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APPROXIMATION FOR GUPTA TYPE GENERAL OPERATORS BASED ON MIHEŞAN BASIS FUNCTIONS

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Abstract. Recently Indian known mathematician V. Gupta introduced a general family of linear positive operators, which produces a large number of well-known linear positive operators as particular cases. The family of operators, proposed by Gupta provides a unified approach. This motivated us to extend the studies and we establish some convergence estimates which include an asymptotic formula and the rate of convergence for the function having derivatives of bounded variation.

1. Introduction

In the year 2008, Miheşan [**19**] introduced a sequence of linear positive operators based on some exponential type operators including non-exponential operators due to Lupaş. These discretely defined operators are not possible to approximate the integrable functions. In this direction Gupta and collaborators [**2**,**11**,**12**,**14**,**15**] proposed several hybrid operators of Durrmeyer type and established many interesting results on convergence. Also, some more important contribution, we refer to the readers, are [**3**, **8**, **13**, **17**, **20**, **21**] etc. Very recently professor Vijay Gupta [**10**] introduced a generalized sequence of linear positive operators, based on Miheşan basis functions, which for $x \in [0, \infty)$, is defined as follows

$$P_{n,\alpha,\beta}(f,x) = \frac{n(\beta-1)}{\beta} \sum_{k=0}^{\infty} m_{n,k}^{\alpha}(x) \int_0^{\infty} m_{n,k}^{\beta}(t) f(t)\, dt \tag{1.1}$$

where

$$m_{n,k}^{\alpha}(x) = \frac{(\alpha)_k}{k!} \frac{\alpha^{\alpha}(nx)^k}{(\alpha+nx)^{\alpha+k}}.$$

Considering different values of α and β, there arise several cases as:

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(1) If $\alpha=\beta=n/c$, $c\in N_0$, we get the well known Heilmann–Müller operators [**16**].
(2) If $\alpha=\beta=n$, we get modified Baskakov operators [**6**].
(3) If $\alpha=\beta=\infty$, we get Szász Durrmeyer operators [**18**].
(4) If $\alpha=\beta=-n$, we get Bernstein–Durrmeyer polynomials [**5**], in the case $x\in[0,1]$ and summation is for $0\leqslant k\leqslant n$.
(5) If $\alpha\neq\beta$ and $\alpha=n$, $\beta=\infty$, we get Baskakov–Szász operators [**7**].
(6) If $\alpha\neq\beta$ and $\alpha=\infty$, $\beta=n$, we get Szász–Baskakov operators [**22**].
(7) If $\alpha\neq\beta$ and $\alpha=nx$, $\beta=n$, we get Lupaş–Baskakov operators [**9**] for the case $d=1$, $c=1$.
(8) If $\alpha\neq\beta$ and $\alpha=nx$, $\beta=\infty$, we get Lupaş–Szász operators [**9**] for the case $d=0$, $c=1$.
(9) If $\alpha\neq\beta$ and $\alpha=nx$, $\beta=nt$, we get Lupaş–Durrmeyer operators [**1, 14**].

The immense properties of operators (1.1) motivate us to establish some convergence estimates for these operators. Here we establish an asymptotic formula and the rate of convergence for these operators while the function have derivatives of bounded variation.

2. Auxiliary results

In this section by simple calculations, we get the boundedness and the estimates for the moments of the operators.

Lemma 2.1. [**10**] *The r^{th}, $r\in N$ order moment with $e_r(t)=t^r$, except above case* (9)*, can be represented as*

$$\mu_{n,\alpha,\beta}(e_r,x)=\frac{\Gamma(\beta-r-1)\Gamma(r+1)}{\Gamma(\beta-1)}\Big(\frac{\beta}{n}\Big)^r{}_2F_1\Big(\alpha,-r;1;\frac{-nx}{\alpha}\Big).$$

Remark 2.1. From above we find that

$$\mu_{n,\alpha,\beta}(e_0,x)=1,\quad \mu_{n,\alpha,\beta}(e_1,x)=\frac{\beta(1+nx)}{n(\beta-2)},$$
$$\mu_{n,\alpha,\beta}(e_2,x)=\frac{\beta^2[2\alpha+4\alpha nx+(\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)}.$$

Remark 2.2. Few central moments are given by

$$P_{n,\alpha,\beta}(e_1-xe_0,x)=\frac{\beta+2nx}{n(\beta-2)},$$
$$P_{n,\alpha,\beta}((e_1-xe_0)^2,x)=\frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2\alpha\beta n[x(\beta+3)+\beta]}{\alpha^2n(\beta-2)(\beta-3)}.$$

Here it is observed that in the definition of the operators the convergence takes place when $\alpha=\alpha(n)$, $\beta=\beta(n)$ or they are constants as indicated in the above cases. Consequently for the above cases $P_{n,\alpha,\beta}((e_1-xe_0)^m,x)=O(n^{-[(m+1)/2]})$.

