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UPPER BOUND ESTIMATES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL WITH RESTRICTED ZEROS

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Abstract. The polar derivative of a polynomial $P(z)$ of degree n with respect to a complex number γ is a polynomial $nP(z) + (\gamma - z)P'(z)$ of degree at most $n-1$ and is denoted by $D_\gamma P(z)$. We consider the class of polynomials $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $\mu \geqslant 1$, of degree n such that $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$ and establish some upper bound estimates for the maximum modulus of $D_\gamma P(z)$ on the unit disk by involving some of the coefficients of $P(z)$. The obtained results refine and generalize some well known polynomial inequalities.

1. Introduction

One of the interesting and fruitful subject in geometry of polynomials is the geometrical relation between the modulus of a complex polynomial on a circle and the position of the zeros of this polynomial inside or outside this circle. Many propositions in the area of polynomial inequalities are presented by Bernstein-type inequalities. We start with the following well known result of Bernstein [**3**]. Let $P(z)$ be a polynomial of degree n, then on $|z| = 1$, we have

$$|P'(z)| \leqslant n \max_{|z|=1} |P(z)|. \tag{1.1}$$

The above inequality (1.1) was proved by Bernstein in 1912 and has been the starting point of a considerable literature in polynomial approximations and, over a period, various versions and generalizations of this inequality are produced by introducing restrictions on the multiplicity of zero at $z = 0$, the modulus of largest root of $P(z)$, restrictions on coefficients etc.

One would expect a relationship between the bound n and the distance of the zeros of the polynomial $P(z)$ from the origin. This fact was observed by Erdös and later proved by Lax [**7**] by proving that, if $P(z)$ is a polynomial of degree n which

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does not vanish in $|z| < 1$, then (1.1) can be replaced by

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1.2}$$

On the other hand Malik [**8**] proved an extension of (1.2) by proving that if $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k} \max_{|z|=1} |P(z)|. \tag{1.3}$$

As a generalization of (1.3), Aziz and Shah [**2**] proved that, if $P(z)$ has no zeros in $|z| < k$, $k \geqslant 1$ with t-fold zeros at the origin, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n+tk}{1+k} \max_{|z|=1} |P(z)|. \tag{1.4}$$

Chan and Malik [**4**] generalized (1.3) in different direction and proved that, if $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^\mu} \max_{|z|=1} |P(z)|. \tag{1.5}$$

As a refinement of (1.5), Pukhta [**13**] proved that, if $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+k^\mu} \Big(\max_{|z|=1} |P(z)| - m_k \Big), \tag{1.6}$$

where

$$m_k = \min_{|z|=k} |P(z)|. \tag{1.7}$$

Further, Kumar and Lal [**6**] generalized inequality (1.6) by proving that, if $P(z) = z^t(a_0 + \sum_{v=\mu}^{n-t} a_v z^v)$, $1 \leqslant \mu \leqslant n-t$, is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$ with t-fold zeros at the origin, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n+tk^\mu}{1+k^\mu} \max_{|z|=1} |P(z)| - \frac{m_k(n-t)}{k^t(1+k^\mu)}, \tag{1.8}$$

where m_k is defined in (1.7). Although the inequality (1.6) sharpens the inequality (1.5) but it has a drawback that if there is even only one zero of $P(z)$ on $|z| = k$, then $m_k = \min_{|z|=k} |P(z)| = 0$, and so it fails to give any improvement over (1.5). So it is natural to ask is there any way to overcome this deficiency, for example can we express the bound in terms of coefficients of a polynomial? By intuition, one may try to answer in different ways at different levels. In this paper, we approach this problem by using the information on some coefficients of the underlying polynomial. In this direction we mention a result due to Qazi [**14**] in which he uses some of the coefficients of a polynomial by proving that, if $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1+A_0(\mu)} \max_{|z|=1} |P(z)|, \tag{1.9}$$

where

$$A_0(\mu) = k^{\mu+1}\left\{\frac{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu-1}+1}{\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^{\mu+1}+1}\right\}.$$

Since if $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$ such that $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then $\frac{\mu}{n}\frac{|a_\mu|}{|a_0|}k^\mu \leqslant 1$, which can be taken as equivalent to $A_0(\mu) \geqslant k^\mu$, $\mu \geqslant 1$. Hence inequality (1.9) is an improvement of (1.5).

