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APPROXIMATION BY GENERALIZED SZÁSZ–MIRAKJAN–KANTOROVICH TYPE OPERATORS

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Abstract. We construct Kantorovich variant of generalized Szász–Mirakjan operators whose construction depends on a continuously differentiable, increasing and unbounded function τ. For these new operators we give weighted approximation, Voronovskaya type theorem, quantitative estimates for the local approximation.

1. Introduction

In 1912, Bernstein [**4**] proposed the famous polynomial, which is constructed by probabilistic method to give the simple, short and most elegant proof of Weierstrass theorem [**20**] as follows:

$$\mathcal{B}_m(g;u)=\sum_{j=0}^{m} b_{m,j}(u)g\Big(\frac{j}{m}\Big),$$

where $u\in[0,1]$, $m=1,2,\dots$, and the basis of Bernstein functions $b_{m,j}$ are defined as follows:

$$b_{m,j}(u)=\binom{m}{j}u^j(1-u)^{m-j}. \tag{1.1}$$

Kantorovich [**12**] gave a new modification of Bernstein polynomials to approximate Lebesgue integrable functions on the interval $[0,1]$, as follows:

$$\mathcal{K}_m^*(g;u)=(m+1)\sum_{j=0}^{m} b_{m,j}(u)\int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} g(y)dy,$$

where $b_{m,j}(u)$ is given by (1.1).

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In 2011, Cárdenas et al. [**5**] defined the Bernstein type operators by

$$g \mapsto B_m(g \circ \tau^{-1}) \circ \tau$$

and also presents a better degree of approximation depending on τ. This type of approximation operators generalizes the Korovkin set from $\{e_0, e_1, e_2\}$ to $\{e_0, \tau, \tau^2\}$.

In 2014, Aral et al. [**3**] defined a similar modification of Szász–Mirakyan type operators by using a suitable function τ, which satisfies following properties

(τ_1) τ be a continuously differentiable function on $[0, \infty)$,

(τ_2) $\tau(0) = 0$ and $\inf_{u \in [0,\infty)} \tau'(u) \geq 1$.

The new operators which are called generalized Szász–Mirakyan operators are defined as

$$\mathcal{S}_m^{\tau}(g; u) = e^{-m\tau(u)} \sum_{j=0}^{\infty} \frac{(m\tau(u))^j}{j!} (g \circ \tau^{-1})\Big(\frac{j}{m}\Big), \tag{1.2}$$

for $m \geq 1$, $u \geq 0$, and suitable functions g defined on $[0, \infty)$. If $\tau(u) = u$ then (1.2) reduces to the Szász–Mirakyan operators defined in [**18**] as

$$\mathcal{S}_m(g; u) = e^{-mu} \sum_{j=0}^{\infty} \frac{(mu)^j}{j!} g\Big(\frac{j}{m}\Big),$$

In this paper, we define Kantorovich variant of operators (1.2) which depends on τ. In Kantorovich type modifications we mainly replace the sample values $\frac{j}{m}$ by the mean values of $(g \circ \tau^{-1})$ in the interval $[\frac{j}{m+1}, \frac{j+1}{m+1}]$. In the recent past, Kantorovich type modifications of Szász–Mirakyan and various other operators were investigated by several researchers (see [**1**, **2**, **11**, **14**–**17**, **19**]).

The present work is organized as follows. In the second section, we define the Kantorovich variant of the generalized Szász–Mirakyan operators and its moments and central moments are calculated. In the third section, we study convergence properties of new constructed operators in the light of weighted space. In section four, we obtain the order of approximation of generalized Szász–Mirakyan–Kantorovich operators associated with the weighted modulus of continuity. In section five, a Voronovskaya type result is obtained. Finally, in the last section, we obtain some local approximation results related to K-functional also we define a Lipschitz-type functions, as well as related results.

2. Construction of the generalized Szász–Mirakyan–Kantorovich operators

Inspired by the above mentioned work, we introduce the Kantorovich variant of operators (1.2) which depend on a suitable function τ as follows:

Definition 2.1. For $g \colon [0, \infty) \to \mathbb{R}$, $u \geq 0$ and $m \in \mathbb{N}$, we define the Kantorovich variant of generalized Szász–Mirakyan operators as

$$\mathcal{K}_{m,\tau}^{*}(g; u) = (m+1) e^{-m\tau(u)} \sum_{j=0}^{\infty} \frac{(m\tau(u))^j}{j!} \int_{\frac{j}{m+1}}^{\frac{j+1}{m+1}} (g \circ \tau^{-1})(y) dy, \tag{2.1}$$

Operators (2.1) are linear and positive. The operator $\mathcal{K}^*_{m,\tau}$ is constructed to obtain results in approximation for discontinuous functions on the basis of integral mean of $(g\circ\tau^{-1})$ over small intervals. The following auxiliary results for $\mathcal{K}^*_{m,\tau}$ can be proved easily which are used to prove our main results.