Corollary 2.1. *Using Cauchy–Schwarz inequality here we can have*

$$P_{n,\alpha,\beta}(|e_1-xe_0|^r,x)\leqslant\sqrt{P_{n,\alpha,\beta}((e_1-xe_0)^{2r},x)}=O(n^{-r/2}).$$

From here

$$P_{n,\alpha,\beta}(|e_1 - xe_0|, x) \leqslant \sqrt{P_{n,\alpha,\beta}((e_1 - xe_0)^2, x)}.$$

DEFINITION 2.1. If $\delta(t)$ is the Dirac delta function, the kernel $K_n^{\alpha,\beta}(x,t)$ of our operators is

$$K_n^{\alpha,\beta}(x,t) = n\sum_{k=1}^{\infty} m_{n,k}^{\alpha}(x) m_{n,k-1}^{\beta-1}(t) + m_{n,0}^{\alpha}(x)\delta(t).$$

LEMMA 2.2. *For fixed $x \geqslant 0$ and n sufficiently large, we have*

$$\begin{aligned}\zeta_n^{\alpha,\beta}(x,y) &= \int_0^y K_n^{\alpha,\beta}(x,t)\,dt, \quad 0 < y < x \\ &\leqslant \frac{\beta^2[2\alpha + 4\alpha n x + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \cdot \frac{1}{(x-y)^2}, \\ 1 - \zeta_n^{\alpha,\beta}(x,z) &= \int_z^{\infty} K_n^{\alpha,\beta}(x,t)\,dt, \quad x < z < \infty \\ &\leqslant \frac{\beta^2[2\alpha + 4\alpha n x + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \cdot \frac{1}{(z-x)^2}.\end{aligned}$$

The proof of this Lemma follows by using Remark 2.2.

3. Direct results

THEOREM 3.1. *Let f be a bounded and integrable function on $[0,\infty)$ such that the second derivative of f exists at a fixed point $x \in [0,\infty)$. Then we have*

$$\lim_{n\to\infty}[P_{n,\alpha,\beta}(f,x) - f(x)] = A(x)f'(x) + \frac{x[B(x)+C]}{2}f''(x),$$

where $A(x), B(x)$ are functions of x, and C is a constant.

PROOF. The Taylor expansion of the function f is

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + R(t,x)(t-x)^2, \tag{3.1}$$

where $\lim_{t\to x} R(t,x) = 0$. Operating with $P_{n,\alpha,\beta}$ on both sides in (3.1), we have

$$\begin{aligned}P_{n,\alpha,\beta}(f,x) - f(x) &= P_{n,\alpha,\beta}(t-x,x)f'(x) \\ &\quad + P_{n,\alpha,\beta}((t-x)^2,x)\frac{f''(x)}{2} + P_{n,\alpha,\beta}(R(t,x)(t-x)^2,x).\end{aligned} \tag{3.2}$$

From Corollary 2.1, we see that

$$P_{n,\alpha,\beta}(R(t,x)(t-x)^2,x) \leqslant \sqrt{P_{n,\alpha,\beta}(R^2(t,x),x)}\sqrt{P_{n,\alpha,\beta}(t-x)^4,x)}.$$