We define $D_\gamma P(z)$, the polar derivative of the polynomial $P(z)$ of degree n with respect to a complex number γ by

$$D_\gamma P(z) := nP(z) + (\gamma - z)P'(z).$$

It is easy to see that polynomial $D_\gamma P(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\gamma\to\infty}\left\{\frac{D_\gamma P(z)}{\gamma}\right\} := P'(z),$$

uniformly with respect to z for $|z| \leqslant R$, $R > 0$. For more information on the polar derivative of a polynomial, one can consult the books of Milovanović et al. [**11**], Rahman and Schmeisser [**15**], and Marden [**9**]. Inequalities between polynomials with restricted zeros in the sup-norm have been extended widely in the literature from ordinary derivative to their polar derivatives. Very recently, Mir and Wani [**12**] obtained the following polar derivative analogue of (1.8) by proving the following result.

THEOREM 1.1. *If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$ with t-fold zeros at the origin, then for every complex number γ with $|\gamma| \geqslant 1$,*

$$\max_{|z|=1}|D_\gamma P(z)| \leqslant \left(\frac{n(|\gamma| + k^\mu) + t(|\gamma| - 1)k^\mu}{1 + k^\mu}\right)\max_{|z|=1}|P(z)| - \frac{(n-t)(|\gamma|-1)m_k}{k^t(1+k^\mu)}, \tag{1.10}$$

where m_k is defined in (1.7).

The authors are intrested to know how the inequality in Theorem 1.1 and other inequalities mentioned in the introduction can be sharpened by using some of the coefficients of $P(z)$. Indeed, this paper is mainly motivated by the desire to establish some new inequalities in the plane that relate the sup-norm of the polar derivative and the polynomial under some conditions. Obtained results in particular give more refined bounds than given by (1.8), (1.9) and (1.10).

2. Lemmas

We need the following lemmas to prove our theorems. The following lemma is a special case of a result due to Aziz and Aliya [**1**].

Lemma 2.1. *If $P(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $1\leqslant\mu\leqslant n$ is a polynomial of degree n such that $P(z)\neq 0$ in $|z|<k$, $k\geqslant 1$ and $0\leqslant\zeta\leqslant 1$, we have for $|z|=1$,*

$$|Q'(z)|-nm_k\zeta\geqslant\psi_0(\mu)|P'(z)|, \tag{2.1}$$

$$\left(\frac{\mu}{n}\right)\frac{|a_\mu|k^\mu}{|a_0|-\zeta m_k}\leqslant 1,$$

where

$$\psi_0(\mu)=k^{\mu+1}\left\{\frac{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\zeta m_k}k^{\mu-1}+1}{\left(\frac{\mu}{n}\right)\frac{|a_\mu|}{|a_0|-\zeta m_k}k^{\mu+1}+1}\right\},$$

$Q(z)=z^n\overline{P(\frac{1}{\bar z})}$ and m_k is defined in (1.7).

The following lemma is a special case of a result due to Govil and Rahman [**5**].

Lemma 2.2. *If $P(z)$ is a polynomial of degree n, then for $|z|=1$,*

$$|P'(z)|+|Q'(z)|\leqslant n\max_{|z|=1}|P(z)|,$$

where $Q(z)=z^n\overline{P(\frac{1}{\bar z})}$.

On combining Lemma 2.2 and inequality (2.1) of Lemma 2.1, we get the following result.

Lemma 2.3. *If $P(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $1\leqslant\mu\leqslant n$ is a polynomial of degree n such that $P(z)\neq 0$ in $|z|<k$, $k\geqslant 1$ and $0\leqslant\zeta\leqslant 1$, then for $|z|=1$,*

$$|P'(z)|\leqslant\frac{n}{1+\psi_0(\mu)}\Big(\max_{|z|=1}|P(z)|-\zeta m_k\Big),$$

where $\psi_0(\mu)$ is defined in Lemma 2.1 and m_k is defined in (1.7).

Lemma 2.4. *If $P(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $1\leqslant\mu\leqslant n$ is a polynomial of degree n such that $P(z)\neq 0$ in $|z|<k$, $k\geqslant 1$ and $0\leqslant\zeta\leqslant 1$, then for any complex number γ with $|\gamma|\geqslant 1$,*

$$\max_{|z|=1}|D_\gamma P(z)|\leqslant\frac{n}{1+\psi_0(\mu)}\Big((|\gamma|+\psi_0(\mu))\max_{|z|=1}|P(z)|-(|\gamma|-1)\zeta m_k\Big),$$

where $\psi_0(\mu)$ is defined in Lemma 2.1 and m_k is defined in (1.7).

