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ON RAMANUJAN'S EISENSTEIN SERIES IDENTITIES

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Abstract. Ramanujan's work was inspiring and distinguishable from those of his predecessors as he wrote many theorems, but he proved very few. In this paper, we make use of P-Q theta function identities written by Ramanujan and we offer new proofs to Eisenstein series identities of level 14 developed by Cooper and Ye, where the proofs were presented using modular forms. Analogously, we derive certain new Eisenstein series identities of level 14.

1. Introduction

Let τ be a complex number satisfying $\mathrm{Im}(\tau) > 0$ and let $q = e^{2\pi i\tau}$. The Dedekind eta-function is defined by $\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$. Let $f(-q) = q^{-1/24}\eta(\tau)$ and $f_n = f(-q^n)$. Let $P(q)$ and $Q(q)$ denote Ramanujan's Eisenstein series, defined by

$$P(q) = 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \quad Q(q) = 1 + 240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}.$$

For any positive integer n, let P_n and Q_n be defined by $P_n = P(q^n)$ and $Q_n = Q(q^n)$. In [**12**], Ramanujan studied the function

$$f(l) = \frac{lP(q^\ell) - P(q)}{l-1},$$

where $l \geqslant 2$ is an integer, called the level. For more detailed study on theta functions and Eisenstein series one may refer to [**1, 10, 11, 14–16**]. Using the theory of modular forms, Cooper and Ye [**9**] have derived the following four identities of level 14 involving Ramanujan's Eisenstein series and Dedekind eta function identities

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and employed them to obtain new series for π:

$$zv = \frac{1}{72}(-P_1 + P_2 + 7P_7 - 7P_{14}) - \frac{1}{3}z, \tag{1.1}$$

$$\frac{z}{v} = \frac{1}{18}(P_1 - 4P_2 - 7P_7 + 28P_{14}) - \frac{8}{3}z, \tag{1.2}$$

$$zw = \frac{1}{144}(5P_1 - 26P_2 + 91P_7 - 70P_{14}) + \frac{5}{6}z, \tag{1.3}$$

$$\frac{z}{w} = \frac{1}{144}(-13P_1 + 10P_2 - 35P_7 + 182P_{14}) + \frac{5}{6}z, \tag{1.4}$$

where

$$z = qf_1f_2f_7f_{14},\ v = q\left(\frac{f_2f_{14}}{f_1f_7}\right)^3 \quad and \quad w = q\left(\frac{f_1f_{14}}{f_2f_7}\right)^4.$$

Recently Vasuki and Veeresha [**18**] obtained proofs of (1.1)–(1.4) using one of the Lambert series due to Bailey [**2**, p. 197, eq. 5], [**4**].

In this paper, using P-Q modular relation identities, we provide a quiet different proofs of four aforementioned identities. Further, we obtain the following new Eisenstein series identities:

$$\frac{1}{240}[Q_1 - Q_2 + 49Q_7 - 49Q_{14}] = z^2\Big(50v^2 + 48v + 13 + \frac{1}{v}\Big), \tag{1.5}$$

$$\frac{1}{30}[Q_1 - 4Q_2 + 49Q_7 - 196Q_{14}] = z^2\Big(8 - \frac{1}{v^2}\Big)(40v^2 + 32v + 5). \tag{1.6}$$

2. Preliminary results

In this section, we state some results which are useful to prove our results. As usual set, for any complex numbers a and q $(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n)$, $|q| < 1$. Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2}, \quad |ab| < 1.$$

By Jacobi's triple product identity [**5**, p. 35, entry 19], we have

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

The most important special cases of $f(a,b)$ are given by

$$\varphi(q) = f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty,$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

and

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

Following Ramanujan, we also define $\chi(q) = (-q;q^2)_\infty$. One can easily show that

$$\varphi(q) = \frac{f_2^5}{f_1^2 f_4^2}, \quad \varphi(-q) = \frac{f_1^2}{f_2}, \quad \psi(q) = \frac{f_2^2}{f_1}, \quad \chi(q) = \frac{f_2^2}{f_1 f_4}, \quad \chi(-q) = \frac{f_1}{f_2}. \tag{2.1}$$