Therefore

$$\lim_{t\to x} R^2(x,x) = 0. \tag{3.3}$$

Since for n sufficiently large, we have $t \to x$ and therefore in view of Remark 2.2 and (3.3) we have $\lim_{n\to\infty} P_{n,\alpha,\beta}(R(t,x)(t-x)^2,x) = 0$. Gathering (3.1), (3.2) and (3.3), we get

$$\begin{aligned}
&\lim_{n\to\infty} n[P_{n,\alpha,\beta}(f,x) - f(x)] \\
&\quad = \lim_{n\to\infty} n[P_{n,\alpha,\beta}(e_1 - xe_0,x)f'(x) + \frac{1}{2}P_{n,\alpha,\beta}((e_1-xe_0)^2,x)f''(x)] \\
&= \lim_{n\to\infty}\Big[\frac{\beta+2nx}{\beta-2}f'(x) + \frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2n\alpha\beta x(\beta+3)+2\alpha\beta^2}{2\alpha^2(\beta-2)(\beta-3)}f''(x)\Big] \\
&\qquad = A(x)f'(x) + \frac{x(B(x)+C)}{2}f''(x).
\end{aligned}$$

This completes the proof of the general Voronovskaja type asymptotic formula. □

Corollary 3.1. *Under the assumption of Theorem 3.1, The conclusions of the asymptotic formulae show cases as following*

(1) *For Heilmann–Müller operators* [**16**]

$$\lim_{n\to\infty} n[P_{n,\alpha,\beta}(f,x) - f(x)]_{\alpha=\beta=\frac{c}{n}} = (1+2cx)f'(x) + x(cx+1)f''(x).$$

(2) *For modified Baskakov operators* [**6**]

$$\lim_{n\to\infty} n[P_{n,\alpha,\beta}(f,x) - f(x)]_{\alpha=\beta=n} = (1+2x)f'(x) + x(x+1)f''(x).$$

(3) *For Szász Durrmeyer operators* [**18**]

$$\lim_{n\to\infty} n[P_{n,\alpha,\beta}(f,x) - f(x)]_{\alpha=\beta=\infty} = f'(x).$$

(4) *For Bernstein–Durrmeyer polynomials* [**5**], *in the case* $x \in [0,1]$ *and summation is for* $0 \leqslant k \leqslant n$

$$\lim_{n\to\infty} n[P_{n,\alpha,\beta}(f,x) - f(x)]_{\alpha=\beta=-n} = (2x-1)f'(x) + x(1-x)f''(x).$$

(5) *For Baskakov**Szász operators* [**7**]

$$\lim_{n\to\infty} n[P_{n,\alpha,\beta}(f,x) - f(x)]_{\alpha=n,\beta=\infty} = f'(x) + \frac{x(x+2)}{2}f''(x).$$

(6) *For Szász**Baskakov operators* [**22**]

$$\lim_{n\to\infty} n[P_{n,\alpha,\beta}(f,x) - f(x)]_{\alpha=\infty,\beta=n} = (1+2x)f'(x) + \frac{x(x+2)}{2}f''(x).$$

(7) *For Lupaş–Baskakov operators* [**9**] *for the case* $d=1$, $c=1$

$$\lim_{n\to\infty} n[P_{n,\alpha,\beta}(f,x) - f(x)]_{\alpha=nx,\beta=n} = (1+2x)f'(x) + \frac{x(x+3)}{2}f''(x).$$

(8) *For Lupaş–Szász operators* [**9**] *for the case* $d=0$, $c=1$

$$\lim_{n\to\infty} n[P_{n,\alpha,\beta}(f,x) - f(x)]_{\alpha=nx,\beta=\infty} = f'(x) + \frac{3x}{2}f''(x).$$

We shall now assert the estimates for approximating the degree of the operators $\hat{K}_n^{(\alpha)}$ towards the identity operator in terms of modulus of continuity.

The first order modulus of continuity satisfies

$$|f(t)-f(x)| \leqslant \omega_1(f,|t-x|) \leqslant \Big(1+\frac{1}{\eta}|t-x|\Big)\omega_1(f,\eta). \tag{3.4}$$

THEOREM 3.2. *Let f be a bounded and continuous function on $[0,\infty)$ and $x \in [0,\infty)$. Then we have $|P_{n,\alpha,\beta}(f,x)-f(x)| \leqslant 2\omega_1(f,\eta)$, where*

$$\eta^2 = \frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2\alpha\beta n[x(\beta+3)+\beta]}{\alpha^2 n(\beta-2)(\beta-3)}.$$

