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SUMMAND INTERSECTION PROPERTY ON THE CLASS OF EXACT SUBMODULES

Figen Takıl Mutlu and Adnan Tercan

Abstract. A module M is said to have the SIP if intersection of each pair of direct summands is also a direct summand of M. In this article, we define a module M to have the SIP^r if and only if intersection of each pair of exact direct summands is also a direct summand of M where r is a left exact preradical for the category of right modules. We investigate structural properties of SIP^r-modules and locate the implications between the other summand intersection properties. We deal with decomposition theory as well as direct summands of SIP^r-modules. We provide examples by looking at special left exact preradicals.

1. Introduction

Throughout this article, all rings are associative with unity and R denotes such a ring. All modules are unital right R-modules and M_R denotes such a module.

Kaplansky [**9**] showed that for a free module over a principal ideal domain intersection of two direct summands is also a direct summand. Later Fuchs [**4**, Problem 9] mentioned a question which asks that characterization of Abelian groups (i.e., $\mathbb{Z}$-modules) which satisfy the aforementioned property. This property is called SIP (*Summand Intersection Property*) and worked out by several authors [**2**, **5**, **8**, **14**].

Recall that a functor r from the category of the right R-modules to itself is called a *left exact preradical* if it has the following properties

(i) $r(M)$ is a submodule of M for every right R-module M,
(ii) $r(N) = N \cap r(M)$ for every submodule N of a right R-module M,
(iii) $\varphi(r(M)) \subseteq r(M')$ for every homomorphism $\varphi\colon M \to M'$ for right R-modules M, M'.

For example, $r = \text{zer}$, i.e., $r(M) = 0$, for all right R-modules M or $r = \text{id}$, i.e., $r(M) = M$, for all right R-modules M are trivial left exact preradicals. Moreover, r is called *radical* if $r(M/r(M)) = 0$ for every right R-module M. It is clear that the singular submodule and the socle are left exact preradicals and the second singular

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submodule (or Goldie torsion submodule) is a radical. For an excellent treatment of the left exact preradicals, please consult [**11**]. A submodule N of M is called exact provided that $r(M/N) = 0$. In this paper, we focus on modules with the SIP in terms of exact direct summands. We call a module have the SIP^r if the intersection of every pair of exact direct summands of M is a direct summand of M.

Let M be a module. Thus $N \leqslant M$, $N \leqslant_e M$, $\text{End}(M)$, $Z(M)$, r and $\text{ann}_r(X)$ will stand for N is a submodule of M, N is an essential submodule of M, the ring of endomorphisms of M, singular submodule of M, the left exact preradical in the category of right modules and the right annihilator of a subset X in M, respectively. For any other terminology or unexplained notions, we refer to [**3**, **6**, **12**].

2. Results

Obviously a module with the SIP has the SIP^r. In general, a module with the SIP^r need not to have the SIP. The following example shows that there exists a module with SIP^r but not have the SIP.

EXAMPLE 2.1. Let $R = \begin{pmatrix} \mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & \mathbb{Z}_4 \end{pmatrix}$ and $M = R$ as a right R-module. Then, all nontrivial idempotents of R are $e_1 = \begin{pmatrix} \bar{0} & \bar{0} \\ 0 & \bar{1} \end{pmatrix}$, $e_2 = \begin{pmatrix} \bar{0} & \bar{2} \\ 0 & \bar{1} \end{pmatrix}$, $e_3 = \begin{pmatrix} \bar{1} & \bar{0} \\ 0 & 0 \end{pmatrix}$, $e_4 = \begin{pmatrix} \bar{1} & \bar{2} \\ 0 & 0 \end{pmatrix}$.

Since $e_1R \cap e_2R$ is not a direct summand of M, M has not the SIP. However, if we take $r = Z$, all exact direct summands of M are e_3R and M. Hence M has the SIP^r.

Next we give some basic properties of the left exact preradicals in the sense of exact submodules.

