\alpha + \beta = 1$, then*
   (a) *If $\alpha = 1$ and $\beta = 0$, then $f = A + c$, $h = A \circ \sigma + A(a)$, where $A\colon S \to E$ is an additive map and c is a constant belonging to E.*
   (b) *If $\alpha = 0$ and $\beta = 1$, then $f = A + c$, $h = A \circ \tau + A(b)$, where $A\colon S \to E$ is an additive map and c is a constant belonging to E.*
   (c) *If $\alpha \neq 0$ and $\beta \neq 0$, then*

$$
\begin{aligned}
f(x) &= Q(x, x) + A(x) + c, \\
h(x) &= \frac{\alpha}{\beta} Q(\sigma(x) + a, \sigma(x) + a) + \alpha A(\sigma(x) + a) + \beta A(\tau(x) + b),
\end{aligned}
\tag{4.3}
$$

where $A : S \to E$ is an additive map, $Q\colon S \times S \to E$ is a symmetric, bi-additive map such that for all $x, y \in S$

(i) $Q(x, \tau(y)) = -\frac{\alpha}{\beta} Q(x, \sigma(y))$, (ii) $Q(x, b) = -\frac{\alpha}{\beta} Q(x, a)$,

and c is a constant belonging to E.

Moreover, if S is a topological semigroup, E is a topological vector space over $\mathbb{R}$ or $\mathbb{C}$, and $f\colon S \to E$ is continuous, then the components A and Q in decomposition (4.3) *are continuous and if σ and τ are continuous, then h is continuous.*

Proof. By putting $\mu = \alpha\delta_a$, $\nu = \beta\delta_b$ in Corollary 4.1 we get the desired result. □

In the following example, we consider a surjective endomorphism which is not bijective.

Example 4.1. Let E be the vector space, over $\mathbb{R}$, of sequences of real numbers. Let φ be the endomorphism of the group $(E, +)$ defined by $\varphi((u_n)_n) = (v_n)_n$ such that for all $n \in \mathbb{N}$, $v_n = u_{n+1} - u_n$.

Note that φ is surjective but not bijective: for $(v_n)_n \in E$, the sequence $(u_n)_n$ defined by $u_n = \sum_{i=0}^{n-1} v_i$ for all $n \geqslant 1$ and u_0 is an arbitrary element of $\mathbb{R}$ satisfies $\varphi((u_n)_n) = (v_n)_n$.

We want to determine the solutions $f, h\colon E \to E$ of the functional equation

$$
f((u_n)_n + (v_n)_n + (a_n)_n) + f((u_n)_n + \varphi((v_n)_n) + (b_n)_n) = 2f((u_n)_n) + 2h((v_n)_n),
\tag{4.4}
$$

for all $(u_n)_n, (v_n)_n \in E$, where $(a_n)_n, (b_n)_n \in E$ are two arbitrary constants. The only symmetric, bi-additive map $Q\colon E \times E \to E$ satisfying $Q((u_n)_n, \varphi((v_n)_n)) = -Q((u_n)_n, (v_n)_n)$ for all $(u_n)_n, (v_n)_n \in E$ is the null map. Indeed, let $(u_n)_n, (v_n)_n \in E$. We have

$$
Q((u_n)_n, \varphi((v_n)_n)) = -Q((u_n)_n, (v_n)_n) \Leftrightarrow Q((u_n)_n, (v_{n+1})_n) = 0.
$$

Then $Q = 0$, because the map $(v_n)_n \to (v_{n+1})_n$ is surjective.

According to Corollary 4.2, the solutions $f, h\colon E \to E$ of (4.4) are

$$f((u_n)_n) = A((u_n)_n) + (c_n)_n, \quad h((u_n)_n) = \frac{1}{2}A((u_{n+1})_n + (a_n)_n + (b_n)_n),$$

where $A\colon E \to E$ is an additive map and $(c_n)_n$ is a constant belonging to E.

Acknowledgement. We wish to express our thanks to the referees for useful comments.

References

1. A. Akkaoui, M. El Fatini, B. Fadli, D. Zeglami, *Nice results about quadratic type functional equations on semigroups*, Aequationes Math. **94**(1) (2020), 83–96.
2. A. Charifi, R. Łukasik, D. Zeglami, *A special class of functional equations*, Math. Slovaca **68**(2) (2018), 397–404.
3. H. Dimou, Y. Aribou, A. Chahbi, S. Kabbaj, *On a quadratic type functional equation on locally compact abelian groups*, Acta Univ. Sapientiae, Math. **10**(1) (2018), 46–55.
4. B. Fadli, D. Zeglami, S. Kabbaj, *On a Gajda's type quadratic equation on a locally compact abelian group*, Indag. Math., New Ser. **26**(4) (2015), 660–668.
5. B. Fadli, A. Chahbi, Iz. EL-Fassi, S. Kabbaj, *On Jensen's and the quadratic functional equations with involutions*, Proyecciones **35**(2) (2016), 213–223.
6. Kh. Sabour, S. Kabbaj, *Jensen's and the quadratic functional equations with an endomorphism*, Proyecciones **36**(1) (2017), 187–196.
7. P. Sinopoulos, *Functional equations on semigroups*, Aequationes Math. **59**(3) (2000), 255–261.
8. H. Stetkær, *Functional Equations on Groups*, World Scientific Publishing Co, Singapore, 2013.
9. ———, *The kernel of the second order Cauchy difference on semigroups*, Aequationes Math. **91**(2) (2017), 279–288.

Department of Mathematics
Faculty of Sciences
Ibn Tofail University
Kenitra
Morocco
ahmed.maths78@gmail.com
melfatini@gmail.com

(Received 08 01 2021)
(Revised 07 06 2021)