PROOF. Using (3.4), we have

$$\begin{aligned}|P_{n,\alpha,\beta}(f,x)-f(x)| &\leqslant \int_0^\infty K_n^{\alpha,\beta}(x,t)|f(t)-f(x)|dt\\ &\leqslant \omega_1(f,\eta)\bigg(1+\frac{1}{\eta}\int_0^\infty K_n^{\alpha,\beta}(x,t)|t-x|dt\bigg).\end{aligned}$$

Applying Holder's inequality, we get

$$|P_{n,\alpha,\beta}(f,x)-f(x)| \leqslant \omega_1(f,\eta)\Big[1+\frac{1}{\eta}P_{n,\alpha,\beta}((e_1-xe_0)^2,x)\Big] = 2\omega_1(f,\eta). \qquad \square$$

THEOREM 3.3. *For $f \in C_B[0,\infty)$, the class of bounded and continuous functions on $[0,\infty)$, there exists a constant $C>0$, such that*

$$\begin{aligned}|P_{n,\alpha,\beta}(f,x)-f(x)| &\leqslant \omega\Big(f,\frac{\beta+2nx}{n(\beta-2)}\Big)\\ &+ C\omega_2\Bigg(f,\sqrt{\Big(\frac{\beta+2nx}{n(\beta-2)}\Big)^2+\frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2\alpha\beta n[x(\beta+3)+\beta]}{\alpha^2 n(\beta-2)(\beta-3)}}\Bigg).\end{aligned}$$

PROOF. We define $(\tilde{K}_n^j f)(x) : C_B[0,\infty) \to C_B[0,\infty)$ an auxiliary operator by

$$\tilde{P}_{n,\alpha,\beta}(g,x) = P_{n,\alpha,\beta}(g,x)+g(x)-g\Big(\frac{\beta(1+nx)}{n(\beta-2)}\Big). \tag{3.5}$$

The above modified form $\tilde{P}_{n,\alpha,\beta}$ preserve linear and constant functions. By using Taylor's theorem, we have for $g \in C_B^2[0,\infty)$

$$g(t) = g(x)+(t-x)g'(x)+\int_x^t (t-u)g''(u)du, \quad x,t \in [0,\infty).$$

Applying the operators $\tilde{P}_{n,\alpha,\beta}$ to above Taylor's expansion, we get

$$\begin{aligned}|\tilde{P}_{n,\alpha,\beta}(g,x)-g(x)| &= \bigg|\tilde{P}_{n,\alpha,\beta}\bigg(\int_x^t (t-u)g''(u)du,x\bigg)\bigg|\\ &\leqslant \bigg|P_{n,\alpha,\beta}\bigg(\int_x^t (t-u)g''(u)du,x\bigg)\bigg|\end{aligned} \tag{3.6}$$

$$+\left|\int_x^{\frac{\beta(1+nx)}{n(\beta-2)}}\Big(\frac{\beta(1+nx)}{n(\beta-2)}-u\Big)g''(u)du\right|.$$

Further, by Remark 2.2, we have

$$\begin{aligned}\Big|P_{n,\alpha,\beta}\Big(\int_x^t (t-u)g''(u)du, x\Big)\Big| \\ &\leqslant P_{n,\alpha,\beta}\Big(\int_x^t |t-u||g''(u)|du, x\Big)\\ &\leqslant \|g''\|P_{n,\alpha,\beta}((t-x)^2,x)\\ &= \|g''\|\frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2\alpha\beta n[x(\beta+3)+\beta]}{\alpha^2 n(\beta-2)(\beta-3)},\end{aligned}$$

and

$$\left|\int_x^{\frac{\beta(1+nx)}{n(\beta-2)}}\Big(\frac{\beta(1+nx)}{n(\beta-2)}-u\Big)g''(u)du\right| \leqslant \|g''\|\Big(\frac{\beta+2nx}{n(\beta-2)}\Big)^2.$$

By the above expressions (3.6) becomes

$$|\tilde{P}_{n,\alpha,\beta}(g,x)-g(x)| \leqslant \|g''\|\Big\{\Big(\frac{\beta+2nx}{n(\beta-2)}\Big)^2 + \frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2\alpha\beta n[x(\beta+3)+\beta]}{\alpha^2 n(\beta-2)(\beta-3)}\Big\}. \tag{3.7}$$