THEOREM 2.1. [**8**, eq. 4.7] *We have*

$$-P_1 + 2P_2 = q^{1/4} f_1^2 f_2^2 \left(\frac{f_1^{12}}{q^{1/2} f_2^{12}} + 64 q^{1/2} \frac{f_2^{12}}{f_1^{12}} \right)^{1/2}.$$

PROOF. From [**18**], we have $-P_1 + 2P_2 = 2\varphi^4(q) - \varphi^4(-q)$. This identity can also be found in [**8**, eq. 4.7]. Using (2.1) in it, we see that

$$-P_1 + 2P_2 = f_1^2 f_2^2 \left\{ 2\frac{\chi^8(-q^2)}{\chi^{10}(-q)} - \chi^6(-q) \right\}. \tag{2.2}$$

From [**5**, p. 123, (11.1)], we have $\varphi^4(q) - \varphi^4(-q) = 16q\psi^4(q^2)$, or

$$\frac{\chi^8(-q^2)}{\chi^8(-q)} - \chi^8(-q) = \frac{16q}{\chi^8(-q^2)}.$$

Solving for $\chi^8(-q^2)$ in the above equation, we find that

$$\chi^8(-q^2) = \frac{\chi^{16}(-q) + \sqrt{\chi^{32}(-q) + 64q\chi^8(-q)}}{2}. \tag{2.3}$$

Since $\chi^8(-q^2) > 0$ and $\chi^{16}(-q) < \sqrt{\chi^{32}(-q) + 64q\chi^8(-q)}$, we have considered positive sign above. Using (2.3) in (2.2), we obtain the required result. □

REMARK. For a slightly different proof of Theorem 2.1, one may refer [**8**].

THEOREM 2.2. [**5**, p. 467], [**13**] *We have*

$$\frac{1}{6}[-P_1 + 7P_7] = q^{2/3} f_1^2 f_7^2 \left(\frac{f_1^4}{q f_7^4} + 13 + 49q \frac{f_7^4}{f_1^4} \right)^{2/3}. \tag{2.4}$$

REMARK. For proof one may refer [**5**, p. 467, entry 5] and also [**18**].

THEOREM 2.3. *Let* $P = f_1/q^{1/4} f_7$ *and* $Q = f_2/q^{1/2} f_{14}$. *Then*

$$P^4 + \frac{49}{P^4} = \frac{1}{v} - 1 + 48v + 64v^2, \tag{2.5}$$

$$Q^4 + \frac{49}{Q^4} = \frac{1}{v^2} + \frac{6}{v} - 1 + 8v. \tag{2.6}$$

REMARK. For the proof of the above identities one may refer Lemma 8.2 of [**7**].

THEOREM 2.4. *We have*

$$Q_1 + 49Q_7 = 50 \left(\frac{-P_1 + 7P_7}{6} \right)^2 - 160 q f_1^3 f_7^3 \, X,$$

where

$$X = \left(\frac{f_1^7}{f_7} + 13 q f_1^3 f_7^3 + 49 q^2 \frac{f_7^7}{f_1} \right)^{1/3}.$$

PROOF. From (13.5.1) and (13.5.2) of [**3**, p. 345], we have

$$Q_1 = \left(\frac{f_1^7}{f_7} + 5\cdot 7^2 q f_1^3 f_7^3 + 7^4 q^2 \frac{f_7^7}{f_1}\right) X, \tag{2.7}$$

$$Q_7 = \left(\frac{f_1^7}{f_7} + 5 q f_1^3 f_7^3 + q^2 \frac{f_7^7}{f_1}\right) X. \tag{2.8}$$

From (2.7) and (2.8), we deduce that

$$\begin{aligned} Q_1 + 49Q_7 &= \left(50\frac{f_1^7}{f_7} + 490 q f_1^3 f_7^3 + 2450 q^2 \frac{f_7^7}{f_1}\right) X \\ &= \left(50\frac{f_1^7}{f_7} + 650 q f_1^3 f_7^3 + 2450 q^2 \frac{f_7^7}{f_1}\right) X - 160 q f_1^3 f_7^3 X \\ &= 50\left(\frac{f_1^7}{f_7} + 13 q f_1^3 f_7^3 + 7^2 q^2 \frac{f_7^7}{f_1}\right)^{4/3} - 160 q f_1^3 f_7^3 X. \end{aligned}$$