PROPOSITION 2.1. *Let M, M_i $(i \in I)$ be right R-modules and r a left exact preradical in the category of right R-modules. Then*

(i) *If N is an exact submodule of M, then $r(M) \leqslant N$.*
(ii) *If $M = \bigoplus_{i \in I} M_i$ then $r(M) = \bigoplus_{i \in I} r(M_i)$.*
(iii) *Let $X \leqslant Y \leqslant M$. If X is exact in Y and Y is exact in M, then X is exact in M.*
(iv) *Let N be a submodule of M. If X is an exact submodule of M, then $X \cap N$ is an exact submodule of N.*

PROOF. (i) and (ii) follows from the definitions (see, for example [**11**]).
(iii) By assumption, $r(Y/X) = r(M/Y) = 0$. Since we have $M/Y \cong \frac{M/X}{Y/X}$, Y/X is exact in M/X. By (i), $r(M/X) \leqslant Y/X$. Since every left exact preradical is idempotent, we have that $r(r(M/X)) = r(M/X) = (M/X) \cap r(Y/X) = 0$. It follows that X is an exact submodule of M.
(iv) Let X be an exact submodule in M. Since $N/(X \cap N) \cong (X + N)/X$, then $r(N/(X \cap N)) = 0$, i.e., $X \cap N$ is an exact submodule of N. □

REMARK 2.1. Let $M = M_1 \oplus M_2$ and N be an exact submodule in M. Since $M_1/(N \cap M_1) \cong (M_1 + N)/N$, $r(M_1/(N \cap M_1)) = 0$ and hence $N \cap M_1$ is exact in M_1. In particular, if N and K are exact direct summands of M, then $N \cap K$ is exact in N and so is exact in M by Proposition 2.1(iii).

PROPOSITION 2.2. *Consider the following statements for a module M.*

(i) *M has the* SIP. (ii) *M has the* SIP^r. (iii) *M has the* SIP^z.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), *but the reverse implications are not true, in general.*

PROOF. (i) $\Rightarrow$ (ii) Follows from the definitions.
(ii) $\Rightarrow$ (iii) Follows by taking $r = Z$.
(ii) $\not\Rightarrow$ (i) By Example 2.1.
(iii) $\not\Rightarrow$ (ii) Let $R = \left(\begin{smallmatrix} \mathbb{Z}_4 & \mathbb{Z}_4 \\ \mathbb{Z}_4 & \mathbb{Z}_4 \end{smallmatrix}\right)$ and $M = R$ as a right R-module. Then, all nontrivial idempotents of R are

$$e_1 = \left\{\begin{pmatrix} \overline{0} & \overline{0} \\ c & \overline{1} \end{pmatrix} \mid c \in \mathbb{Z}_4\right\}, \quad e_2 = \left\{\begin{pmatrix} \overline{0} & b \\ \overline{0} & \overline{1} \end{pmatrix} \mid b \in \mathbb{Z}_4\right\}, \quad e_3 = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{2} & \overline{1} \end{pmatrix},$$

$$e_4 = \left\{\begin{pmatrix} \overline{1} & 0 \\ c & \overline{0} \end{pmatrix} \mid c \in \mathbb{Z}_4\right\}, \quad e_5 = \begin{pmatrix} \overline{1} & \overline{2} \\ \overline{2} & \overline{0} \end{pmatrix}, \quad e_6 = \left\{\begin{pmatrix} \overline{1} & b \\ \overline{0} & \overline{0} \end{pmatrix} \mid b \in \mathbb{Z}_4\right\},$$

$$e_7 = \begin{pmatrix} \overline{3} & \overline{1} \\ \overline{2} & \overline{2} \end{pmatrix}, \quad e_8 = \begin{pmatrix} \overline{3} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix}, \quad e_9 = \begin{pmatrix} \overline{3} & \overline{2} \\ \overline{3} & \overline{2} \end{pmatrix}, \quad e_{10} = \begin{pmatrix} \overline{3} & \overline{3} \\ \overline{2} & \overline{2} \end{pmatrix},$$

$$e_{11} = \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{1} & \overline{3} \end{pmatrix}, \quad e_{12} = \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{3} & \overline{3} \end{pmatrix}, \quad e_{13} = \begin{pmatrix} \overline{2} & \overline{3} \\ \overline{2} & \overline{3} \end{pmatrix}, \quad e_{14} = \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{2} & \overline{3} \end{pmatrix}.$$

By routine calculations, all z-closed direct summands (i.e., D is a direct summand with $Z(M/D) = 0$) of M are 0 and M. Hence M has the SIP^z. Now, let $r =$ zer, then every submodule of M_R is an exact submodule of M. However, $e_3R \cap \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{2} & \overline{0} \end{pmatrix} R = \left(\begin{smallmatrix} 2\mathbb{Z}_4 & 2\mathbb{Z}_4 \\ 0 & 0 \end{smallmatrix}\right)$ is not a direct summand of M. It follows that M has not the SIP^r. □

The following theorem gives a characterization of modules with the SIP^r as well as the SSP^r in terms of certain kind of homomorphisms. Recall that a module have the SSP^r if the sum of every pair of exact direct summands of M is a direct summand of M.

