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TABLEAU FOR THE LOGIC ILP

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Abstract. A tableau system for a logic suitable for intuitionistic reasoning about probabilities is presented. Soundness and completeness of the system are proved. A decision procedure based on the tableau system is given.

1. Introduction

The formal system called (semantic, analytic) tableau arose from the works of Beth, Hintikka and Smullyan [**1**, **5**, **11**]. It is particularly popular in the framework of modal and intuitionistic logics thanks to Fitting's papers [**3**, **4**], where a proof procedure closely related to Kripke models [**6**, **7**] for those logics is prsented. A tableau for a formula is a tree whose nodes are labeled by subformulas of the considered formula, constructed by rules that reflect semantic properties of logical operators. The tableau proof procedure is a refutation method which specifies the order of applications of rules which modify a tableau to try to construct a counter model of the considered formula. If the refutation does not succeed, the formula should be valid.

In this paper we adapt the prefixed tableau system for intuitionistic logic [**3**, **4**] and propose a system for the logic ILP which formalizes intuitionistic reasoning about probabilities. The paper [**2**] presents an intuitionistic logic with probabilistic operators and a complete axiom system for intuitionistic Kripke models in which each possible world is equipped with two partial functions representing inner and outer probability measures with a finite range. Probabilistic logics based on intuitionistic logic, but with probabilistic operators obeying laws of classical logic (e.g., the probability of an uncertain proposition should be either greater or equal to some r or less than r) are proposed in [**8**–**10**]. In ILP atomic formulas are of the form $P_{*r}\alpha$, where $* \in \{\geqslant, \leqslant, >, <\}$ and α is a classical formula. The intended meaning of the probabilistic operators $P_{\geqslant r}$ is "It is proven that the probability is

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at least r", and similarly for $P_{\leqslant r}$, $P_{>r}$ and $P_{<r}$. The associated semantics consists of intuitionistic Kripke models equipped with partial probabilistic measures of classical formulas. Furthermore, in ILP:

- in contrast to [**2**] probability measures (and not inner and outer probabilities) are considered, and the range of probability measures is the unit interval of rational numbers (and not a finite subset of that interval), and
- probabilistic operators, in contrast to [**8**–**10**], follow the laws of intuitionistic logic.

To the best of our knowledge this is the first paper discussing tableau system for logics with probabilistic operators. The presence of these operators in the formal language requires that new rules relating to probabilistic operators be added to the set of tableau rules, as well as that the new (in comparison with [**3**, **4**]) criterion for closing of branches be defined. These are the main novelties from the point of view of the development of the tableau method.

The paper is organized as follows. In Section 2 we provide basic definitions about syntax and semantics of ILP, while Section 3 presents our tableau system. In Sections 4 and 5 soundness and completeness are proved. A proof of decidability of satisfiability for ILP is sketched in Section 6 relying on the tableau proof procedure.

2. The logic ILP

2.1. Syntax. Let Var be a countably infinite set of propositional letters. Variables for Var are p and q. For_C denotes the set of classical formulas inductively defined in the following way:

$$p \mid \neg\alpha \mid \alpha \wedge \beta \mid \alpha \vee \beta.$$

Variables for For_C are α, β, γ and δ, indexed if necessary. We abbreviate $\neg(\alpha \to \alpha)$ by $\bot$ and $\alpha \to \alpha$ by $\top$.

Let $[0,1]_{\mathbb{Q}}$ denote the set all rational numbers from the unit interval. The set of probabilistic formulas For_P is inductively defined as follows:

$$P_{*r}\alpha \mid \neg\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \to \psi.$$

Here $* \in \{\geqslant, \leqslant, >, <\}$ and $r \in [0,1]_{\mathbb{Q}}$. Formulas of the form $P_{*r}\alpha$ are called atomic probabilistic formulas. Variables for For_P are ϕ and ψ, indexed if necessary. We abbreviate: $P_{\geqslant r}\alpha \wedge P_{\leqslant r}\alpha$ by $P_{=r}\alpha$.