Lemma 2.1. *Let $\mathcal{K}^*_{m,\tau}$ be given by (2.1). Then for each $u \geq 0$ and $m \in \mathbb{N}$ we have*

(i) $\mathcal{K}^*_{m,\tau}(1;u) = 1$,
(ii) $\mathcal{K}^*_{m,\tau}(\tau;u) = \frac{1}{2(m+1)}(2m\tau(u)+1)$,
(iii) $\mathcal{K}^*_{m,\tau}(\tau^2;u) = \frac{1}{3(m+1)^2}(3m^2\tau(u)^2+6m\tau(u)+1)$,
(iv) $\mathcal{K}^*_{m,\tau}(\tau^3;u) = \frac{1}{4(m+1)^3}(4m^3\tau(u)^3+18m^2\tau(u)^2+14m\tau(u)+1)$,
(v) $\mathcal{K}^*_{m,\tau}(\tau^4;u) = \frac{1}{5(m+1)^4}(5m^4\tau(u)^4+40m^3\tau(u)^3+75m^2\tau(u)^2+30m\tau(u)+1)$,
(vi) $\mathcal{K}^*_{m,\tau}(\tau^5;u) = \frac{1}{6(m+1)^5}(6m^5\tau(u)^5+75m^4\tau(u)^4+260m^3\tau(u)^3+270m^2\tau(u)^2$ $+62m\tau(u)+1)$.

Corollary 2.1. *By using Lemma 2.1 and by linearity of operators $\mathcal{K}^*_{m,\tau}$, we can acquire the central moments as*

(i) $\mathcal{K}^*_{m,\tau}(\tau(\zeta)-\tau(u);u) = \frac{1}{2(m+1)}[-2\tau(u)+1]$,
(ii) $\mathcal{K}^*_{m,\tau}((\tau(\zeta)-\tau(u))^2;u) = \frac{1}{3(m+1)^2}[3\tau(u)^2+3(m-1)\tau(u)+1]$,
(iii) $\mathcal{K}^*_{m,\tau}((\tau(\zeta)-\tau(u))^3;u) = \frac{1}{4(m+1)^3}[-4\tau(u)^3-6(2m-1)\tau(u)^2+2(5m-2)\tau(u)+1]$,
(iv) $\mathcal{K}^*_{m,\tau}((\tau(\zeta)-\tau(u))^4;u) = \frac{1}{5(m+1)^4}[5\tau(u)^4+10(3m-1)\tau(u)^3+5(3m^2-10m+2)\tau(u)^2+5(5m-1)\tau(u)+1]$,
(v) $\mathcal{K}^*_{m,\tau}((\tau(\zeta)-\tau(u))^5;u) = \frac{1}{6(m+1)^5}[-6\tau(u)^5-15(4m-1)\tau(u)^4-10(9m^2-15m+2)\tau(u)^3+15(7m^2-10m+1)\tau(u)^2+2(28m-3)\tau(u)+1]$.

3. Weighted approximation

In this section we prove the convergence properties of newly constructed operators $\mathcal{K}^*_{m,\tau}$ in the light of weighted space.

Let $\Psi(u)$ be a function satisfying the conditions (τ_1) and (τ_2) given above. Also, we take the weight function $\Psi(u) = 1+\tau^2(u)$ and we define the weighted spaces as follows: $\mathcal{B}_\Psi[0,\infty) = \{g\colon [0,\infty)\to\mathbb{R} \mid |g(u)| \leq \mathcal{M}_f\Psi(u), u\geq 0\}$, where $\mathcal{M}_g$ is a constant which depends only on g. $\mathcal{B}_\Psi[0,\infty)$ is a normed linear space equipped with the norm $\|g\|_\Psi = \sup_{u\in[0,\infty)}\frac{|g(u)|}{\Psi(u)}$. Also, the subspaces $\mathcal{C}_\Psi[0,\infty)$, $U_\Psi[0,\infty)$ and $U_\Psi[0,\infty)$ of $\mathcal{B}_\Psi[0,\infty)$ are defined as

$$\mathcal{C}_\Psi[0,\infty) = \{g\in\mathcal{B}_\Psi[0,\infty):\ g \text{ is continuous on } [0,\infty)\},$$

$$\mathcal{C}^*_\Psi[0,\infty) = \Big\{g\in\mathcal{C}_\Psi[0,\infty):\ \lim_{u\to\infty}\frac{g(u)}{\Psi(u)} = \mathcal{M}_g = \text{const.}\Big\},$$

$$U_\Psi[0,\infty) = \Big\{g\in\mathcal{C}_\Psi[0,\infty):\ \frac{g(u)}{\Psi(u)} \text{ is uniformly continuous on } [0,\infty)\Big\}.$$