Also, we have

$$\|\tilde{P}_{n,\alpha,\beta}(f,x)\| \leqslant 3\|f\|. \tag{3.8}$$

By using (3.5), (3.7) and (3.8), for each $g \in C_B^2[0,\infty)$ and in the last step using the connection between K-functional and modulus of continuity (see [**4**]), we have

$$\begin{aligned}|P_{n,\alpha,\beta}(f,x)-f(x)| &= |\tilde{P}_{n,\alpha,\beta}(f-g,x)-(f-g)(x)| + \Big|f\Big(\frac{\beta(1+nx)}{n(\beta-2)}\Big)-f(x)\Big|\\ &\quad + |(\tilde{P}_{n,\alpha,\beta}(g,x)-g(x)|\\ &\leqslant 4\|f-g\| + \Big|f\Big(\frac{\beta(1+nx)}{n(\beta-2)}\Big)-f(x)\Big|\\ &\quad + \Big(\Big(\frac{\beta+2nx}{n(\beta-2)}\Big)^2 + \frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2\alpha\beta n[x(\beta+3)+\beta}{\alpha^2 n(\beta-2)(\beta-3)}\Big)\|g''\|\\ &\leqslant \omega\Big(f,\frac{\beta+2nx}{n(\beta-2)}\Big)\\ &\quad + K_2\Big(f,\Big(\frac{\beta+2nx}{n(\beta-2)}\Big)^2 + \frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2\alpha\beta n[x(\beta+3)+\beta]}{\alpha^2 n(\beta-2)(\beta-3)}\Big)\\ &\leqslant \omega\Big(f,\frac{\beta+2nx}{n(\beta-2)}\Big)\\ &\quad + C\omega_2\Big(f,\sqrt{\Big(\frac{\beta+2nx}{n(\beta-2)}\Big)^2 + \frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2\alpha\beta n[x(\beta+3)+\beta]}{\alpha^2 n(\beta-2)(\beta-3)}}\Big) \quad \square\end{aligned}$$

DEFINITION 3.1. We describe the class BD_φ of entirely continuous function f having a derivative of bounded variation on the interval $[0,\infty)$ as

$$BD_\varphi = \Big\{ f : f(x) = f(c) + \int_c^x \varphi(t)\,dt \Big\}$$

for all $t \to \infty$ and $|f(t)| \leqslant Mt^\gamma$, $\gamma \geqslant 0$.

The following result is the rate of convergence for functions with derivatives of bounded variation.

THEOREM 3.4. *L Let $f \in BD_\varphi, f(t) = O(t^\gamma)$ for all $t \in [0,\infty)$ with the condition that $\alpha \to \infty$ as $n \to \infty$ yields $\lim_{n\to\infty} n/\alpha = l$ and $\beta \to \infty$ as $n \to \infty$ yields $\lim_{n\to\infty} n/\beta = m$ then for absolutely large n, we have*

$$\begin{aligned}|P_{n,\alpha,\beta}(f,x) - f(x)| \leqslant &\Big\{ \frac{|\varphi(x+) + \beta\varphi(x-)|}{\beta+1} + \frac{\beta|\varphi(x+) - \varphi(x-)|}{\beta+1} \Big\} \\ &\times \sqrt{\frac{n^2x^2(\beta^2+\alpha\beta+6\alpha) + 2n\alpha\beta x(\beta+3) + 2\alpha\beta^2}{\alpha^2 n(\beta-2)(\beta-3)}} \\ &+ \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} \varphi_x + \frac{\beta^2[2\alpha + 4\alpha n x + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \sum_{i=1}^{\sqrt{n}} \bigvee_{x-x/\sqrt{i}}^{x+x/\sqrt{i}} \varphi_x \\ &+ \frac{\beta^2[2\alpha + 4\alpha n x + (\alpha+1)n^2x^2]}{\alpha n^2x^2(\beta-2)(\beta-3)} \Big\{ |f(2x) - f(x) - x\varphi(x+)| \\ &\qquad + \frac{|f(x)|}{x^2} + |\varphi(x+)| \Big\} + M2^\gamma O(n^{-\gamma/2}).\end{aligned}$$

PROOF. By Definition 2.1 of operators (1.1), we have

$$\begin{aligned}P_{n,\alpha,\beta}(f,x) - f(x) &= \int_0^\infty K_n^{\alpha,\beta}(x,t)[f(t) - f(x)]dt \\ &= \int_0^\infty K_n^{\alpha,\beta}(x,t)\Big(\int_x^t \varphi(u)du \Big) dt\end{aligned} \tag{3.9}$$