Definition 2.1. A prefixed signed formula is an expression of the form:

- σ T ϕ, or
- σ F ϕ,

where σ is a nonempty finite sequence of positive integers (called prefix) and $\phi \in \mathrm{For}_P$.

A prefixed signed atomic probabilistic formula is an expression of the form σ T $P_{*r}\alpha$, or σ F $P_{*r}\alpha$, where $* \in \{\geqslant, \leqslant, >, <\}$ and σ is a prefix. □

2.2. Semantics. We consider a class of Kripke models to give semantics to For_P-formulas.

DEFINITION 2.2. A model is a structure $\mathbb{M} = \langle W, \{H_w : w \in W\}, \{\mu_w \ : \ w \in W\}, \leqslant\rangle$ with the following properties:

- W is a non-empty set of possible worlds,
- $\langle W, \leqslant\rangle$ is a partially ordered set (poset) called a frame;
- every H_w is a subset of For_C which satisfies:
  - $\bot \in H_w$, $\top \in H_w$;
  - If $\alpha \in H_w$ and α is equivalent to β (i.e. $\alpha \leftrightarrow \beta$ is a classical tautology), then $\beta \in H_w$;
  - $\alpha \in H_w$ iff $\neg\alpha \in H_w$;
- For all $w_1, w_2 \in W$, if $w_1 \leqslant w_2$, then $H_{w_1} \subseteq H_{w_2}$;
- every μ_w is a mapping from H_w to $[0,1]_{\mathbb{Q}}$ which satisfies:
  - $\mu_w(\bot) = 0$, $\mu_w(\top) = 1$;
  - If $w_1 \leqslant w_2$, then μ_{w_1} is a restriction of μ_{w_2} on H_{w_1};
  - If $\alpha, \alpha \vee \beta \in H_w$, then $\mu_w(\alpha) \leqslant \mu_w(\alpha \vee \beta)$;
  - If α, β, $\alpha \wedge \beta$, $\alpha \vee \beta \in H_w$ and $\mu_w(\alpha \wedge \beta) = 0$, then $\mu_w(\alpha \vee \beta) = \mu_w(\alpha) + \mu_w(\beta)$. □

Note that μ_w's are (partially defined) probabilistic measures, and that it might be that some formulas are not measurable in some of the possible worlds from a model. However, if w_1 and w_2 are possible worlds from a model $\mathbb{M}$, such that $w_1 \leqslant w_2$, and $\mu_{w_1}(\alpha)$ is defined, then it must be that $\mu_{w_2}(\alpha)$ is also defined such that $\mu_{w_2}(\alpha) = \mu_{w_1}(\alpha)$.

The forcing (i.e., intuitionistic satisfiability) relation is recursively defined as follows:

DEFINITION 2.3. Let $\mathbb{M} = \langle W, \{H_w : w \in W\}, \{\mu_w \ : \ w \in W\}, \leqslant\rangle$ be a model and $w \in W$. The forcing relation $\Vdash_{\mathbb{M}}$ between possible worlds and For_P-formulas satisfies:

- $w \Vdash P_{\geqslant r}\alpha$ iff $\alpha \in H_w$ and $\mu_w(\alpha) \geqslant r$;
- $w \Vdash P_{\leqslant r}\alpha$ iff $\alpha \in H_w$ and $\mu_w(\alpha) \leqslant r$;
- $w \Vdash P_{>r}\alpha$ iff $\alpha \in H_w$ and $\mu_w(\alpha) > r$;
- $w \Vdash P_{<r}\alpha$ iff $r > 0$, $\alpha \in H_w$ and $\mu_w(\alpha) < r$;
- $w \Vdash P_{<0}\alpha$ iff $\alpha \notin H_w$;
- $w \Vdash \neg\phi$ iff, for all $v \geqslant w$, $v \nVdash \phi$;
- $w \Vdash \phi \wedge \psi$ iff $w \Vdash \phi$ and $w \Vdash \psi$;
- $w \Vdash \phi \vee \psi$ iff $w \Vdash \phi$ or $w \Vdash \psi$;
- $w \Vdash \phi \to \psi$ iff, for all $v \geqslant w$, either $v \nVdash \phi$, or $v \Vdash \phi \wedge \psi$. □

Note the particular case $w \Vdash P_{<0}\alpha$ which we use to denote that α is not measurable ($\alpha \notin H_w$).