It is obvious that $\mathcal{C}^*_\Psi[0,\infty)\subset U_\Psi[0,\infty)\subset\mathcal{C}_\Psi[0,\infty)\subset\mathcal{B}_\Psi[0,\infty)$.

In [**8**], Gadjiev prove the following results for the weighted Korovkin type theorems.

We consider $\{\mathcal{Q}_m\}_{m\geq 1}$ be a sequence of positive linear operators which acts from $\mathcal{C}_\Psi[0,\infty)$ to $\mathcal{B}_\Psi[0,\infty)$.

LEMMA 3.1. [**8**] *The positive linear operators $\mathcal{Q}_m$, $m \geq 1$, acts from $\mathcal{C}_\Psi[0,\infty)$ to $\mathcal{B}_\Psi[0,\infty)$ if and only if the inequality $|\mathcal{Q}_m(\Psi;u)| \leq \mathcal{M}_m\Psi(u)$, $u \geq 0$, holds, where $\mathcal{M}_m > 0$ is a constant depending on m.*

THEOREM 3.1. [**8**] *Let the sequence of positive linear operators $\mathcal{Q}_m$, $m \geq 1$ acting from $\mathcal{C}_\Psi[0,\infty)$ to $\mathcal{B}_\Psi[0,\infty)$ and satisfying*

$$\lim_{m\to\infty} \|\mathcal{Q}_m\tau^i - \tau^i\|_\Psi = 0, \quad i = 0,1,2.$$

Then for any function $g \in C^*_\Psi[0,\infty)$ we have $\lim_{m\to\infty} \|\mathcal{Q}_m(g) - g\|_\Psi = 0$.

Therefore, we can prove the following results.

THEOREM 3.2. *For each function $g \in C^*_\Psi[0,\infty)$ we have*

$$\lim_{m\to\infty} \|\mathcal{K}^*_{m,\tau}(g) - g\|_\Psi = 0.$$

PROOF. By Lemma 2.1 (i) and (ii), it is clear that $\|\mathcal{K}^*_{m,\tau}(1;u) - 1\|_\Psi = 0.$ and

$$\|\mathcal{K}^*_{m,\tau}(\tau;u) - \tau\|_\Psi \leq \Big|\frac{m}{m+1} - 1\Big| \sup_{u\in[0,\infty)} \frac{\tau(u)}{1+\tau^2(u)} + \frac{1}{2(m+1)} \leq \frac{3}{2(m+1)}.$$

Again by Lemma 2.1 (iii), we have

$$\begin{aligned}\|\mathcal{K}^*_{m,\tau}(\tau^2;u) - \tau^2\|_\Psi &\leq \Big|\frac{m^2}{(m+1)^2} - 1\Big| \sup_{u\in[0,\infty)} \frac{\tau^2(u)}{1+\tau^2(u)} \\ &\quad + \frac{2m}{(m+1)^2} \sup_{u\in[0,\infty)} \frac{\tau(u)}{1+\tau^2(u)} + \frac{1}{3(m+1)^2} \\ &\leq \frac{12m+4}{3(m+1)^2}.\end{aligned} \tag{3.1}$$

Then from Lemma 2.1 and (3.1) we get $\lim_{m\to\infty} \|\mathcal{K}^*_{m,\tau}(\tau^i) - \tau^i\|_\Psi = 0$, $i = 0,1,2$. Hence, the proof is completed. □

4. Rate of convergence

In this section, we determine the rate of convergence for $\mathcal{K}^*_{m,\tau}$ by weighted modulus of continuity $\omega_\tau(g;\delta)$ which was recently considered by Holhoş [**10**] as follows:

$$\omega_\tau(g;\delta) = \sup_{u,\zeta\in[0,\infty),|\tau(\zeta)-\tau(u)|\leq\delta} \frac{|g(\zeta) - g(u)|}{\Psi(\zeta) + \Psi(u)}, \quad \delta > 0, \tag{4.1}$$

where $g \in \mathcal{C}_\Psi[0,\infty)$, with the following properties:

(i) $\omega_\tau(g;0) = 0$,
(ii) $\omega_\tau(g;\delta) \geq 0$, $\delta \geq 0$ for $g \in \mathcal{C}_\Psi[0,\infty)$,
(iii) $\lim_{\delta\to 0} \omega_\tau(g;\delta) = 0$, for each $g \in U_\Psi[0,\infty)$.

