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ON SOME SUFFICIENT CONDITIONS FOR STARLIKENESS

Mamoru Nunokawa and Janusz Sokół

Abstract. We determine the sufficient conditions for function $f(z) = z + a_2z^2 + \cdots$ to be starlike of order $1/2$, which shows also the starlikeness of f.

1. Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} = \{z : \ |z| < 1\}$ on the complex plane $\mathbb{C}$. We will use the following notations:

$$J_{CV}(f;z) := 1 + \frac{zf''(z)}{f'(z)}, \qquad J_{ST}(f;z) := \frac{zf'(z)}{f(z)}.$$

Let the function $f \in \mathcal{H}$ be univalent in the unit disc $\mathbb{D}$ with the normalization $f(0) = 0$. Then f maps $\mathbb{D}$ onto a starlike domain with respect to $w_0 = 0$ if and only if [**5**]

$$\text{(1.1)} \qquad \mathrm{Re}\{J_{ST}(f;z)\} > 0, \quad (z \in \mathbb{D}).$$

Such function f is said to be starlike in $\mathbb{D}$ with respect to $w_0 = 0$ (or briefly starlike). Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the linear segment joining w_0 to every other point $w \in E$ lies entirely in E, while a set E is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any two points of E lies entirely in E. A function f maps $\mathbb{D}$ onto a convex domain E if and only if [**16**]

$$\text{(1.2)} \qquad \mathrm{Re}\{J_{CV}(f;z)\} > 0, \quad (z \in \mathbb{D})$$

and then f is said to be convex in $\mathbb{D}$ (or briefly convex). It is well known that if an analytic function f satisfies (1.1) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{D}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{D}$ will be denoted by $\mathcal{S}^*$. The set of all functions $f \in \mathcal{A}$ that are convex univalent in $\mathbb{D}$ by $\mathcal{K}$. It is known that for $f \in \mathcal{A}$ condition (1.2) is sufficient for starlikeness

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of f. Also the condition $|J_{CV}(f;z)-1|<2$, $(z\in\mathbb{D})$ is sufficient for starlikeness of f. In this paper we shall consider certain sufficient conditions for starlikeness of order $1/2$. The class $\mathcal{S}^*_\alpha$ of starlike functions of order $\alpha<1$ may be defined as

$$\mathcal{S}^*_\alpha := \{f\in\mathcal{A}:\ \mathrm{Re}\{J_{ST}(f;z)\}>\alpha,\ z\in\mathbb{D}\}.$$

The class $\mathcal{S}^*_\alpha$ and the class $\mathcal{K}_\alpha$ of convex functions of order $\alpha<1$

$$\mathcal{K}_\alpha := \{f\in\mathcal{A}:\mathrm{Re}\{J_{CV}(f;z)\}>\alpha,\ z\in\mathbb{D}\} = \{f\in\mathbb{D}: zf'\in\mathcal{S}^*_\alpha\}$$

were introduced by Robertson in [**11**]. It is known from the old Strohhäcker result [**13**] that $\mathcal{K}_0\subset\mathcal{S}^*_{1/2}\subset\mathcal{S}^*_0$. Furthermore, note that if $f\in\mathcal{K}_\alpha$ then $f\in\mathcal{S}^*_{\delta(\alpha)}$, see [**14**], where

$$\delta(\alpha)=\begin{cases}\frac{1-2\alpha}{2^{2-2\alpha}-2} & \text{for } \alpha\neq 1/2,\\ \frac{1}{2\log 2} & \text{for } \alpha=1/2.\end{cases}$$

Robertson [**12**] proved that if $f\in\mathcal{A}$ with $f(z)/z\neq 0$ and if there exists a k, $0<k\leqslant 2$, such that $|J_{CV}(f;z)-1|\leqslant k\,|J_{ST}(f;z)|$, $z\in\mathbb{D}$, then $f\in\mathcal{S}^*_{2/(2+k)}$. In [**4**] it was proved that if $f\in\mathcal{A}$ with $f(z)f'(z)/z\neq 0$ and satisfies

$$|J_{CV}(f;z)|\leqslant\sqrt{2}\,|J_{ST}(f;z)+1|,\quad (z\in\mathbb{D}),$$

then $f\in\mathcal{S}^*$. Several more complicated sufficient conditions for starlikeness and for convexity are collected in the book [**3**, Chapter 5].

In this paper we shall determine the new sufficient conditions for the starlikeness of order $1/2$. The key in proving is Nunokawa's lemma [**6**] and the following lemma which generalizes it [**6**, **7**], see also [**1**]. Note, that the geometric interpretation of Nunokawa's lemma is similar to the geometric interpretation of Jack's one [**2**].