THEOREM 2.1. *Let M be a module. Then*

(i) *M has the* SIP^r *if and only if for every decomposition $M = A \oplus B$ with an exact direct summand A of M and for every homomorphism $f\colon A \to B$,* Ker f *is a direct summand of M.*
(ii) *M has the* SSP^r *if and only if for every decomposition $M = A \oplus B$ with an exact direct summand A of M and for every homomorphism $f\colon A \to B$, Imf is a direct summand of M.*

PROOF. (i) Assume that M has the SIP^r. Let $M = A \oplus B$, A exact in M and $f\colon A \to B$ a homomorphism. Let $X = \{a + f(a) \mid a \in A\}$ and $m \in M$. Then, it can be seen that $M = X \oplus B$ and $\operatorname{Ker} f = X \cap A$. Since A is exact in M, $r(B) = 0$. Hence X is exact in M. Then by assumption, $\operatorname{Ker} f$ is a direct summand of M.

Conversely, assume that for every decompositon $M = A \oplus B$ with an exact direct summand A of M and for every homomorphism $f\colon A \to B$, $\operatorname{Ker} f$ is a direct summand of M. Let N and K be exact direct summands of M. Then $M = N \oplus N_1$ and $M = K \oplus K_1$ for some $N_1, K_1 \leqslant M$. Let $\pi_{N_1}\colon M \to N_1$ and $\pi_K\colon M \to K$ be the canonical projections. Now, define $h = (\pi_{N_1} \circ \pi_K)|_N\colon N \to N_1$. Then

$\operatorname{Ker} h = (N \cap K) \oplus (N \cap K_1)$ is a direct summand of M by assumption. Since $N \cap K$ is a direct summand of $\operatorname{Ker} h$, it is a direct summand of M. Hence M has the SIP^r.

(ii) Assume that M has the SSP^r. Let $M = A \oplus B$, A exact in M and $f\colon A \to B$ a homomorphism. Let $X = \{a + f(a) \mid a \in A\}$ and $m \in M$. Then, $M = X \oplus B$ and X is exact in M as in (i). Then by assumption, $M = (A + T) \oplus L$ for some $L \leqslant M$. Since $A \cap Imf = 0$, $A + T = A \oplus Imf$. Hence Imf is a direct summand of M. The converse follows from [**1**, Theorem 8]. □

The following result shows that the SIP^r property is inherited by direct summands of a module which satisfies SIP^r.

THEOREM 2.2. *If M has the* SIP^r*, then any direct summand of M has also the* SIP^r*.*

PROOF. Let M be an SIP^r-module and M_1 a direct summand of M. Assume that N and K be exact direct summands of M_1. Then $M = M_1 \oplus M_2$ for some submodule M_2 of M and $M_1 = N \oplus N' = K \oplus K'$ for some submodules N', K' of M_1. By assumption, $r(N') = r(K') = 0$ and hence $N \oplus M_2$ and $K \oplus M_2$ are exact submodules of M. By hypothesis, $(N \oplus M_2) \cap (K \oplus M_2)$ is a direct summand of M. Now, applying the modular law, we have $(N \oplus M_2) \cap (K \oplus M_2) = [N \cap (K \oplus M_2)] \oplus M_2$. Then, it can be checked that $[N \cap (K \oplus M_2)] \oplus M_2 = (N \cap K) \oplus M_2$. Thus $(N \cap K) \oplus M_2$ is a direct summand of M and hence $(N \cap K)$ is a direct summand of M_1. It follows that M_1 has the SIP^r. □

3. Decompositions

In this section, we focus on direct sums and decompositions of SIP^r-modules. To this end, we obtain several results when a direct sum and various kind of SIP^r-modules enjoy the property. Moreover, we obtain characterizations of this new class of modules.

THEOREM 3.1. *Let $M = \bigoplus M_i$ be a direct sum of fully invariant submodules M_i of M. Then M has the* SIP^r *if and only if each M_i has the* SIP^r*.*

PROOF. Assume that each M_i has the SIP^r. Let S and T be exact direct summands of M. Since each M_i is fully invariant, $S = \bigoplus (S \cap M_i)$ and $T = \bigoplus (T \cap M_i)$. Hence

$$S \cap T = \big[\bigoplus (S \cap M_i)\big] \cap \big[\bigoplus (T \cap M_i)\big] = \bigoplus \big[(S \cap M_i) \cap (T \cap M_i)\big].$$

By Proposition 2.1(iv), $S \cap M_i$ and $T \cap M_i$ are exact direct summands of M_i, and hence so also does $(S \cap M_i) \cap (T \cap M_i)$ by assumption. It follows that M has the SIP^r. The converse is clear by Theorem 2.2. □

To prove our next theorem, we need to have the following lemma which appears in [**7**, Proposition 3.9].