Proof. By hypothesis $P(z)=a_0+\sum_{v=\mu}^{n}a_vz^v$, $1\leqslant\mu\leqslant n$ is a polynomial of degree n which does not vanish in $|z|<k$, $k\geqslant 1$. Applying Lemma 2.3 to the polynomial $P(z)$, we have for $|z|=1$,

$$|P'(z)|\leqslant\frac{n}{1+\psi_0(\mu)}\Big(\max_{|z|=1}|P(z)|-\zeta m_k\Big). \tag{2.2}$$

Now for $\gamma\in\mathbb{C}$, with $|\gamma|\geqslant 1$

$$\begin{aligned}|D_\gamma P(z)|&=|nP(z)+(\gamma-z)P'(z)|\\&=|nP(z)+\gamma P'(z)-zP'(z)|\\&\leqslant|nP(z)-zP'(z)|+|\gamma||P'(z)|.\end{aligned}$$

It can be easily seen that

$$|nP(z) - zP'(z)| = |Q'(z)| \quad \text{for} \quad |z| = 1.$$

Therefore we have for $|z| = 1$,

$$|D_\gamma P(z)| \leqslant |Q'(z)| + |\gamma||P'(z)| = |Q'(z)| + |P'(z)| - |P'(z)| + |\gamma||P'(z)|.$$

Using Lemma 2.2, we get for $|z| = 1$,

$$|D_\gamma P(z)| \leqslant n|P(z)| + (|\gamma| - 1)|P'(z)|,$$

which on using (2.2), we get

$$\max_{|z|=1} |D_\gamma P(z)| \leqslant \frac{n}{1 + \psi_0(\mu)} \Big((|\gamma| + \psi_0(\mu)) \max_{|z|=1} |P(z)| - (|\gamma| - 1)\zeta m_k \Big),$$

which is the desired inequality and hence proved Lemma 2.4. □

LEMMA 2.5. *If* $P(z) = z^t(a_0 + \sum_{v=\mu}^{n-t} a_v z^v)$, $1 \leqslant \mu \leqslant n - t$, $0 \leqslant t \leqslant n - 1$ *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$ *with t-fold zeros at the origin, then for* $0 \leqslant \zeta \leqslant 1$,

$$\psi_t(\mu) = k^{\mu+1} \left\{ \frac{\left(\frac{\mu}{n-t}\right) \frac{|a_\mu|}{k^t|a_0| - \zeta m_k} k^{\mu+t-1} + 1}{\left(\frac{\mu}{n-t}\right) \frac{|a_\mu|}{k^t|a_0| - \zeta m_k} k^{\mu+t+1} + 1} \right\} \geqslant k^\mu, \quad \mu \geqslant 1,$$

where m_k *is defined in* (1.7).

The above lemma is due to Milovanović and Mir [**10**, Lemma 3.5].

3. Main results and proofs

THEOREM 3.1. *If* $P(z) = z^t(a_0 + \sum_{v=\mu}^{n-t} a_v z^v)$, $1 \leqslant \mu \leqslant n - t$, $0 \leqslant t \leqslant n - 1$, *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$ *with t-fold zeros at the origin, then for every complex number* γ *with* $|\gamma| \geqslant 1$ *and* $0 \leqslant \zeta \leqslant 1$,

$$\max_{|z|=1} |D_\gamma P(z)| \leqslant \Big(\frac{n(|\gamma| + \psi_t(\mu)) + t(|\gamma| - 1)\psi_t(\mu)}{1 + \psi_t(\mu)} \Big) \max_{|z|=1} |P(z)| - \frac{(n-t)(|\gamma| - 1)\zeta m_k}{k^t(1 + \psi_t(\mu))}, \tag{3.1}$$

where

$$\psi_t(\mu) = k^{\mu+1} \left\{ \frac{\left(\frac{\mu}{n-t}\right) \frac{|a_\mu|}{k^t|a_0| - \zeta m_k} k^{\mu+t-1} + 1}{\left(\frac{\mu}{n-t}\right) \frac{|a_\mu|}{k^t|a_0| - \zeta m_k} k^{\mu+t+1} + 1} \right\} \tag{3.2}$$

and m_k *is defined in* (1.7).