Using (2.4) above, we obtain the required result. □

3. Main results

PROOF OF (1.1). Changing q to q^2 in (2.4) and subtracting the resulting identity from (2.4), we find that

$$\begin{aligned} &\frac{1}{6}[-P_1 + P_2 + 7P_7 - 7P_{14}] \\ &\qquad = q^{2/3} f_1^2 f_7^2 \Big(P^4 + 13 + \frac{49}{P^4}\Big)^{2/3} - q^{4/3} f_2^2 f_{14}^2 \Big(Q^4 + 13 + \frac{49}{Q^4}\Big)^{2/3}, \end{aligned}$$

Using Theorem 2.3 in the above, we find that

$$\begin{aligned} \frac{1}{6}[-P_1 + P_2 + 7P_7 - 7P_{14}] &= \frac{q^{2/3} f_1^2 f_7^2}{v^{2/3}} \left(1 + 12v + 48v^2 + 64v^3\right)^{2/3} \\ &\quad - \frac{q^{4/3} f_2^2 f_{14}^2}{v^{4/3}} \left(1 + 6v + 12v^2 + 8v^3\right)^{2/3} \\ &= \frac{q^{4/3} f_2^2 f_{14}^2}{v^{4/3}} (4v + 12v^2) \\ &= 12\left(\frac{q^2 f_2^4 f_{14}^4}{f_1^2 f_7^2} + \frac{q f_1 f_2 f_7 f_{14}}{3}\right), \end{aligned}$$

which is the required result. □

PROOF OF (1.2). Changing q to q^2 in (2.4) and subtracting 4 times the resulting identity from (2.4), we find that

$$\begin{aligned} &\frac{1}{6}[P_1 - 4P_2 - 7P_7 + 28P_{14}] \\ &\qquad = 4q^{4/3} f_2^2 f_{14}^2 \left(Q^4 + 13 + \frac{49}{Q^4}\right)^{2/3} - q^{2/3} f_1^2 f_7^2 \left(P^4 + 13 + \frac{49}{P^4}\right)^{2/3}. \end{aligned}$$

Using Theorem 2.2 in the above, we find that

$$\begin{aligned}\frac{1}{6}[P_1-4P_2-7P_7+28P_{14}] &= 4\frac{q^{4/3}f_2^2f_{14}^2}{v^{4/3}}\left(1+6v+12v^2+8v^3\right)^{2/3}\\ &\quad -\frac{q^{2/3}f_1^2f_7^2}{v^{2/3}}\left(1+12v+48v^2+64v^3\right)^{2/3}\\ &= \frac{q^{2/3}f_1^2f_7^2}{v^{2/3}}(8v+3)\\ &= 3\left(\frac{f_1^4f_7^4}{f_2^2f_{14}^2}+\frac{8qf_1f_2f_7f_{14}}{3}\right),\end{aligned}$$

which is the required result. □

THEOREM 3.1. *We have*

$$\frac{1}{18}(-P_1-2P_2+7P_7+14P_{14}) = z\Big(w+\frac{1}{w}-\frac{5}{3}\Big). \tag{3.1}$$

PROOF. Multiplying (1.1) by 2 and adding with (1.2), we obtain

$$\frac{1}{18}(-P_1-2P_2+7P_7+14P_{14}) = z\Big(8v+\frac{1}{v}+\frac{16}{3}\Big). \tag{3.2}$$

From [**17**], we have

$$8v+\frac{1}{v} = w+\frac{1}{w}-7. \tag{3.3}$$

Using (3.3) in (3.2), we obtain the required result. □

THEOREM 3.2. *We have*

$$\frac{1}{8}(-P_1+2P_2-7P_7+14P_{14}) = z\Big(\frac{1}{w}-w\Big). \tag{3.4}$$

PROOF. Changing q to q^7 in Theorem 2.1 and adding 7 times the resulting identity with Theorem 2.1, we find that

$$-P_1+2P_2-7P_7+14P_{14} = z\left\{u\Big(P^{12}+\frac{64}{P^{12}}\Big)^{1/2}+\frac{7}{u}\Big(Q^{12}+\frac{64}{Q^{12}}\Big)^{1/2}\right\}, \tag{3.5}$$

where

$$P=\frac{f_1}{q^{1/12}f_2},\quad Q=\frac{f_7}{q^{7/12}f_{14}},\quad u=\frac{f_1f_2}{q^{3/4}f_7f_{14}}.$$