LEMMA 3.1. *Let $M = M_1 \oplus M_2$ be an R-module. If $\operatorname{ann}_r(M_1) + \operatorname{ann}_r(M_2) = R$, then every submodule N of M can be written as $N = N_1 \oplus N_2$, where $N_1 \leqslant M_1$ and $N_2 \leqslant M_2$.*

Theorem 3.2. *Let M and N be modules with the* SIP^r. *If* $\mathrm{ann}_r(M)+\mathrm{ann}_r(N) = R$, *then $M \oplus N$ has the* SIP^r.

Proof. Let A and B be exact direct summands of $M \oplus N$. By Lemma 3.1, there exist $M_1, M_2 \leqslant M$ and $N_1, N_2 \leqslant N$ such that $A = M_1 \oplus N_1$ and $B = M_2 \oplus N_2$. It is easy to check that M_1 and M_2 are direct summands of M, N_1 and N_2 are direct summands of N. Since $(M\oplus N)/A = (M\oplus N)/(M_1\oplus N_1) \cong M/M_1 \oplus N/N_1$ and A is an exact direct summand of $M \oplus N$, M_1 is exact in M and N_1 is exact in N. Similarly, M_2 is exact in M and N_2 is exact in N. By assumption, $M_1 \cap M_2$ and $N_1 \cap N_2$ are direct summands of M and N, respectively. Therefore, $(M_1 \cap M_2) \oplus (N_1 \cap N_2)$ is a direct summand of $M \oplus N$. Now,

$$(M_1 \cap M_2) \oplus (N_1 \cap N_2) = (M_1 \oplus N_1) \cap (M_2 \oplus N_2) = A \cap B.$$

Thus $A \cap B$ is a direct summand of $M \oplus N$, and hence $M \oplus N$ has the SIP^r. □

Now, recall the conditions (C_3) and (D_3):

(C_3) If A and B are direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

(D_3) If A and B are direct summands of M with $A + B = M$, then $A \cap B$ is a direct summand of M.

We consider modules with (C_3) and (D_3) in terms of exact direct summands and call the conditions (C_3^r) and (D_3^r), respectively:

(C_3^r) If A and B are exact direct summands of M with $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

(D_3^r) If A and B are exact direct summands of M with $A + B = M$, then $A \cap B$ is a direct summand of M.

Theorem 3.3. *Let M be a module. If for any two exact direct summands A and B of M, $A + B$ has (D_3^r), then M has the* SIP^r.

Proof. Assume that for any two exact direct summands A and B of M, $A+B$ has (D_3^r). Note that A and B are also direct summands of $A+B$. Since $A+B$ has (D_3^r), $A \cap B$ is a direct summand of $A + B$. Then $A + B = (A \cap B) \oplus L$ for some $L \leqslant A + B$. Thus $A = (A \cap B) \oplus (A \cap L)$, and hence $A \cap B$ is a direct summand of M. □

Proposition 3.1. *Let M be a module. If for any two exact direct summands A and B of M, $A + B$ is a quasi-projective module, then M has the* SIP^r.

Proof. It is well known that any quasi-projective module has (D_2). If $A+B$ is a quasi-projective module then it has (D_3^r). Thus M has the SIP^r by Theorem 3.3. □

The following result and its corollary provide the link between the SIP^r and the SSP^r conditions.

Proposition 3.2. *Let M be a module with (C_3^r). If M has the* SIP^r *then M has the* SSP^r.

PROOF. The proof follows similarly to [**1**, Lemma 19 (1)]. □

COROLLARY 3.1. *Let M be a module with the* SIP^r*. Then M has (C_3^r) if and only if M has the* SSP^r*.*

PROOF. Let M be a module with the SIP^r. Assume that M has (C_3^r). Then by Proposition 3.2, M has the SSP^r. The converse is clear since every SSP^r module has (C_3^r). □

PROPOSITION 3.3. *Let M be a module with indecomposable radical submodule $r(M)$ and N be a module. If $r(M) \oplus N$ has the* SIP^r*, then every nonzero homomorphism from $r(M)$ to N is a monomorphism.*

PROOF. Assume that $r(M) \oplus N$ has the SIP^r. Let $f\colon r(M) \to N$ be a nonzero homomorphism. Since $r(M)$ is exact in M, $\operatorname{Ker} f$ is a direct summand of $r(M) \oplus N$ by Theorem 2.1. So, $\operatorname{Ker} f$ is a direct summand of $r(M)$. By assumption, $\operatorname{Ker} f = 0$, i.e., f is a monomorphism. □

Recall that a module M is *quasi-Dedekind* if for every $0 \neq f \in \operatorname{End}(M)$, f is a monomorphism (see [**10**]).