PROOF. Since $P(z) = z^t \rho(z)$ where $\rho(z) = a_0 + \sum_{v=\mu}^{n-t} a_v z^v$, $1 \leqslant \mu \leqslant n - t$ is a polynomial of degree $n - t$ which does not vanish in $|z| < k$, $k \geqslant 1$. Applying Lemma 2.4 to the polynomial $\rho(z)$, we get for $|\gamma| \geqslant 1$,

$$\max_{|z|=1} |D_\gamma \rho(z)| \leqslant \frac{n-t}{1 + \psi_0'(\mu)} \Big((|\gamma| + \psi_0'(\mu)) \max_{|z|=1} |\rho(z)| - (|\gamma| - 1)\zeta m_k' \Big), \tag{3.3}$$

where

$$\psi_0'(\mu) = k^{\mu+1}\left\{\frac{\left(\frac{\mu}{n-t}\right)\frac{|a_\mu|}{|a_0|-\zeta m_k'}k^{\mu-1}+1}{\left(\frac{\mu}{n-t}\right)\frac{|a_\mu|}{|a_0|-\zeta m_k'}k^{\mu+1}+1}\right\}$$

and $m_k' = \min_{|z|=k}|\rho(z)| = \frac{1}{k^t}\min_{|z|=k}|P(z)| = \frac{m_k}{k^t}$.
Also for $|z|=1$, $\max_{|z|=1}|\rho(z)| = \max_{|z|=1}|P(z)|$ and therefore

$$\begin{aligned}\psi_0'(\mu) &= k^{\mu+1}\left\{\frac{\left(\frac{\mu}{n-t}\right)\frac{|a_\mu|}{|a_0|-\frac{\zeta m_k}{k^t}}k^{\mu-1}+1}{\left(\frac{\mu}{n-t}\right)\frac{|a_\mu|}{|a_0|-\frac{\zeta m_k}{k^t}}k^{\mu+1}+1}\right\}\\ &= k^{\mu+1}\left\{\frac{\left(\frac{\mu}{n-t}\right)\frac{|a_\mu|}{k^t|a_0|-\zeta m_k}k^{\mu+t-1}+1}{\left(\frac{\mu}{n-t}\right)\frac{|a_\mu|}{k^t|a_0|-\zeta m_k}k^{\mu+t+1}+1}\right\} = \psi_t(\mu).\end{aligned}$$

Using these observations in (3.3), we get

$$\max_{|z|=1}|D_\gamma\rho(z)| \leqslant \frac{n-t}{1+\psi_t(\mu)}\Big((|\gamma|+\psi_t(\mu))\max_{|z|=1}|P(z)| - \frac{m_k}{k^t}(|\gamma|-1)\zeta\Big). \tag{3.4}$$

Now for any complex number γ with $|\gamma|\geqslant 1$, we have

$$\begin{aligned}D_\gamma P(z) &= nP(z) + (\gamma - z)P'(z)\\ &= nz^t\rho(z) + (\gamma - z)[tz^{t-1}\rho(z) + z^t\rho'(z)]\\ &= z^t D_\gamma\rho(z) + \gamma t z^{t-1}\rho(z),\end{aligned}$$

which gives

$$zD_\gamma P(z) = z^{t+1}D_\gamma\rho(z) + \gamma t P(z). \tag{3.5}$$

Hence for $|z|=1$, we get from above inequality (3.5) that

$$|D_\gamma P(z)| \leqslant |D_\gamma\rho(z)| + t|\gamma||P(z)|,$$

which in particular implies,

$$\max_{|z|=1}|D_\gamma P(z)| \leqslant \max_{|z|=1}|D_\gamma\rho(z)| + t|\gamma|\max_{|z|=1}|P(z)|. \tag{3.6}$$

Inequality (3.6) when combined with (3.4), gives

$$\begin{aligned}\max_{|z|=1}|D_\gamma P(z)| \leqslant \Big(&\frac{n(|\gamma|+\psi_t(\mu)) + t(|\gamma|-1)\psi_t(\mu)}{1+\psi_t(\mu)}\Big)\max_{|z|=1}|P(z)|\\ &- \frac{(n-t)(|\gamma|-1)\zeta m_k}{k^t(1+\psi_t(\mu))},\end{aligned}$$

which is (3.1) and this completes the proof of Theorem 3.1. □

If we do not have the knowledge of $m_k = \min_{|z|=k}|P(z)|$, we can use the following result, whose proof is similar to that of Theorem 3.1.