Set

$$\sqrt{w}+\frac{1}{\sqrt{w}} = k. \tag{3.6}$$

The above identity (3.6) can be written as

$$\left(\frac{Q}{P}\right)^6+\left(\frac{P}{Q}\right)^6 = k^3-3k, \tag{3.7}$$

$$\left(\frac{Q}{P}\right)^6-\left(\frac{P}{Q}\right)^6 = (k^2-1)\sqrt{k^2-4}. \tag{3.8}$$

From (3.3), we deduce that

$$(PQ)^6 + \frac{64}{(PQ)^6} = (k^2-5)(k^2-13), \tag{3.9}$$

$$(PQ)^6 - \frac{64}{(PQ)^6} = (k^2-9)\sqrt{k^4-18k^2+49}. \tag{3.10}$$

From (3.7) and (3.9), we find that

$$P^{12} + \frac{64}{P^{12}} + Q^{12} + \frac{64}{Q^{12}} = k(k^2-3)(k^2-5)(k^2-13).$$

From (3.8) and (3.10), we find that

$$-P^{12} - \frac{64}{P^{12}} + Q^{12} + \frac{64}{Q^{12}} = (k^2-1)(k^2-9)\sqrt{(k^2-4)(k^4-18k^2+49)}.$$

From the above two equations, we find that

$$P^{12} + \frac{64}{P^{12}} = \frac{1}{2}\Big\{k(k^2-3)\left(k^2-5\right)\left(k^2-13\right) - (k^2-1)(k^2-9)\sqrt{(k^2-4)(k^4-18k^2+49)}\Big\}, \tag{3.11}$$

$$Q^{12} + \frac{64}{Q^{12}} = \frac{1}{2}\Big\{k(k^2-3)\left(k^2-5\right)\left(k^2-13\right) + (k^2-1)(k^2-9)\sqrt{(k^2-4)(k^4-18k^2+49)}\Big\}. \tag{3.12}$$

From [**6**, p. 327, entry 55], we have

$$u^2 + \frac{49}{u^2} = k^3 - 8k,$$

which implies

$$u = \frac{1}{2}\Big\{\sqrt{k^3-8k+14} + \sqrt{k^3-8k-14}\Big\}, \tag{3.13}$$

$$\frac{7}{u} = \frac{1}{2}\Big\{\sqrt{k^3-8k+14} - \sqrt{k^3-8k-14}\Big\}. \tag{3.14}$$

From (3.11) and (3.13), we deduce that

$$u\Big(P^{12} + \frac{64}{P^{12}}\Big)^{1/2} = \frac{1}{4}\left\{16k\sqrt{k^2-4} - 12\sqrt{(k^2+2k-7)(k^2-2k-7)}\right\}. \tag{3.15}$$

From (3.12) and (3.14), we deduce that

$$\frac{7}{u}\Big(Q^{12} + \frac{64}{Q^{12}}\Big)^{1/2} = \frac{1}{4}\left\{16k\sqrt{k^2-4} + 12\sqrt{(k^2+2k-7)(k^2-2k-7)}\right\}. \tag{3.16}$$

Using (3.15) and (3.16) in (3.5) and then using (3.6), we get the required result. □

Proof of (1.3) and (1.4). Subtracting (3.1) from (3.4), we obtain (1.3). Adding (3.1) and (3.4), we obtain (1.4). □

PROOF OF (1.5). From (1.1) and (1.2), we find that

$$\frac{-P_1+7P_7}{6}=z\Big(16v+\frac{1}{v}+8\Big), \tag{3.17}$$

$$\frac{-P_2+7P_{14}}{6}=z\Big(4v+\frac{1}{v}+4\Big). \tag{3.18}$$

Using (3.17) in Theorem 2.4, we find that

$$Q_1+49Q_7=50z^2\Big(16v+\frac{1}{v}+8\Big)^2-160qf_1^3f_7^3\ X.$$

Using (2.5) in the above to simplify X, we obtain

$$Q_1+49Q_7=z^2\Big\{50\Big(16v+\frac{1}{v}+8\Big)^2-160\frac{(4v+1)}{v}\Big\}. \tag{3.19}$$