Theorem 4.1. [**10**] *Let* $\mathcal{Q}_m\colon \mathcal{C}_\Psi[0,\infty) \to \mathcal{B}_\Psi[0,\infty)$ *be a sequence of positive linear operators with*

$$\begin{aligned} \|\mathcal{Q}_m(\tau^0)-\tau^0\|_{\Psi^0} &= a_m, & \|\mathcal{Q}_m(\tau^2)-\tau^2\|_{\Psi} &= c_m,\\ \|\mathcal{Q}_m(\tau)-\tau\|_{\Psi^{1/2}} &= b_m, & \|\mathcal{Q}_m(\tau^3)-\tau^3\|_{\Psi^{3/2}} &= d_m, \end{aligned} \tag{4.2}$$

where the sequences a_m, b_m, c_m *and* d_m *converge to zero as* $m\to\infty$. *Then*

$$\|\mathcal{Q}_m(g)-g\|_{\Psi^{3/2}} \le (7+4a_m+2c_m)\omega_\tau(g;\delta_m)+\|g\|_\Psi a_m,$$

for all $g\in\mathcal{C}_\Psi[0,\infty)$, *where*

$$\delta_m = 2\sqrt{(a_m+2b_m+c_m)(1+a_m)}+a_m+3b_m+3c_m+d_m.$$

Theorem 4.2. *Let for each* $g\in\mathcal{C}_\Psi[0,\infty)$ *we have*

$$\|\mathcal{K}^*_{m,\tau}(g)-g\|_{\Psi^{3/2}} \le \Big(\frac{21m^2+66m+29}{3(m+1)^2}\Big)\omega_\tau(g;\delta_m),$$

where

$$\delta_m = 2\sqrt{\frac{21m+13}{3(m+1)^2}}+\frac{78m^2+140m+27}{4(m+1)^3}.$$

Proof. If we calculate the sequences (a_m), (b_m), (c_m) and (d_m), then by using Lemma 2.1, clearly we have

$$\begin{aligned} \|\mathcal{K}^*_{m,\tau}(\tau^0)-\tau^0\|_{\Psi^0} &= 0 = a_m,\\ \|\mathcal{K}^*_{m,\tau}(\tau)-\tau\|_{\Psi^{1/2}} &\le \frac{3}{2(m+1)} = b_{m,q},\\ \|\mathcal{K}^*_{m,\tau}(\tau^2)-\tau^2\|_{\Psi} &\le \frac{12m+4}{3(m+1)^2} = c_m. \end{aligned}$$

Finally,

$$\|\mathcal{K}^*_{m,\tau}(\tau^3)-\tau^3\|_{\Psi^{3/2}} \le \frac{48m^2+88m+5}{4(m+1)^3} = d_{m,q}.$$

Thus, conditions (4.1)–(4.2) are satisfied. Now by Theorem 4.1, we obtain the desired result. □

Remark 4.1. From property (iii) of $\omega_\tau(g;\delta)$ and Theorem 4.2, we have

$$\lim_{m\to\infty}\|\mathcal{K}^*_{m,\tau}(g)-g\|_{\Psi^{3/2}}=0, \quad \text{for } \ g\in U_\Psi[0,\infty).$$

5. Voronovskaya type theorem

In this section, we prove pointwise convergence of $\mathcal{K}^*_{m,\tau}$ by using a technique which is developed in [**5**] by Cardenas–Morales, Garrancho and Raşa.