For $\varphi \in BD_\varphi$, using (3.9) and applying the identity

$$\begin{aligned}\varphi(u) = \varphi_x(u) &+ \frac{\varphi(x+) + \beta\varphi(x-)}{\beta+1} \\ &+ \frac{\varphi(x+) - \varphi(x-)}{2}\Big(\operatorname{sgn}(u-x) + \frac{\beta-1}{\beta+1} \Big) \\ &+ \Big(\varphi(u) - \frac{\varphi(x+) - \beta\varphi(x-)}{2} \Big)\chi_x(u),\end{aligned} \tag{3.10}$$

where

$$\varphi_x(t) = \begin{cases} \varphi(t) - \varphi(x-), & 0 \leqslant t < x; \\ 0, & t = x. \\ \varphi(t) - \varphi(x+), & x < t < \infty \end{cases} \quad \text{and} \quad \chi_x(u) = \begin{cases} 1, & u = x \\ 0, & u \neq x. \end{cases}$$

From (3.9) and (3.10), we have

$$P_{n,\alpha,\beta}(f,x)-f(x) = -\,J_1^{n,\alpha,\beta}(\varphi_x,x)+J_2^{n,\alpha,\beta}(\varphi_x,x) + J_3^{n,\alpha,\beta}(\varphi_x,x)+J_4^{n,\alpha,\beta}+J_5^{n,\alpha,\beta}+J_6^{n,\alpha,\beta},$$

where

$$J_1^{n,\alpha,\beta}(\varphi_x,x) = \int_0^x K_n^{\alpha,\beta}(x,t)\bigg(\int_t^x \varphi_x(u)du\bigg)dt$$
$$J_2^{n,\alpha,\beta}(\varphi_x,x) = \int_x^{2x} K_n^{\alpha,\beta}(x,t)\bigg(\int_x^t \varphi_x(u)du\bigg)dt$$
$$J_3^{n,\alpha,\beta}(\varphi_x,x) = \int_{2x}^{\infty} K_n^{\alpha,\beta}(x,t)\bigg(\int_x^t \varphi_x(u)du\bigg)dt$$
$$J_4^{n,\alpha,\beta} = \int_0^{\infty} K_n^{\alpha,\beta}(x,t)\bigg(\int_x^t \frac{\varphi(x+)+\beta\varphi(x-)}{\beta+1}du\bigg)dt$$
$$J_5^{n,\alpha,\beta} = \int_0^{\infty} K_n^{\alpha,\beta}(x,t)\int_x^t \frac{\varphi(x+)-\varphi(x-)}{2}\bigg(\operatorname{sgn}(u-x)+\frac{\beta-1}{\beta+1}\bigg)du\,dt$$
$$J_6^{n,\alpha,\beta} = \int_0^{\infty} K_n^{\alpha,\beta}(x,t)\int_x^t \varphi(u)-\frac{\varphi(x+)-\beta\varphi(x-)}{2}\bigg)\chi_x(u)\,du\,dt.$$

From Lemma 2.2, we obtain

$$\begin{aligned} J_4^{n,\alpha,\beta} &= \frac{\varphi(x+)+\beta\varphi(x-)}{\beta+1}\int_0^{\infty}(t-x)K_n^{\alpha,\beta}(x,t)\,dt \\ &= \frac{\varphi(x+)+\beta\varphi(x-)}{\beta+1}V_{n,\alpha,\beta}(e_1-xe_0,x) \\ &= \frac{\varphi(x+)+\beta\varphi(x-)}{\beta+1}\sqrt{\frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2n\alpha\beta x(\beta+3)+2\alpha\beta^2}{\alpha^2 n(\beta-2)(\beta-3)}}\,. \end{aligned}$$