DEFINITION 2.4. A formula ϕ is ILP–valid iff for all $\mathbb{M} = \langle W, \dots\rangle$, and all $w \in W$, $w \Vdash \phi$.

A formula ϕ is ILP–satisfiable iff there are a model $\mathbb{M} = \langle W, \dots\rangle$, and a possible world $w \in W$ such that $w \Vdash \phi$. □

DEFINITION 2.5. A set S of prefixed signed formulas is satisfiable if there is a model $\mathbb{M} = \langle W, \{H_w : w \in W\}, \{\mu_w \ : \ w \in W\}, \leqslant\rangle$ and a mapping π from the set of prefixes occurring in S to the set W of possible worlds such that:

- if σ and $\sigma.n$ are prefixes of formulas that occur in S, then $\pi(\sigma) \leqslant \pi(\sigma.n)$,
- if σ T $\phi \in S$, then $\pi(\sigma) \Vdash \phi$, and
- if σ F $\phi \in S$, then $\pi(\sigma) \nVdash \phi$. □

3. Tableau

A tableau is a tree. Every node of a tableau is labeled with a prefixed signed formula. The tableau system is a set of rules for constructing a tableau. There are the following groups of rules:

- the branch extending rules,
- the prefixes handling rules, and
- the atomic probabilistic formulas handling rules.

The former two groups are from [**4**], while the latter group is the new one. The need for those new rules emerges because of probabilistic formulas that are not present in intuitionistic logic. Similarly, the standard notion of a closed tableau branch must be changed to represent semantics of (probabilistic) atomic formulas.

3.1. Tableau rules.

The tableau construction rules are as follows:

3.1.1. *The branch extending rules.*

$$T\wedge \quad \frac{\sigma \text{ T } \phi\wedge\psi}{\begin{array}{c}\sigma \text{ T } \phi\\ \sigma \text{ T } \psi\end{array}} \qquad\qquad F\vee \quad \frac{\sigma \text{ F } \phi\vee\psi}{\begin{array}{c}\sigma \text{ F } \phi\\ \sigma \text{ F } \psi\end{array}}$$

$$F\wedge \quad \frac{\sigma \text{ F } \phi\wedge\psi}{\sigma \text{ F } \phi \quad | \quad \sigma \text{ F } \psi} \qquad\qquad T\vee \quad \frac{\sigma \text{ T } \phi\vee\psi}{\sigma \text{ T } \phi \quad | \quad \sigma \text{ T } \psi}$$

$$T\rightarrow \quad \frac{\sigma \text{ T } \phi\rightarrow\psi}{\sigma \text{ F } \phi \quad | \quad \sigma \text{ T } \psi} \qquad\qquad T\neg \quad \frac{\sigma \text{ T } \neg\phi}{\sigma \text{ F } \phi}$$

3.1.2. *The prefixes handling rules.* A prefix is new on a branch if it does not occur as an initial segment of any prefix on the branch where the rule is being applied, while a prefix is already introduced on a branch if it occurs on the branch.

The rules for introducing a new prefix $\sigma.n$:

$$F\neg \quad \frac{\sigma \text{ F } \neg\phi}{\sigma.n \text{ T } \phi} \qquad\qquad F\rightarrow \quad \frac{\sigma \text{ F } \phi\rightarrow\psi}{\begin{array}{c}\sigma.n \text{ T } \phi\\ \sigma.n \text{ F } \psi\end{array}}$$