Theorem 5.1. *Let* $g\in\mathcal{C}_\Psi[0,\infty)$, $u\in[0,\infty)$ *and suppose that* $(g\circ\tau^{-1})'$ *and* $(g\circ\tau^{-1})''$ *exist at* $\tau(u)$. *If* $(g\circ\tau^{-1})''$ *is bounded on* $[0,\infty)$, *then we have*

$$\lim_{m\to\infty} m[\mathcal{K}^*_{m,\tau}(g;u)-g(u)] = \tau(u)(g\circ\tau^{-1})'\Big\{\frac12-\tau(u)\Big\}+\frac12\tau(u)(g\circ\tau^{-1})''\tau(u).$$

PROOF. By using Taylor expansion of $(g \circ \tau^{-1})$ at $\tau(u) \in [0,\infty)$, there exists a point ζ lying between u and z, so we have

$$g(\zeta) = (g \circ \tau^{-1})(\tau(\zeta)) = (g \circ \tau^{-1})(\tau(u)) + (g \circ \tau^{-1})'(\tau(u))(\tau(\zeta) - \tau(u)) + \frac{1}{2}(g \circ \tau^{-1})''(\tau(u))(\tau(\zeta) - \tau(u))^2 + \lambda_u(\zeta)(\tau(\zeta) - \tau(u))^2, \tag{5.1}$$

where

$$\lambda_u(\zeta) = \frac{(g \circ \tau^{-1})''(\tau(\zeta)) - (g \circ \tau^{-1})''(\tau(u))}{2}. \tag{5.2}$$

Therefore, by the assumption on g and (5.2) ensures that $|\lambda_u(\zeta)| \leq \mathcal{K}$, for all $\zeta \in [0,\infty)$ and $\lim_{\zeta\to u} \lambda_u(\zeta) = 0$. Now by applying operators (2.1) to equality (5.1), we get

$$[\mathcal{K}^*_{m,\tau}(g;u) - g(u)] = (g \circ \tau^{-1})'(\tau(u))\mathcal{K}^*_{m,\tau}((\tau(\zeta) - \tau(u));u) + \frac{(g \circ \tau^{-1})''(\tau(u))\mathcal{K}^*_{m,\tau}((\tau(\zeta) - \tau(y))^2;u)}{2} + \mathcal{K}^*_{m,\tau}(\lambda^u(\zeta)((\tau(\zeta) - \tau(u))^2;u))\zeta \tag{5.3}$$

From Lemma 2.1 and Corollary 2.1, we obtain

$$\lim_{m\to\infty} m\mathcal{K}^*_{m,\tau}((\tau(\zeta) - \tau(u));u) = \frac{1 - 2\tau(u)}{2}, \tag{5.4}$$

$$\lim_{m\to\infty} m\mathcal{K}^*_{m,\tau}((\tau(\zeta) - \tau(u))^2;u) = \tau(u). \tag{5.5}$$

By estimating the last term on the right hand side of equality (5.3), we will get the proof.

Since from (5.2), for every $\epsilon > 0$, $\lim_{\zeta\to u} \lambda_u(\zeta) = 0$. Let $\delta > 0$ such that $|\lambda_u(\zeta)| < \varepsilon$ for every $\zeta \geq 0$. By the Cauchy–Schwartz inequality, we get

$$\lim_{m\to\infty} m\mathcal{K}^*_{m,\tau}(|\lambda_u(\zeta)|(\tau(\zeta) - \tau(u))^2;u) \leq \varepsilon \lim_{m\to\infty} m\mathcal{K}^*_{m,\tau}((\tau(\zeta) - \tau(u))^2;u) + \frac{\mathcal{K}^*}{\delta^2} \lim_{m\to\infty} \mathcal{K}^*{}_{m,\tau}((\tau(w) - \tau(u))^4;u).$$

Since $\lim_{m\to\infty} m\mathcal{K}^*_{m,\tau}((\tau(\zeta) - \tau(u))^4;u) = 0$, we obtain

$$\lim_{m\to\infty} m\mathcal{K}^*_{m,\tau}(|\lambda_u(\zeta)|(\tau(\zeta) - \tau(u))^2;u) = 0. \tag{5.6}$$

Thus, by taking into account equations (5.4), (5.5) and (5.6) to equation (5.3) this proof the theorem. □

6. Local and global approximation

In order to prove local approximation theorems for the operators $\mathcal{K}^*_{m,\tau}$, let us recall some basic concepts and results concerning modulus of continuity and K-functional. Let $\mathcal{C}_B[0,\infty)$, be the space of real-valued continuous and bounded functions g defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $\mathcal{C}_B[0,\infty)$ is given by $\|g\| = \sup_{0\leq u<\infty} |g(u)|$. Further, K-functional is defined as:

$$K_2(g,\delta) = \inf_{r\in W^2}\{\| g - r \| + \delta\|g''\|\},$$

where $\delta > 0$ and $W^2 = \{s \in \mathcal{C}_B[0,\infty) : r', r'' \in \mathcal{C}_B[0,\infty)\}$. Then, in view of known result [**6**], there exists an absolute constant $\mathcal{D} > 0$ such that $\mathcal{K}^*(g,\delta) \leq \mathcal{D}\omega_2(g,\sqrt{\delta})$. For $g \in C_B[0,\infty)$ the second order modulus of smoothness is defined as follows

$$\omega_2(g,\sqrt{\delta}) = \sup_{0<h\leq\sqrt{\delta}} \sup_{u\in[0,\infty)} |g(u+2h) - 2g(u+h) + g(u)|$$

and the usual modulus of continuity is defined as

$$\omega(g,\delta) = \sup_{0<h\leq\delta} \sup_{u\in[0,\infty)} |g(u+h) - g(u)|.$$