From Lemma 2.2 and Corollary 2.1 for sufficiently large n, we have

$$\begin{aligned} J_5^{n,\alpha,\beta} &= \frac{\varphi(x+)-\varphi(x-)}{2}\bigg[-\int_0^x K_n^{\alpha,\beta}(x,t)\int_t^x \Big(\operatorname{sgn}(u-x)+\frac{\beta-1}{\beta+1}\Big)du\,dt \\ &\qquad -\int_x^{\infty} K_n^{\alpha,\beta}(x,t)\bigg(\int_x^t \Big(\operatorname{sgn}(u-x)+\frac{\beta-1}{\beta+1}\Big)du\bigg)dt\bigg] \\ &\leqslant \frac{\beta}{\beta+1}|\varphi(x+)-\varphi(x-)|V_{n,\alpha,\beta}(|e_1-xe_0|,x) \\ &= \frac{\beta|\varphi(x+)-\varphi(x-)|}{\beta+1}\sqrt{\frac{n^2x^2(\beta^2+\alpha\beta+6\alpha)+2n\alpha\beta x(\beta+3)+2\alpha\beta^2}{\alpha^2 n(\beta-2)(\beta-3)}}\,. \end{aligned}$$

$J_6^{n,\alpha,\beta}$ is obviously zero by definition of $\chi_x(u)$. Now by the Stieltjes integral and integration by part, for $y=x-x/\sqrt{n}$, we have

$$J_1^{n,\alpha,\beta}(\varphi_x,x) = \int_0^x \bigg(\int_t^x \varphi_x(u)du\bigg)d_t\zeta_n^{\alpha,\beta}(x,t)\,dt$$

$$= \int_0^x \varphi_x(t)\zeta_n^{\alpha,\beta}(x,t)\,dt$$
$$= \int_0^{x-x/\sqrt{n}} \varphi_x(t)\zeta_n^{\alpha,\beta}(x,t)\,dt + \int_{x-x/\sqrt{n}}^x \varphi_x(t)\zeta_n^{\alpha,\beta}(x,t)\,dt$$

Since $\zeta_n^{\alpha,\beta}(x,t) \leqslant 1$ and $\varphi_x(x) = 0$, we get

$$\left|\int_{x-x/\sqrt{n}}^x (\varphi_x(t) - \varphi_x(x))\zeta_n^{\alpha,\beta}(x,t)\,dt\right| \leqslant \int_{x-x/\sqrt{n}}^x \bigvee_t^x \varphi_x dt \leqslant \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^x \varphi_x,$$

where$\bigvee_a^b \varphi_x$ is the total variation of φ_x on $[a,b]$. Again from Lemma 2.2 and taking $h = \frac{x}{x-t}$

$$\left|\int_0^{x-x/\sqrt{n}} \varphi_x(t)\zeta_n^{\alpha,\beta}(x,t)\,dt\right| \leqslant \frac{\beta^2[2\alpha + 4\alpha nx + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \int_0^{x-x/\sqrt{n}} \bigvee_t^x \varphi_x \frac{1}{(x-t)^2}\,dt$$
$$= \frac{\beta^2[2\alpha + 4\alpha nx + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \int_1^{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^x \varphi_x\,dh$$
$$\leqslant \frac{\beta^2[2\alpha + 4\alpha nx + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \sum_{i=1}^{\sqrt{n}} \bigvee_{x-x/\sqrt{i}}^x \varphi_x.$$

Thus we have

$$\left|J_1^{n,\alpha,\beta}(\varphi_x, x)\right| \leqslant \frac{\beta^2[2\alpha + 4\alpha nx + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \sum_{i=1}^{\sqrt{n}} \bigvee_{x-x/\sqrt{i}}^x \varphi_x + \frac{x}{\sqrt{n}} \bigvee_{x-x/\sqrt{n}}^x \varphi_x.$$

In the similar way we find the estimate $J_2^{n,\alpha,\beta}(\varphi_x, x)$, using Lemma 2.2

$$|J_2^{n,\alpha,\beta}(\varphi_x, x)| = \int_x^{2x} \left(\int_t^x \varphi_x(u)du\right) d_t\zeta_n^{\alpha,\beta}(x,t)\,dt$$
$$= -\int_x^{2x} \left(\int_t^x \varphi_x(u)du\right) d_t(1 - \zeta_n^{\alpha,\beta}(x,t))\,dt$$
$$= -\int_x^{2x} \varphi_x(u)du(1 - \zeta_n^{\alpha,\beta}(x,2x))$$
$$+ \int_x^{2x} \varphi_x(t)(1 - \zeta_n^{\alpha,\beta}(x,2x))\,dt := J_2^1 + J_2^2$$