The rule for adding prefixed signed formulas with prefixes already introduced on the branch:

$$\text{Lower} \quad \frac{\sigma \text{ T } \phi}{\sigma.n \text{ T } \phi}$$

3.1.3. *The atomic probabilistic formulas handling rules.*

$$F\ P_{\geqslant r}\ \frac{\sigma\ \mathrm{F}\ P_{\geqslant r}\alpha}{\begin{array}{l|l}\sigma\ \mathrm{T}\ P_{\geqslant 0}\alpha & \sigma\ \mathrm{T}\ P_{<0}\alpha \\ \sigma\ \mathrm{T}\ P_{<r}\alpha & \end{array}} \qquad F\ P_{\leqslant r}\ \frac{\sigma\ \mathrm{F}\ P_{\leqslant r}\alpha}{\begin{array}{l|l}\sigma\ \mathrm{T}\ P_{\geqslant 0}\alpha & \sigma\ \mathrm{T}\ P_{<0}\alpha \\ \sigma\ \mathrm{T}\ P_{>r}\alpha & \end{array}}$$

$$F\ P_{>r}\ \frac{\sigma\ \mathrm{F}\ P_{>r}\alpha}{\begin{array}{l|l}\sigma\ \mathrm{T}\ P_{\geqslant 0}\alpha & \sigma\ \mathrm{T}\ P_{<0}\alpha \\ \sigma\ \mathrm{T}\ P_{\leqslant r}\alpha & \end{array}} \qquad F\ P_{<r}\ \frac{\sigma\ \mathrm{F}\ P_{<r}\alpha}{\begin{array}{l|l}\sigma\ \mathrm{T}\ P_{\geqslant 0}\alpha & \sigma\ \mathrm{T}\ P_{<0}\alpha \\ \sigma\ \mathrm{T}\ P_{\geqslant r}\alpha & \end{array}}$$

3.2. Intuition behind the rules. Fitting proposes to think of σ T ϕ as saying that ϕ is forced in the possible world denoted by σ, and to think of σ F ϕ as saying that ϕ is not forced in σ. Having this in mind, the above rules are easy to understand. For example, if a node is labeled by σ T $\phi \wedge \psi$, then following Definition 2.3 both T ϕ and Tψ are forced in σ, which is obtained by Rule $T\wedge$. The prefixes handling rules are related to two situations: introducing a new prefix and adding prefixed signed formulas with already introduced prefixes. Consider Rule $F \rightarrow$ and suppose that a node is labeled by σ F $\phi \rightarrow \psi$. Then there must be a world accessible from σ in which T ϕ and F ψ are forced. On the other hand, if a node is labeled by σ T $\phi \rightarrow \psi$, then in each possible world accessible from σ, if T ϕ is forced, so must be T ψ. Finally, concerning the atomic probabilistic formulas handling rules, recall that σ F $P_{\geqslant r}\alpha$ means that $\sigma \not\Vdash P_{\geqslant r}\alpha$, which in turn can happen because:

- either the measure of α in w, $\mu_w(\alpha)$, is lesser than r, or
- α is not measurable in w, i.e., $\alpha \notin H_w$.

This is formally expressed by the $F\ P_{\geqslant r}$-rule. Similar explanations hold for σ F $P_{\leqslant r}\alpha$, σ F $P_{>r}\alpha$ and σ F $P_{<r}\alpha$.

3.3. Tableau proof. In [**3**, **4**] the notion of a closed tableau branch is rather simple: a branch is closed if it contains both σ T ψ and σ F ψ for some atomic ψ. Here, since atomic formulas are probabilistic with more complicated semantics, the notion of a closed branch should be more elaborated.

Let us consider a branch and let

$$\begin{aligned}\Lambda(\sigma) = &\{P_{\geqslant r_i}\alpha_i \ : \ i = 1, \dots, m_0\} \\ &\cup \{P_{<s_j}\beta_j \ : \ j = 1, \dots, m_1\} \\ &\cup \{P_{\leqslant t_k}\gamma_k \ : \ k = 1, \dots, m_2\} \\ &\cup \{P_{>o_l}\delta_l \ : \ l = 1, \dots, m_3\}.\end{aligned}$$

be the set of all atomic probabilistic formulas such that σ T ψ, for $\psi \in \Lambda(\sigma)$, appears on the branch. According to Definition 2.3 of the forcing relation, if the set of formulas $\Lambda(\sigma)$ is satisfied in a possible world denoted by σ, then the corresponding system $S(\Lambda(\sigma))$ of linear (in)equalities:

$$\bigwedge_{i=1}^{m_0} x^\sigma_{\alpha_i} \geqslant r_i \ \wedge \ \bigwedge_{i=1}^{m_0} 0 \leqslant x^\sigma_{\alpha_i} \leqslant 1 \tag{3.1}$$

$$\wedge \bigwedge_{j=1}^{m_1} x^\sigma_{\beta_j} < s_j \ \wedge \ \bigwedge_{j=1}^{m_1} 0 \leqslant x^\sigma_{\beta_j} \leqslant 1 \tag{3.2}$$

$$\wedge \bigwedge_{k=1}^{m_2} x^\sigma_{\gamma_k} \leqslant t_k \ \wedge \ \bigwedge_{k=1}^{m_2} 0 \leqslant x^\sigma_{\gamma_k} \leqslant 1 \tag{3.3}$$

$$\wedge \bigwedge_{l=1}^{m_3} x^\sigma_{\delta_l} > o_l \ \wedge \ \bigwedge_{l=1}^{m_3} 0 \leqslant x^\sigma_{\delta_l} \leqslant 1 \tag{3.4}$$

$$\wedge \bigwedge_{y} x^\sigma_y + x^\sigma_{\neg y} = 1 \tag{3.5}$$

$$\wedge \bigwedge_{\models y \to z} x^\sigma_y \leqslant x^\sigma_z \tag{3.6}$$

$$\wedge \bigwedge_{\substack{\models f \leftrightarrow y \wedge z \\ \models g \leftrightarrow y \vee z}} x^\sigma_g = x^\sigma_y + x^\sigma_z - x^\sigma_f \tag{3.7}$$

has a solution. Here the variables

- $x^\sigma_{\alpha_i}$, $x^\sigma_{\neg\alpha_i}$ for $i = 1, \ldots, m_0$,
- $x^\sigma_{\beta_j}$, $x^\sigma_{\neg\beta_j}$ for $j = 1, \ldots, m_1$,
- $x^\sigma_{\gamma_k}$, $x^\sigma_{\neg\gamma_k}$ for $k = 1, \ldots, m_2$, and
- $x^\sigma_{\delta_l}$, $x^\sigma_{\neg\delta_l}$ for $l = 1, \ldots, m_3$

represent the measures $\mu_\sigma(\alpha_1)$, $\mu_\sigma(\neg\alpha_1)$, ..., $\mu_\sigma(\delta_{m_3})$, $\mu_\sigma(\neg\delta_{m_3})$ in the possible world labeled by σ. The inequalities 3.1 – 3.4 correspond to the atomic probabilistic formulas from the set $\Lambda(\sigma)$. The equalities 3.5 mean that if a formula is measurable, so is its negation, and the sum of their measures is 1. The inequalities 3.6 express monotonicity, while the equalities 3.7 correspond to additivity, where

$$y, z, f, g \in \{\alpha_i, \neg\alpha_i, \beta_j, \neg\beta_j, \gamma_k, \neg\gamma_k, \delta_l, \neg\delta_l \ : \ i = 1, \ldots, m_0, j = 1, \ldots, m_1, \\ k = 1, \ldots, m_2, l = 1, \ldots, m_3\}.$$

Note that, since coefficients in $S(\Lambda(\sigma))$ are rational numbers, the system is solvable iff it has rational solutions.

On the other hand, solvability of the above system $S(\Lambda(\sigma))$ does not imply satisfiability of $\Lambda(\sigma)$ in the possible world labeled by σ, since measures of formulas must also satisfy the condition from Definition 2.2 of models that:

- If σ and $\sigma.n$ appear on a branch, the measure μ_σ is a restriction of the measure $\mu_{\sigma.n}$ on H_σ.