Lemma 1.1. *[8] Let $p(z)$ be of the form*

$$p(z)=1+\sum_{n=m\geqslant 1}^{\infty}a_nz^n,\qquad a_m\neq 0,\quad (z\in\mathbb{D}),$$

with $p(z)\neq 0$ in $|z|<1$. If there exists a point z_0, $|z_0|<1$, such that

$$|\arg\{p(z)\}|<\pi\alpha/2\quad in\quad |z|<|z_0|$$

and $|\arg\{p(z_0)\}|=\pi\alpha/2$ for some $\alpha>0$, then we have

$$\frac{z_0p'(z_0)}{p(z_0)}=ik\alpha,$$

where

$$k\geqslant m\left(a^2+1\right)/(2a)\quad when\quad \arg\{p(z_0)\}=\pi\alpha/2$$

and

$$k\leqslant -m\left(a^2+1\right)/(2a)\quad when\quad \arg\{p(z_0)\}=-\pi\alpha/2,$$

where $\{p(z_0)\}^{1/\alpha}=\pm ia$, $a>0$.

2. Main result

THEOREM 2.1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic and* $J_{CV}(f;z) \neq 1/2$, $J_{ST}(f;z) \neq 0$ *in* $\mathbb{D}$. *Suppose also that*

$$\left|\arg\left\{J_{CV}(f;z) - \tfrac{1}{2}\right\}\right| < \pi - \tan^{-1}\left\{4\left|\operatorname{Im}\left\{J_{ST}(f;z)\right\}\right|\right\}, \quad (z \in \mathbb{D}), \tag{2.1}$$

then $\operatorname{Re}\{J_{ST}(f;z)\} > \frac{1}{2}$, $(z \in \mathbb{D})$, *or* f *is starlike of order* $1/2$, $f \in \mathcal{S}^*_{1/2}$.

PROOF. Let us write

$$p(z) = \frac{2zf'(z)}{f(z)} - 1 = 1 + 2a_2 z + \cdots . \tag{2.2}$$

If there exists a point z_0, $|z_0| < 1$, such that

$$|\arg\{p(z)\}| < \tfrac{\pi}{2}, \quad |z| < |z_0| \quad \text{and} \quad |\arg\{p(z_0)\}| = \tfrac{\pi}{2},$$

then, from Nunokawa's lemma 1.1 with $\alpha = 1$, we have $z_0 p'(z_0)/p(z_0) = ik$, where

$$\begin{aligned} k &\geqslant (a^2+1)/(2a) \ \text{ when } \ \arg\{p(z_0)\} = \pi/2, \\ k &\geqslant -(a^2+1)/(2a) \ \text{ when } \ \arg\{p(z_0)\} = -\pi/2, \\ &p(z_0) = \pm ia, \quad a > 0. \end{aligned}$$

For the case $p(z_0) = ia$, $a > 0$, we have from (2.2)

$$\begin{aligned} \left|\arg\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{1}{2}\right\}\right| &= \left|\arg\left\{\frac{z_0 p'(z_0)}{1 + p(z_0)} + \frac{p(z_0)}{2}\right\}\right| = \left|\arg\left\{\frac{-ka}{ia+1} + \frac{ia}{2}\right\}\right| \\ &= \left|\arg\left\{\frac{2ak}{2(a^2+1)}\left(-1 + i\frac{1+a^2+2ak}{2k}\right)\right\}\right| \\ &= \left|\arg\left\{-1 + i\frac{1+a^2+2ak}{2k}\right\}\right|. \end{aligned}$$

In the considered case we have $a > 0$, $k \geqslant (a^2+1)/(2a)$ and so, we have

$$\frac{\pi}{2} < \arg\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{1}{2}\right\} < \pi$$

and we can write

$$\left|\arg\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{1}{2}\right\}\right| = \pi - \tan^{-1}\left\{\frac{1+a^2+2ak}{2k}\right\}.$$

Observe that for $a > 0$, $k \geqslant (a^2+1)/(2a)$ we also have

$$\frac{1+a^2+2ak}{2k} = a + \frac{1+a^2}{2k} \leqslant a + \frac{1+a^2}{2(1+a^2)/(2a)} = a + a = 2a. \tag{2.3}$$

Moreover, applying (2.2) with $p(z_0) = ia$, we have

$$a = \operatorname{Im}\left\{\frac{2z_0 f'(z_0)}{f(z_0)}\right\}.$$

Therefore, we obtain

$$\left|\arg\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)} - \frac{1}{2}\right\}\right| \geqslant \pi - \tan^{-1}\{2a\} = \pi - \tan^{-1}\left\{\operatorname{Im}\left\{\frac{4z_0 f'(z_0)}{f(z_0)}\right\}\right\}.$$

This is the contradiction with (2.1) and for the case $\arg\{p(z_0)\} = -\pi/2$, applying the same method as the above, we can also get the contradiction. $\square$

Since $\mathcal{K}_0 \subset \mathcal{S}^*_{1/2} \subset \mathcal{S}^*_0$, then (2.1) is a sufficient condition for f to be starlike in $\mathbb{D}$.

