PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 111(125) (2022), 129–134 DOI: https://doi.org/10.2298/PIM2225127D

CERTAIN COEFFICIENT INEQUALITIES ASSOCIATED WITH HANKEL DETERMINANT FOR A SPECIFIC SUBFAMILY OF HOLOMORPHIC MAPPINGS

Vamshee Krishna Deekonda and Shalini Dasumahanthi

Abstract. We introduce a new subfamily of holomorphic functions and attempt to estimate an upper bound for the Hankel determinant of the second and third kind for the normalized regular mapping f, a member of this class.

1. Introduction

Let $\mathcal{A}$ represent the family of mappings f of the type

$$f(z) = z + \sum_{t=2}^{\infty} a_t z^t \tag{1.1}$$

in $\mathcal{U} = \{z \in \mathcal{C} : 1 > |z|\}$, denotes the open unit disc and S is the subfamily of $\mathcal{A}$, possessing univalent (schlit) mappings. Pommerenke **[15]** characterized the r^{th}-Hankel determinant of order n, for f with r, $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ namely

$$H_{r,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+r-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+r-1} & a_{n+r} & \cdots & a_{n+2r-2} \end{vmatrix}, \quad (a_1 = 1). \tag{1.2}$$

The Fekete–Szegö functional is obtained for $r = 2$ and $n = 1$ in (1.2), denoted by $H_2(1)$. Further, sharp bounds to the functional $|H_{2,2}(f)|$, obtained for $r = 2$ and $n = 2$ in (1.2), called as Hankel determinant of order two, given by

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2,$$

In recent years, research on estimation of an upper bound (UB) to $|H_{2,2}(f)|$ has been focused on by many authors. The exact estimates of $|H_{2,2}(f)|$ for the functions namely, bounded turning, starlike and convex functions, subfamilies of

---

2020 *Mathematics Subject Classification:* Primary 30C45; Secondary 30C50.
Key words and phrases: holomorphic function, upper bound, functionals connected with Hankel determinant, positive real part function.
Communicated by Gradimir Milovanović.

S, symbolized as Re, S^* and $\mathcal{K}$ respectively fulfilling the conditions $\operatorname{Re} f'(z) > 0$, $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0$ and $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0$ in the unit disc $\mathcal{U}$, were proved by Janteng et al. [**9**,**10**] and derived the bounds as 4/9, 1, and 1/8 respectively. Choosing $r = 2$ and $n = p + 1$ in (1.2), we obtain the Hankel determinant of second order for the p-valent function (see [**20**]), given by

$$H_{2,(p+1)}(f) = \begin{vmatrix} a_{p+1} & a_{p+2} \\ a_{p+2} & a_{p+3} \end{vmatrix} = a_{p+1}a_{p+3} - a_{p+2}^2,$$

The case $r = 3$ seems to be much tough than $r = 2$. A small number of papers have been dedicated to the study of the third order Hankel determinant denoted by $H_{3,1}(f)$, obtained for $r = 3$ and $n = 1$ in (1.2), namely

$$(1.3)\quad H_{3,1}(f) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_1(a_3a_5 - a_4^2) + a_2(a_3a_4 - a_2a_5) + a_3(a_2a_4 - a_3^2),$$

The concept of estimation of an upper bound for $|H_{3,1}(f)|$ was firstly introduced and studied by Babalola [**4**], who tried to estimate for this functional to the classes $\mathcal{R}$, S^* and $\mathcal{K}$, obtained as follows.

(i) $f \in S^* \Rightarrow |H_{3,1}(f)| \leqslant 16$.
(ii) $f \in \mathcal{K} \Rightarrow |H_{3,1}(f)| \leqslant 0.714$.
(iii) $f \in \mathcal{R} \Rightarrow |H_{3,1}(f)| \leqslant 0.742$.

As a result of the paper from Babalola [**4**], many articles containing results associated with the Hankel determinant of order 3 and 4 for specific subfamilies of holomorphic functions were obtained (see [**1**–**3**, **5**, **13**, **16**, **18**]). Motivated with the results obtained by the authors specified here, those who are working in this direction, for our study here, we are attempting to introduce and interpret a new subfamily of holomorphic functions, derive an upper bound (UB) to the functionals $H_{2,3}(f) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_5 \end{vmatrix} = a_3a_5 - a_4^2$ and $H_{3,1}(f)$ for the mapping f belongs to the class defined as below.

Definition 1.1. A mapping $f \in \mathcal{A}$ to be in $S^*\mathcal{K}_s(\beta)$ $(0 \leqslant \beta \leqslant 1)$, if

$$(1.4)\qquad \operatorname{Re}\Big[\frac{2\{zf'(z) + \beta z^2 f''(z)\}}{(1-\beta)\{f(z) - f(-z)\} + \beta\{zf'(z) + zf'(-z)\}}\Big] > 0, \quad z \in \mathcal{U}.$$

For $\beta = 0$ and $\beta = 1$ in (1.4), we get $S^*\mathcal{K}_s(0) = S_s^*$, consisting of starlike f ets. with regard to symmetric points, interpreted and studied by Sakaguchi [**17**] and $S^*\mathcal{K}_s(1) = \mathcal{K}_s$, consisting of convex f ets. about symmetric points, interpreted and studied by Das and Singh [**6**] respectively.