Note that this prevents that the set of prefixed signed atomic formulas

$$\{1 \text{ T } P_{\geqslant\frac{1}{3}}\alpha, 1.1 \text{ T } P_{\geqslant\frac{1}{3}}\alpha, 1.1 \text{ T } P_{<\frac{2}{3}}\alpha, 1.2 \text{ T } P_{\geqslant\frac{1}{3}}\alpha, 1.2 \text{ T } P_{>\frac{2}{3}}\alpha\}$$

from a branch is satisfiable since:

- $\mu_1(\alpha) \in [\frac{1}{3}, 1]$,
- $\mu_{1.1}(\alpha) \in [\frac{1}{3}, \frac{2}{3})$, and
- $\mu_{1.2}(\alpha) \in (\frac{2}{3}, 1]$.

Construction of a tableau of ϕ begins with a single node tree. That node, the root, is labeled by 1 F ϕ. Next, we apply the above tableau rules so that no rule for introducing a new prefix ($F\neg$, $F \rightarrow$) is applied more than once to an occurrence of a prefixed signed formula on a tableau branch.

DEFINITION 3.1. Let $\sigma_1, \dots, \sigma_n$ be all prefixes on a tableau branch B. B is open if the following is fulfilled:

- all systems $S(\Lambda(\sigma))$, for $\sigma \in \{\sigma_1, \dots, \sigma_n\}$, are solvable, and
- for all σ and $\sigma.n$ appearing on B, μ_σ is a restriction of $\mu_{\sigma.n}$ on the set of formulas $\{\alpha_i, \neg\alpha_i, \beta_j, \neg\beta_j, \gamma_k, \neg\gamma_k, \delta_l, \neg\delta_l \,:\, i = 1, \dots, m_0, j = 1, \dots, m_1, k = 1, \dots, m_2, l = 1, \dots, m_3\}$.

Otherwise, B is closed.

The set of all prefixed signed formulas from a tableau branch B is open (closed) if B is open (closed).

A tableau is closed if each of its branches is closed. Otherwise, the tableau is open.

A closed tableau with the root labeled by 1 F ϕ is a proof of ϕ. ϕ is a theorem, denoted $\vdash \phi$, if it has a proof. □

The following example illustrates that there are infinite tableaux.

EXAMPLE 3.1. This is a tableau for $\neg\neg\neg\phi$:

$$\begin{array}{c} 1 \text{ F } \neg\neg\neg\phi \quad (1) \\ | \\ 1.1 \text{ T } \neg\neg\phi \quad (2) \\ | \\ 1.1 \text{ F } \neg\phi \quad (3) \\ | \\ 1.1.1 \text{ T } \phi \quad (4) \\ | \\ 1.1.1 \text{ T } \neg\neg\phi \quad (5) \\ | \\ \dots \end{array}$$

The node (2) is obtained from (1) by $F\neg$, (3) is from (2) by $T\neg$, (4) is from (3) by $F\neg$, and (5) is from (2) by Lower. □

4. Soundness

A tableau branch B is satisfiable if the set of all prefixed signed formula on it is satisfiable (according to Definition 2.5). A tableau is satisfiable if at least one of its branches is satisfiable.

The above tableau rules are defined so that, applied to a satisfiable tableau, produce a new satisfiable tableau. For the rules $T\wedge$, $F\vee$, $F\wedge$, $T\vee$, $T \rightarrow$, $T\neg$, $F\neg$,

$F \rightarrow$ and Lower that is proved in [**3**, **4**], while to analyze the rules $F\ P_{\geqslant r}$, $F\ P_{\leqslant r}$, $F\ P_{>r}$ and $F\ P_{<r}$ we can consider $F\ P_{\geqslant r}$ and the other cases follow similarly.

So, let us consider a branch B, the corresponding set $S(\mathrm{B})$ of all prefixed signed formula on B, and the formula σ F $P_{\geqslant r}\alpha \in S(\mathrm{B})$. Assume that $S(\mathrm{B})$ is satisfiable under a mapping π, which maps prefixes occurring in S to the set W of possible worlds from the model $\mathbb{M} = \langle W, \{H_w : w \in W\}, \{\mu_w\ :\ w \in W\}, \leqslant\rangle$. It means that

- $\pi(\sigma) \nVdash P_{\geqslant r}\alpha$.