COROLLARY 3.2. *Let M be a module with indecomposable radical submodule $r(M)$ and N any module such that $Hom(r(M), N) \neq 0$. If $r(M) \oplus N$ has the* SIP^r*, then $r(M)$ is quasi-Dedekind. In particular, if $r(M) \oplus r(M)$ has the* SIP^r*, then $r(M)$ is quasi-Dedekind.*

PROOF. By Proposition 3.3, there is a monomorphism $f\colon r(M) \to N$. Assume that $r(M)$ is not quasi-Dedekind. Then, there exists a nonzero endomorphism $g\colon r(M) \to r(M)$ such that $\operatorname{Ker} g \neq 0$. Since f is a monomorphism, $f \circ g\colon r(M) \to N$ is a monomorphism with $\operatorname{Ker} f \circ g \neq 0$, which is a contradiction. Thus, $r(M)$ is quasi-Dedekind. □

PROPOSITION 3.4. *Let M be a module with injective indecomposable radical submodule $r(M)$, and N be an indecomposable module such that $Hom(r(M), N) \neq 0$. If $r(M) \oplus N$ has the* SIP^r*, then $r(M)$ is isomorphic to N and $r(M)$ is quasi-Dedekind.*

PROOF. By Proposition 3.3, $r(M)$ is isomorphic to a submodule of N, and by Corollary 3.2, $r(M)$ is quasi-Dedekind. Since $r(M)$ is injective, there is an injective submodule N_1 of N. By the injectivity of N_1, N_1 is a direct summand of N. Since N is indecomposable, $N_1 = N$. Thus $r(M)$ is isomorphic to N. □

LEMMA 3.2. *Let M be a module and $L \leqslant N \leqslant M$. If L is essential in N, then $r(L)$ is essential in $r(N)$.*

PROOF. Let $0 \neq x \in r(N)$. Then $x \in N$ and hence there exists $t \in R$ such that $0 \neq xt \in L$. Also $xt \in r(N)$ since $r(N) \leqslant M$. Thus $0 \neq xt \in r(N) \cap L = r(L)$. It follows that $r(L)$ is essential in $r(N)$. □

PROPOSITION 3.5. *Let M be a right R-module. Then the following statements are equivalent.*

(i) *E(M) has the* SIP^r,

(ii) *for every T and S exact direct summands of M, there is the equality*

$$E(T) \cap E(S) = E(T \cap S).$$

PROOF. (i) ⇒ (ii) Let S and T be exact direct summands of M. Then there exist submodules S' and T' of M such that $M = S \oplus S' = T \oplus T'$. Since S' is essential in $E(S')$ and T' is essential in $E(T')$, $r(S')$ is essential in $r(E(S'))$ and $r(T')$ is essential in $r(E(T'))$ by Lemma 3.2. It follows that $E(S)$ and $E(T)$ are exact direct summands of $E(M)$. Then by assumption, $E(T) \cap E(S)$ is a direct summand of $E(M)$. On the other hand, $E(S \cap T)$ is a direct summand in both $E(S)$ and $E(T)$, and hence $E(S \cap T)$ is a direct summand in $E(S) \cap E(T)$. So, $E(S \cap T)$ is a direct summand in $E(M)$. It follows that

$$E(S) \cap E(T) = E(S \cap T) \oplus K \tag{*}$$

for some $K \in E(M)$. Then $K = E(X)$ for some $X \in M$ and $X \leqslant S \cap T$ by [**13**, Lemma 4.14]. Hence $K = E(X)$ is a direct summand in $E(S \cap T)$, which contradicts the equality (*). It follows that $K = E(X) = 0$ and the equality in the statement of Theorem is provided.

(ii) ⇒ (i) It is clear from [**13**, Lemma 4.14]. □

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Department of Mathematics
Eskisehir Technical University
Eskisehir
Turkey
figent@eskisehir.edu.tr

Department of Mathematics
Hacettepe University
Ankara
Turkey
tercan@hacettepe.edu.tr

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