THEOREM 3.2. *If* $P(z) = z^t\big(a_0 + \sum_{v=\mu}^{n-t} a_v z^v\big)$, $1 \leqslant \mu \leqslant n-t$, $0 \leqslant t \leqslant n-1$, *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$ *with* t*-fold zeros at the origin, then for every complex number* γ *with* $|\gamma| \geqslant 1$,

$$\max_{|z|=1} |D_\gamma P(z)| \leqslant \Big(\frac{n(|\gamma| + A_t(\mu)) + t(|\gamma|-1)A_t(\mu)}{1 + A_t(\mu)}\Big) \max_{|z|=1} |P(z)|, \tag{3.7}$$

where

$$A_t(\mu) = k^{\mu+1}\left\{\frac{\big(\frac{\mu}{n-t}\big)\frac{|a_\mu|}{k^t|a_0|}k^{\mu+t-1} + 1}{\big(\frac{\mu}{n-t}\big)\frac{|a_\mu|}{k^t|a_0|}k^{\mu+t+1} + 1}\right\}.$$

REMARK 3.1. Independently inequality (3.7) was recently established by Milovanović and Mir [**10**, Remark 2.12], by first proving an integral norm estimate of $D_\gamma P(z)$ with $|\gamma| \geqslant 1$ and then the result so obtained produced (3.7) as a special case. Dividing both sides of inequality (3.7) by $|\gamma|$ and let $|\gamma| \to \infty$, we get the following generalization of (1.9).

COROLLARY 3.1. *If* $P(z) = z^t\big(a_0 + \sum_{v=\mu}^{n-t} a_v z^v\big)$, $1 \leqslant \mu \leqslant n-t$, $0 \leqslant t \leqslant n-1$, *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$ *with* t*-fold zeros at the origin, then*

$$\max_{|z|=1} |P'(z)| \leqslant \Big(\frac{n + tA_t(\mu)}{1 + A_t(\mu)}\Big) \max_{|z|=1} |P(z)|, \tag{3.8}$$

where $A_t(\mu)$ *is defined in Theorem* 3.2.

REMARK 3.2. By taking $t = 0$ in (3.8), yields $A_t(\mu) = A_0(\mu)$ and hence (3.8) reduces to (1.9). Several other interesting results easily follows from Theorem 3.1. Here, we mention few of these. It is easy to verify that for every complex number γ with $|\gamma| \geqslant 1$, the function

$$x \to \Big(\frac{n(|\gamma| + x) + t(|\gamma|-1)x}{1+x}\Big) \max_{|z|=1} |P(z)| - \Big(\frac{(n-t)(|\gamma|-1)\zeta}{k^t(1+x)}\Big) m_k$$

is a non-increasing function of x. If we combine this fact with Lemma 2.5 according to which $\psi_t(\mu) \geqslant k^\mu$, $\mu \geqslant 1$, we observe that the right hand side of (3.1) does not exceed the right hand side of (1.10). Thus Theorem 3.1 represents a refinement of Theorem 1.1. Dividing both sides of inequality (3.1) by $|\gamma|$ and let $|\gamma| \to \infty$, we get the following result.

COROLLARY 3.2. *If* $P(z) = z^t\big(a_0 + \sum_{v=\mu}^{n-t} a_v z^v\big)$, $1 \leqslant \mu \leqslant n-t$, $0 \leqslant t \leqslant n-1$, *is a polynomial of degree* n *such that* $P(z) \neq 0$ *in* $|z| < k$, $k \geqslant 1$ *with* t*-fold zeros at the origin, then for* $0 \leqslant \zeta \leqslant 1$,

$$\max_{|z|=1} |P'(z)| \leqslant \Big(\frac{n + t\psi_t(\mu)}{1 + \psi_t(\mu)}\Big) \max_{|z|=1} |P(z)| - \frac{(n-t)\zeta m_k}{k^t(1 + \psi_t(\mu))}, \tag{3.9}$$

where $\psi_t(\mu)$ *is defined in* (3.2) *and* m_k *is defined in* (1.7). *The above corollary represents a refinement of inequality* (1.8). *Setting* $t = 0$ *in* (3.9), *we get the following refinement of* (1.6).

COROLLARY 3.3. *If $P(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$, $1 \leqslant \mu \leqslant n$, is a polynomial of degree n such that $P(z) \neq 0$ in $|z| < k$, $k \geqslant 1$, then for $0 \leqslant \zeta \leqslant 1$,*

$$\max_{|z|=1} |P'(z)| \leqslant \left(\frac{n}{1+\psi_0(\mu)}\right) \max_{|z|=1} |P(z)| - \frac{n\zeta m_k}{1+\psi_0(\mu)},$$

where $\psi_0(\mu)$ is defined in Lemma 2.1 and m_k is defined in (1.7).

REMARK 3.3. For $\mu = 1$ and $\zeta = 0$, inequality (3.9) provides a refinement to (1.4).

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