THEOREM 6.1. *Let $g \in \mathcal{C}_B[0,\infty)$ and τ be a function satisfying the conditions (τ_1), (τ_2) and $\|\tau''\|$ is finite. Then, there exists an absolute constant $\mathcal{D} > 0$ such that $|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \leq \mathcal{D}K(g,\delta_m(u))$ where*

$$\delta_m(u) = \frac{1}{3(m+1)^2}[3\tau(u)^2 + 3(m-1)\tau(u) + 1].$$

PROOF. Let $r \in W^2$ and $u, y \in [0,\infty)$. By using Taylor's formula we have

$$r(y) = r(u) + (r\circ\tau^{-1})'(\tau(u))(\tau(y)-\tau(u)) + \int_{\tau(u)}^{\tau(y)} (\tau(y)-v)(r\circ\tau^{-1})''(v)dv. \tag{6.1}$$

By using the equality

$$(r\circ\tau^{-1})''(\tau(u)) = \frac{r''(u)}{(\tau'(u))^2} - r''(u)\frac{\tau''(u)}{(\tau'(u))^3}.$$

Now, put $v = \tau(z)$ in the last term in equality (6.1), we get

$$\int_{\tau(u)}^{\tau(y)} (\tau(y)-v)(r\circ\tau^{-1})''(v)dv = \int_u^y (\tau(y)-\tau(z))\Big[\frac{r''(z)\tau'(z) - r'(z)\tau''(v)}{(\tau'(z))^2}\Big]dz,$$

i.e.,

$$\begin{aligned}\int_{\tau(u)}^{\tau(y)} (\tau(y)-v)(r\circ\tau^{-1})''(v)dv &= \int_{\tau(u)}^{\tau(y)} (\tau(y)-v)\frac{r''(\tau^{-1}(v))}{(\tau'(\tau^{-1}(v)))^2}dv \\ &\quad - \int_{\tau(u)}^{\tau(y)} (\tau(y)-v)\frac{r'(\tau^{-1}(v))\tau''(\tau^{-1}(v))}{(\tau'(\tau^{-1}(v)))^3}dv.\end{aligned} \tag{6.2}$$

By applying operator (2.1) to both sides of equality (6.1) and by using Lemma 2.1 and (6.2), we deduce

$$\begin{aligned}\mathcal{K}^*_{m,\tau}(r;u) &= r(u) + \mathcal{K}^*_{m,\tau}\bigg(\int_{\tau(u)}^{\tau(y)} (\tau(y)-v)\frac{r''(\tau^{-1}(v))}{(\tau'(\tau^{-1}(v)))^2}dv;u\bigg) \\ &\quad - \mathcal{K}^*_{m,\tau}\bigg(\int_{\tau(u)}^{\tau(y)} (\tau(y)-v)\frac{r'(\tau^{-1}(v))\tau''(\tau^{-1}(v))}{(\tau'(\tau^{-1}(v)))^3}dv;u\bigg).\end{aligned}$$

As we know τ is strictly increasing on $[0,\infty)$ and with condition (τ_2), we get

$$|\,\mathcal{K}^*_{m,\tau}(r;u) - r(u)| \leq \mathcal{M}^\tau_{m,2}(u)(\|r''\| + \|r'\|\|\tau''\|),$$

where $\mathcal{M}^\tau_{m,2}(u) = \mathcal{K}^*_{m,\tau}((\tau(y)-\tau(u))^2;u)$.