COROLLARY 2.1. *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be analytic and* $J_{CV}(f;z) \neq 1/2$, $J_{ST}(f;z) \neq 0$ *in* $\mathbb{D}$. *Suppose also that*

$$|\mathrm{Im}\,\{J_{ST}(f;z)\}| < \frac{1}{4}, \quad (z \in \mathbb{D}), \tag{2.4}$$

$$\left|\arg\left\{J_{CV}(f;z) - \frac{1}{2}\right\}\right| < \frac{3\pi}{4}, \quad (z \in \mathbb{D}); \tag{2.5}$$

then $\mathrm{Re}\{J_{ST}(f;z)\} > 1/2$ *in* $\mathbb{D}$ *or* $f \in \mathcal{S}^*_{1/2}$.

Condition (2.4) is equivalent to

$$J_{ST}(f;z) \prec F(z) = 1 + \frac{1}{2\pi} \log \frac{1+z}{1-z}$$

because $F(z)$ maps the unit disc onto horizontal strip $|\mathrm{Im}\,\{w\}| < \frac{1}{4}$. Condition (2.5) says that $f(z)$ is not necessary convex.

THEOREM 2.2. *Let* $f(z) = z + \sum_{n=m\geqslant 2}^{\infty} a_n z^n$, $a_m \neq 0$, *be analytic and* $J_{CV}(f;z) \neq 1/2$, $J_{ST}(f;z) \neq 0$ *in* $\mathbb{D}$. *Suppose also that*

$$\left|\arg\left\{J_{CV}(f;z) - \frac{1}{2}\right\}\right| < \pi - \tan^{-1}\left\{2\left(1 + \frac{1}{m-1}\right)\left|\mathrm{Im}\left\{J_{ST}(f;z)\right\}\right|\right\}$$

in the unit disc $\mathbb{D}$, *then* $\mathrm{Re}\{J_{ST}(f;z)\} > 1/2$ *in* $\mathbb{D}$ *or* $f \in \mathcal{S}^*_{1/2}$.

PROOF. If we put

$$p(z) = \frac{2zf'(z)}{f(z)} - 1,$$

then

$$p(z) = 1 + 2a_m(m-1)z^{m-1} + \cdots, \quad a_m \neq 0, \quad (z \in \mathbb{D}).$$

Now the proof runs in the same way as the proof of Theorem 2.1. By Lemma 1.1, instead of (2.3), we have here

$$\frac{1 + a^2 + 2ak}{2k} = a + \frac{1 + a^2}{2k} \leqslant a + a/(m-1) = a\left(1 + \frac{1}{m-1}\right). \qquad \square$$

COROLLARY 2.2. *Let* $f(z) = z + \sum_{n=3}^{\infty} a_n z^n$ *be analytic and* $J_{CV}(f;z) \neq 1/2$, $J_{ST}(f;z) \neq 0$ *in* $\mathbb{D}$. *Suppose also that*

$$\left|\arg\left\{J_{CV}(f;z) - \frac{1}{2}\right\}\right| < \pi - \tan^{-1}\left\{3\left|\mathrm{Im}\left\{J_{ST}(f;z)\right\}\right|\right\}, \quad (z \in \mathbb{D}),$$

then $\mathrm{Re}\{J_{ST}(f;z)\} > 1/2$ *in* $\mathbb{D}$ *or* $f \in \mathcal{S}^*_{1/2}$.

3. Applications for the convolution

Since $\mathcal{K}_0 \subset \mathcal{S}^*_{1/2} \subset \mathcal{S}^*_0$, then each of the above proved sufficient conditions for $f \in \mathcal{S}^*_{1/2}$ is also the sufficient condition for f to be starlike in $\mathbb{D}$. Recall the convolution:

$$\sum_{n=0}^{\infty} a_n z^n * \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} a_n b_n z^n$$

It is known that $f \in \mathcal{K}_0$, $g \in \mathcal{K}_0$ implies $f * g \in \mathcal{K}_0$. Such implication does not hold for starlike functions. However, if $f \in \mathcal{S}^*_{1/2}$, $g \in \mathcal{S}^*_{1/2}$, then $f * g \in \mathcal{S}^*_{1/2}$. Moreover, if $F \in \mathcal{H}$, then

$$\frac{f(z) * g(z)F(z)}{f(z) * g(z)} \in \overline{\mathrm{co}}F(\mathbb{D}), \quad (z \in \mathbb{D}),$$

where $\overline{\mathrm{co}}F(\mathbb{D})$ denotes the closed convex hull of $F(\mathbb{D})$. For these reasons new sufficient conditions for the starlikeness of order $1/2$ may be valuable.

For other results on the several problems connected to the starlikeness, we refer also to the recent papers [**9**, **10**, **15**].

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University of Gunma
Chiba, Japan
mamorununokawa1983@gmail.com

University of Rzeszów, College of Natural Sciences
Rzeszów, Poland
jsokol@ur.edu.pl

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