According to Definition 2.3:

- either $\alpha \in H_w$ (i.e., $\pi(\sigma) \Vdash P_{\geqslant 0}\alpha$) and $\pi(\sigma) \Vdash P_{<r}\alpha$, or
- $\alpha \notin H_w$, i.e., $\pi(\sigma) \Vdash P_{<0}\alpha$.

It follows that, after the rule $F\ P_{\geqslant r}$ is applied on σ F $P_{\geqslant r}\alpha \in S(\mathrm{B})$, one of the enlarged branches $\mathrm{B}, \{\sigma \text{ T } P_{\geqslant 0}\alpha, \sigma \text{ T } P_{<r}\alpha\}$ or $\mathrm{B}, \{\sigma \text{ T } P_{<0}\alpha\}$ is satisfiable.

Thus, the following hold:

- if a rule is applied on a satisfiable tableau, then the resulting tableau is also satisfiable, and
- since satisfiable tableaux are not closed, every satisfiable formula has only not closed tableaux,

so we have:

THEOREM 4.1 (Soundness). *Every theorem is a valid formula.*

5. Completeness

The paper [**3**] presents an approach to proving completeness of the prefixed tableau systems for modal logics which is based on a systematic proof procedure and uses König's lemma. Here we uses the same idea for ILP. The systematic proof procedure essentially guarantees:

- that rules are applied on all formulas in a tableau, and
- that a tableau proof must be produced for valid formulas.

While constructing a tableau, there are situations that some formulas must be considered more than once. For example, Rule Lower might be applied to σ T ϕ many times, i.e., for every accessible $\sigma.n$ appearing on the same branch. It is necessary to keep information about such formulas. This can be performed using the following inessential extension of rules:

- rules are applied on each occurrence of a formula only once,
- after the application of a rule the corresponding node becomes finished, and
- whenever Rule Lower is applied on a node on a branch, a new copy of the node is added at the end of the branch.

Having this modification, the systematic procedure starts with the root 1 F ϕ and then in a loop:

- uses the breadth-first strategy and applies the corresponding rule on the leftmost not yet finished node as close to the root as possible, and

- declares that node finished (and adds a copy of the node if Rule Lower is applied).

The procedure stops:

- if the tableau is closed, or
- if every occurrence of a prefixed formula is finished, except possibly some nodes of the form σ T ψ but without prefixes accessible from σ for which Rule Lower might be applied.

Now, to prove completeness of the prefixed tableau system for ILP, it is showed that if the systematic procedure does not produce a proof for ϕ, it generates enough information to construct a counter-model for ϕ.

First we introduce the notion of downward saturated sets of prefixed signed formulas and show that each such set is satisfiable. Then, it is proved that the set of all of prefixed signed formulas from a finished open branch is downward saturated, which implies completeness.

Definition 5.1. A set S of prefixed signed formulas is downward saturated if the following holds:

- S is open,
- if σ T $\phi \wedge \psi \in S$, then σ T $\phi \in S$, and σ T $\psi \in S$,
- if σ F $\phi \vee \psi \in S$, then σ F $\phi \in S$, and σ F $\psi \in S$,
- if σ F $\phi \wedge \psi \in S$, then σ F $\phi \in S$, or σ F $\psi \in S$,
- if σ T $\phi \vee \psi \in S$, then σ T $\phi \in S$, or σ T $\psi \in S$,
- if σ T $\phi \rightarrow \psi \in S$, then σ F $\phi \in S$, or σ T $\psi \in S$,
- if σ T $\neg\phi \in S$, then σ F $\phi \in S$,
- if σ F $\neg\phi \in S$, then $\sigma.n$ T $\phi \in S$ for some n,
- if σ F $\phi \rightarrow \psi \in S$, then $\sigma.n$ T $\phi \in S$ and $\sigma.n$ F $\psi \in S$ for some $\sigma.n$,
- if σ T $\phi \in S$, then $\sigma.n$ T $\phi \in S$ for every $\sigma.n$ which occurs in S,
- if σ F $P_{\geqslant r}\alpha \in S$, then σ T $P_{\geqslant 0}\alpha, \sigma$ T $P_{<r}\alpha \in S$, or σ T $P_{<0}\alpha \in S$,
- if σ F $P_{\leqslant r}\alpha \in S$, then σ T $P_{\geqslant 0}\alpha, \sigma$ T $P_{>r}\alpha \in S$, or σ T $P_{<0}\alpha \in S$,
- if σ F $P_{>r}\alpha \in S$, then σ T $P_{\geqslant 0}\alpha, \sigma$ T $P_{\leqslant r}\alpha \in S$, or σ T $P_{<0}\alpha \in S$, and
- if σ F $P_{<r}\alpha \in S$, then σ T $P_{\geqslant 0}\alpha, \sigma$ T $P_{\geqslant r}\alpha \in S$, or σ T $P_{<0}\alpha \in S$.