Changing q to q^2 in Theorem 2.4 and then using (3.18), we find that

$$Q_2+49Q_{14}=50z^2\Big(4v+\frac{1}{v}+4\Big)^2-160qf_2^4f_{14}^4\Big(Q^4+13+\frac{49}{Q^4}\Big)^{1/3}.$$

Using (2.6) in the above, we obtain

$$Q_2+49Q_{14}=z^2\Big\{50\Big(4v+\frac{1}{v}+4\Big)^2-160(2v+1)\Big\}. \tag{3.20}$$

Subtracting (3.20) from (3.19), we deduce that

$$Q_1-Q_2+49Q_7-49Q_{14}=240z^2\Big(50v^2+48v+13+\frac{1}{v}\Big).$$

Which is the required result. □

PROOF OF (1.6). Subtracting 4 times of (3.20) from (3.19), we obtain the required result. □

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References

1. C. Adiga, N. A. S. Bulkhali, Y. Simsek, H. M. Srivastava, *A continued fraction of ramanujan and some ramanujan-weber class invariants*, Filomat, 31:3975–3997, 2017.
2. R. P. Agarwal, *Resonance of Ramanujan's mathematics. Vol. I*, New Age International Publishers, New Delhi, 1996.
3. G. E. Andrews, B. C. Berndt, *Ramanujan's lost notebook. Part II*, Springer, New York, 2009.
4. W. N. Bailey, *A further note on two of Ramanujan's formulae*, Quart. J. Math. Oxford Ser. (2), 3:158–160, 1952.
5. B. C. Berndt, *Ramanujan's Notebooks: Part III*, Springer New York, 1991.
6. ———, *Ramanujan's Notebooks: Part IV*, Springer New York, 1994.
7. B. C. Berndt, H. H. Chan, S.-S. Huang, *Incomplete elliptic integrals in Ramanujan's lost notebook*, 254:79–126, 2000.
8. S. Cooper, *Inversion formulas for elliptic functions*, Proc. Lond. Math. Soc. (3), 99(2):461–483, 2009.
9. S. Cooper, D. Ye, *Level 14 and 15 analogues of Ramanujan's elliptic functions to alternative bases*, Trans. Amer. Math. Soc. 368:7883–7910, 2016.

10. D. Kim, J. Kyung, Y. Simsek, *Arithmetic of infinite products and Rogers–Ramanujan continued fractions*, Commun. Korean Math. Soc. 22(3):331–351, 2007.
11. M. S. M. Naika, Y. Simsek, M. C. Maheshkumar, *Generalized Dedekind eta function*, South East Asian J. Math. Math. Sc. 8(3):33–41, 2010.
12. S. Ramanujan, *Modular equations and approximations to* π, Quart. J. Math. 45:350–372, 1914.
13. ______, *Notebooks (2 volumes)*, Tata Institute of Fundamental Research, Bombay, 1957.
14. Y. Simsek, *Relations between theta-functions Hardy sums Eisenstein and Lambert series in the transformation formula of* $\log \eta g$, *h(z)*, J. Number Theory 99(2):338–360, 2003.
15. ______, *On normalized Eisenstein series and new theta functions*, Proc. Jangjeon Math. Soc. 8(1):25–34, 2005.
16. H. M. Srivastava, M. P. Chaudhary, F. K. Wakene, *A family of theta-function identities based upon q-binomial theorem and Heine's transformations*, Montes Taurus J. Pure Appl. Math. 2(2):1–6, 2020.
17. K. R. Vasuki, T. G. Sreeramamurthy, *A note on P-Q modular equations*, Tamsui Oxf. J. Math. Sci. 21(2):109–120, 2005.
18. K. R. Vasuki, R. G. Veeresha, *Ramanujan's Eisenstein series of level* 7 *and* 14, J. Number Theory 159:59–75, 2016.

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