For all $g \in \mathcal{C}_B[0,\infty)$, we have

$$|\mathcal{K}^*_{m,\tau}(r;u)| \leq \|g \circ \tau^{-1}\| e^{-m\tau(u)} \sum_{j=0}^{\infty} \frac{(m\tau(u))^j}{j!} \leq \|g\| \mathcal{K}^*_{m,\tau}(1;u) = \|g\|.$$

Hence we have

$$\begin{aligned}|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| &\leq |\mathcal{K}^*_{m,\tau}(g-r;u)| + |\mathcal{K}^*_{m,\tau}(r;u) - r(u)| + |r(u) - g(u)| \\ &\leq 2\|g-r\| + \frac{3\tau^2(u) + 6m\tau(u) + 1}{3(m+1)^2}(\|r''\| + \|r'\|\|\tau''\|),\end{aligned}$$

if we choose $\mathcal{D}$=max$\{2, \|\tau''\|\}$, then

$$|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \leq \mathcal{D}\Big(2\|g-r\| + \frac{3\tau^2(u) + 3(m-1)\tau(u) + 1}{3(m+1)^2}\|r''\|_{W^2}\Big).$$

Taking infimum over all $r \in W^2$ we obtain $|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \leq \mathcal{D}K(g, \delta_m(u))$. □

Now, we recall the Lipschitz class given in **[9]**. Let τ be a function satisfying the conditions (τ_1), (τ_2), $0 < \alpha \leq 1$ and $\mathrm{Lip}_{\mathcal{H}}(\tau(u);\alpha)$, $\mathcal{H} \geq 0$ is the set of functions g satisfying the inequality

$$|g(y) - g(u)| \leq \mathcal{H}|\tau(u) - \tau(u)|^\alpha, \quad u, y \geq 0.$$

Moreover, for a bounded subset $\mathcal{Y} \subset [0,\infty)$, we say that the function $g \in \mathcal{C}_B[0,\infty)$ belongs to $\mathrm{Lip}_{\mathcal{H}}(\tau(u);\alpha)$, $0 < \alpha \leq 1$ on $\mathcal{Y}$ if

$$|g(z) - g(u)| \leq \mathcal{H}_{\alpha,g}|\tau(y) - \tau(u)|^\alpha, \qquad u \in \mathcal{Y} \quad \text{and} \quad y \geq 0,$$

where $\mathcal{H}_{\alpha,g}$ is a constant depending on α and g.

THEOREM 6.2. *Let τ be a function satisfying the conditions (τ_1), (τ_2). Then for any $g \in \mathrm{Lip}_{\mathcal{H}}(\tau(u);\alpha)$, $0 < \alpha \leq 1$ and for every $u \in (0,\infty)$, $m \in \mathbb{N}$, we have*

$$|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \leq \mathcal{H}(\delta_m(u))^{\alpha/2}, \tag{6.3}$$

where

$$\delta_m(u) = \frac{1}{3(m+1)^2}[3\tau(u)^2 + 3(m-1)\tau(u) + 1].$$

PROOF. Assume that $\alpha = 1$. Then, for $g \in \mathrm{Lip}_{\mathcal{H}}(\alpha;1)$ and $u \in (0,\infty)$, we have

$$\begin{aligned}|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| &\leq \mathcal{K}^*_{m,\tau}(|g(y) - g(u)|;u) \\ &\leq \mathcal{H}\mathcal{K}^*_{m,\tau}(|\tau(y) - g(u)|;u).\end{aligned}$$

By applying the Cauchy Schwartz inequality, we get

$$\begin{aligned}|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| &\leq \mathcal{H}[\mathcal{K}^*_{m,\tau}((\tau(y) - \tau(u))^2;u)]^{1/2} \\ &\leq \mathcal{H}\sqrt{\delta_m(u)}.\end{aligned}$$

Let us assume that $\alpha \in (0,1)$. Then, for $g \in \mathrm{Lip}_{\mathcal{H}}(\alpha;1)$ and $u \in (0,\infty)$, we have

$$\begin{aligned}|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| &\leq \mathcal{K}^*_{m,\tau}(|g(y) - g(u)|;u) \\ &\leq \mathcal{H}\mathcal{K}^*_{m,\tau}(|\tau(y) - g(u)|^\alpha;u).\end{aligned}$$

By taking $p = 1/\alpha$ and $q = 1/(1-\alpha)$, $g \in \mathrm{Lip}_{\mathcal{H}}(\tau(u);\alpha)$, and applying Hölder's inequality we have

$$|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \le \mathcal{H}[\mathcal{K}^*_{m,\tau}(|\tau(y)-\tau(u)|;u)]^{\alpha}.$$

Finally, by applying the Cauchy–Schwartz inequality, we get

$$|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \le \mathcal{H}(\delta_m(u))^{\alpha/2}. \qquad \square$$