□

Lemma 5.1. *Every downward saturated set S is satisfiable in a model whose possible worlds are prefixes occurring in S.*

Proof. Let W be the set of all prefixes appearing in the set S. We assume that $\sigma \leqslant \sigma$ and $\sigma \leqslant \sigma.n$ always holds and that $\leqslant$ is transitive. Then, $\langle W, \leqslant\rangle$ is a frame.

Since S is open there are measures μ_σ, for $\sigma \in W$, such that:

- H_σ consists of all For_C-formulas whose measures are obtained as solutions of the system $S(\Lambda(\sigma))$ defined above, and
- if $\sigma_1 \leqslant \sigma_2$, then μ_{σ_1} is a restriction of μ_{σ_2} on H_{σ_1}.

Obviously, all atomic prefixed signed probabilistic formulas with the prefix σ are forced in σ.

By a straightforward induction on complexity of formulas it follows that for every signed formula ϕ and every prefix σ:

- if σ T $\phi \in S$, then $\sigma \Vdash \phi$, and
- if σ F $\phi \in S$, then $\sigma \nVdash \phi$.

So, the statement follows. □

THEOREM 5.1 (Completeness). *Every valid formula has a tableau proof.*

PROOF. Let ϕ be not provable. Then the systematic procedure which starts with 1 F ϕ does not produce a closed tableau. If the procedure stops, the resulting tableau is finite and open and there is at least one finite open branch. If the procedure does not stop, the resulting tableau is infinite. By König's lemma the tableau has an infinite open branch. Whether the open branch is finite or not, the set S of all prefixed signed formulas from the branch is downward saturated. By Lemma 5.1, S is satisfiable and since 1 F $\phi \in S$, it follows that

$$1 \nVdash \phi.$$

Thus, ϕ is not valid. □

6. Decidability

Since there are infinite tableaux (e.g., in Example 3.1) we have to show that we can modify the systematic procedure such that it becomes a decision procedure. This can be done as in [**3**] for transitive modal logics. Here we just sketch the main ideas that guarantees that tableau construction must terminate which gives decidability:

- Let B be a branch of a tableau and σ a prefix which appears on B; then $S(B,\sigma)$ denotes the set of all signed formulas Z such that σ Z appears on B. Let as call $S(B,\sigma)$ the set of signed formulas associated with σ.
- Since there are only finitely many signed formulas (without prefixes) that appear in any tableau for ϕ, for every branch B there are only finitely many different sets of the form $S(B,\sigma)$.
- On any infinite branch B there must be two prefixes with the same set of associated signed formulas. Let σ and ρ ($\sigma \leqslant \rho$) be the first two prefixes such that $S(B,\sigma) = S(B,\rho)$.
- Obviously, after ρ there a periodic behavior is repeated and the corresponding successors of σ and ρ (e.g. $\sigma.1$ and $\rho.1$) have the same sets of associated signed formulas. It means that no essentially new information will appear after ρ.

So, even on an infinite brunch when such σ and ρ are detected the construction can be stopped without affecting possibility to construct a counter-model for the considered formula. It means that the procedure is finite which implies decidability.

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