From Lemma 2.1

$$J_2^1 \leqslant \left|\int_x^{2x} \varphi_x(u)du\right| |1 - \zeta_n^{\alpha,\beta}(x,2x)|$$
$$\leqslant \frac{\beta^2[2\alpha + 4\alpha nx + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \left|\int_x^{2x} [\varphi(u) - \varphi(x+)]du\right|$$
$$= \frac{\beta^2[2\alpha + 4\alpha nx + (\alpha+1)n^2x^2]}{\alpha n^2x^2(\beta-2)(\beta-3)} [f(2x) - f(x) - x\varphi(x+)]$$

and

$$J_2^2 \leqslant \left| \int_x^{x+x/\sqrt{n}} \varphi_x(t)\, dt \right| + \frac{\beta^2[2\alpha + 4\alpha nx + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \left| \int_{x+x/\sqrt{n}}^{2x} \frac{\varphi_x(t)}{(t-x)^2} dt \right|$$
$$= \frac{x}{\sqrt{n}} \bigvee_x^{x+x/\sqrt{n}} \varphi_x + \frac{\beta^2[2\alpha + 4\alpha nx + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \sum_{i=1}^{\sqrt{n}} \bigvee_x^{x+x/\sqrt{i}} \varphi_x.$$

Next to estimate $J_3^{n,\alpha,\beta}(\varphi_x, x)$, we use $f(t) = O(t^\gamma)$ as $t \to \infty$. Then for some positive constant C depending on f, x and r, we obtain

$$\begin{aligned}
|J_3^{n,\alpha,\beta}(\varphi_x, x)| &= \left| \int_{2x}^{\infty} \left(\int_x^t (\varphi(t) - \varphi(x+)du \right) K_n^{\alpha,\beta}(x,t)\, dt \right| \\
&\leqslant \left| \int_{2x}^{\infty} \left(\int_x^t \varphi(t)du \right) K_n^{\alpha,\beta}(x,t)\, dt \right| + |\varphi(x+)| \left| \int_{2x}^{\infty} K_n^{\alpha,\beta}(x,t)\, dt \right| \\
&= \left| \int_{2x}^{\infty} [f(t) - f(x)] K_n^{\alpha,\beta}(x,t)\, dt \right| + |\varphi(x+)| \left| \int_{2x}^{\infty} K_n^{\alpha,\beta}(x,t)\, dt \right| \\
&\leqslant M \left| \int_{2x}^{\infty} [t^\gamma K_n^{\alpha,\beta}(x,t)\, dt \right| + |f(x)| \left| \int_{2x}^{\infty} K_n^{\alpha,\beta}(x,t)\, dt \right| \\
&\quad + |\varphi(x+)| \left| \int_{2x}^{\infty} K_n^{\alpha,\beta}(x,t)\, dt \right|
\end{aligned}$$

Using the inequality $t \leqslant 2(t-x)$ for $t > 2x$, we have

$$\begin{aligned}
|J_3^{n,\alpha,\beta}(\varphi_x, x)| &\leqslant M \left| \int_{2x}^{\infty} 2^\gamma (t-x)^2 K_n^{\alpha,\beta}(x,t)\, dt \right| \\
&\quad + \frac{|f(x)|}{x^2} \left| \int_{2x}^{\infty} (t-x)^2 K_n^{\alpha,\beta}(x,t)\, dt \right| + |\varphi(x+)| \left| \int_{2x}^{\infty} K_n^{\alpha,\beta}(x,t)\, dt \right|
\end{aligned}$$

Using Corollary 2.1 and Lemma 2.1, we get

$$\begin{aligned}
|J_3^{n,\alpha,\beta}(\varphi_x, x)| &\leqslant M 2^\gamma O(n^{-\gamma/2}) + \frac{|f(x)|}{x^2} \frac{\beta^2[2\alpha + 4\alpha nx + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)} \\
&\quad + |\varphi(x+)| \frac{\beta^2[2\alpha + 4\alpha nx + (\alpha+1)n^2x^2]}{\alpha n^2(\beta-2)(\beta-3)}.
\end{aligned}$$

Collecting all the estimates $J_i^{n,\alpha,\beta}(\varphi_x, x)$, $i = 1, 2, \dots 6$, we obtain the required result. □

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