THEOREM 6.3. *Let τ be a function satisfying the conditions (τ_1), (τ_2) and $\mathcal{Y}$ be a bounded subset of $[0,\infty)$. Then for any $g \in \mathrm{Lip}_{\mathcal{H}}(\tau(u);\alpha)$, $0 < \alpha \le 1$ on $\mathcal{Y}$ $\alpha \in (0,1]$, we have*

$$|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \le \mathcal{H}_{\alpha,g}\{(\delta_m(u))^{\alpha/2} + 2[\tau'(u)]^{\alpha} d^{\alpha}(u,\mathcal{Y})\}, \quad u \in [0,\infty),\ m \in \mathbb{N},$$

where $d(u,\mathcal{Y}) = \inf\{\|u-z\| : z \in \mathcal{Y}\}$ and $\mathcal{M}_{\alpha,g}$ is a constant depending on α and g, where

$$\delta_m(u) = \frac{1}{3(m+1)^2}[3\tau(u)^2 + 3(m-1)\tau(u) + 1].$$

PROOF. Let $\bar{\mathcal{Y}}$ be the closure of $\mathcal{Y}$ in $[0,\infty)$. Then, there exists a point $u_0 \in \bar{\mathcal{Y}}$ such that $d(u,\mathcal{Y}) = |u-u_0|$.

Using the monotonicity of $\mathcal{K}^*_{m,\tau}$ and the hypothesis of g, we obtain

$$\begin{aligned}|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| &\le \mathcal{K}^*_{m,\tau}(|g(y)-g(u_0)|;u) + \mathcal{K}^*_{m,\tau}(|g(u)-g(u_0)|;u)\\ &\le \mathcal{M}_{\alpha,g}\{\mathcal{K}^*_{m,\tau}(|\tau(y)-\tau(u_0)|^{\alpha};u) + |\tau(u)-\tau(u_0)|^{\alpha}\}\\ &\le \mathcal{H}_{\alpha,g}\{\mathcal{K}^*_{m,\tau}(|\tau(y)-\tau(u)|^{\alpha};u) + 2|\tau(u)-\tau(u_0)|^{\alpha}\}.\end{aligned}$$

By using Hölder's inequality for $p = 2/\alpha$ and $q = 2/(2-\alpha)$, as well as the fact $|\tau(u)-\tau(u_0)| = \tau'(u)|\tau(u)-\tau(u_0)|$ in the last inequality we get

$$|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \le \mathcal{H}_{\alpha,g}\big\{[\mathcal{K}^*_{m,\tau}((\tau(y)-\tau(u))^2;u)]^{\frac{1}{2}} + 2[\tau'(u)|\tau(u)-\tau(u_0)|]^{\alpha}\big\}.$$

Hence, by Corollary 2.1 we get the proof. $\square$

Now, we recall the local approximation given in **[13]** for $g \in \mathcal{C}_B[0,\infty)$, given as

$$\tilde{\omega}^{\tau}_{\alpha}(g;u) = \sup_{y\ne u, y\in(0,\infty)} \frac{|g(y)-g(u)|}{|y-u|^{\alpha}}, \quad u \in [0,\infty) \text{ and } \alpha \in (0,1]. \tag{6.4}$$

Then we get the following result:

THEOREM 6.4. *Let $g \in \mathcal{C}_B[0,\infty)$ and $\alpha \in (0,1]$. Then, for all $u \in [0,\infty)$, we have*

$$|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \le \tilde{\omega}^{\tau}_{\alpha}(g;u)(\delta_m(u))^{\alpha/2},$$

where

$$\delta_m(u) = \frac{1}{3(m+1)^2}[3\tau(u)^2 + 3(m-1)\tau(u) + 1]$$

PROOF. We know that $|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \le \mathcal{K}^*_{m,\tau}(|g(y)-g(u)|;u)$. From equation (6.4), we have

$$|\mathcal{K}^*_{m,\tau}(g;u) - g(u)| \le \tilde{\omega}^{\tau}_{\alpha}(g;u)\mathcal{K}^*_{m,\tau}(|\tau(y)-\tau(u)|^{\alpha};u).$$

By applying Hölder's inequality with $p = 2/\alpha$ and $q = 2/(2-\alpha)$, we have

$$\begin{aligned} |\mathcal{K}^*_{m,\tau}(g;u) - g(u)| &\leq \tilde{\omega}^{\tau}_{\alpha}(g;u)[\mathcal{K}^*_{m,\tau}((\tau(y)-\tau(u))^2;u)]^{\alpha/2} \\ &\leq \tilde{\omega}^{\tau}_{\alpha}(g;u)(\delta_m(u))^{\alpha}/2. \end{aligned}$$

which proves the desired result □

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