In proving our results, the required sharp estimates specified below, given in the form of Lemmas, which holds suitable for f ets. possessing positive real part.

The collection $\mathcal{P}$, of all functions g, each is called as Caratheodòry function [**7**] of the form, $g(z) = 1 + \sum_{t=1}^{\infty} c_t z^t$, holomorphic in $\mathcal{U}$ and $\operatorname{Re} g(z) > 0$ for $z \in \mathcal{U}$.

LEMMA 1.1. [8] *If $g \in \mathcal{P}$, then the estimate $|c_i - \mu c_j c_{i-j}| \leqslant 2$, holds for $i, j \in \mathbb{N}$, with $i > j$ and $\mu \in [0,1]$.*

LEMMA 1.2. [12] *If $g \in \mathcal{P}$, then the estimate $|c_i - c_j c_{i-j}| \leqslant 2$, holds for $i, j \in \mathbb{N}$, with $i > j$.*

LEMMA 1.3. [14] *If $g \in \mathcal{P}$, then $|c_t| \leqslant 2$, for $t \in \mathbb{N}$; equality occurs for the function $h(z) = \frac{1+z}{1-z}$, $z \in \mathcal{U}$.*

LEMMA 1.4. [21] *If $g \in \mathcal{P}$, then $|c_2 c_4 - c_3^2| \leqslant 4 - \frac{1}{2}|c_2|^2 + \frac{1}{4}|c_2|^3$.*

We prove our results, following the procedure from Libera and Zlotkiewicz [**11**].

2. Important Outcomes

THEOREM 2.1. *If $f \in S^*\mathcal{K}_s(\beta)$ $(0 \leqslant \beta \leqslant 1)$, then*

$$|H_{3,1}(f)| \leqslant \Big[\frac{-8\beta^6 + 240\beta^5 + 734\beta^4 + 782\beta^3 + 377\beta^2 + 84\beta + 7}{4(1+\beta)^2(1+2\beta)^3(1+3\beta)^2(1+4\beta)}\Big].$$

PROOF. For $f \in S^*\mathcal{K}_s(\beta)$, there exists a function $g \in \mathcal{P}$ such that

$$\Big[\frac{2\{zf'(z) + \beta z^2 f''(z)\}}{(1-\beta)\{f(z) - f(-z)\} + \beta\{zf'(z) + zf'(-z)\}}\Big] = g(z). \tag{2.1}$$

Equivalently

$$2\{zf'(z) + \beta z^2 f''(z)\} = [(1-\beta)\{f(z) - f(-z)\} + \beta\{zf'(z) + zf'(-z)\}]g(z).$$

Putting the values for f, f', f'' and g, this simplifies to

$$\begin{aligned}[2(1+\beta)a_2 + 3(1+2\beta)a_3 z + 4(1+3\beta)a_4 z^2 + 5(1+4\beta)a_5 z^3 + \cdots] \\ = [c_1 + \{c_2 + (1+2\beta)a_3\}z + \{c_3 + (1+2\beta)c_1 a_3\}z^2 \\ + \{c_4 + (1+2\beta)c_2 a_3 + (1+4\beta)a_5\}z^3 + \cdots].\end{aligned} \tag{2.2}$$

Comparing the coefficients of z^0, z^1, $z^2$and z^3 respectively in (2.2), we obtain

$$a_2 = \frac{c_1}{2(1+\beta)}; \quad a_3 = \frac{c_2}{2(1+2\beta)}; \quad a_4 = \frac{(2c_3 + c_1 c_2)}{8(1+3\beta)}; \quad a_5 = \frac{(2c_4 + c_2^2)}{8(1+4\beta)} \tag{2.3}$$

and substituting these values into (1.3), after simplifying, we get

$$\begin{aligned}H_{3,1}(f) = \Big[&\frac{c_2 c_4}{8(1+2\beta)(1+4\beta)} - \frac{(-4\beta^2 + 4\beta + 1)c_2^3}{16(1+2\beta)^3(1+4\beta)} - \frac{c_3^2}{16(1+3\beta)^2} \\ &+ \frac{(-2\beta^2 + 3\beta + 1)c_1 c_2 c_3}{16(1+\beta)(1+2\beta)(1+3\beta)^2} - \frac{c_1^2 c_4}{16(1+\beta)^2(1+4\beta)} \\ &+ \frac{(-8\beta^4 - 10\beta^3 + 13\beta^2 + 8\beta + 1)c_1^2 c_2^2}{64(1+\beta)^2(1+2\beta)(1+3\beta)^2(1+4\beta)}\Big].\end{aligned} \tag{2.4}$$

On grouping the suitable terms in (2.4), we have

(2.5)

$$H_{3,1}(f) = \Big[\frac{c_4(c_2 - c_1^2)}{16(1+\beta)^2(1+4\beta)} - \frac{c_3}{16(1+3\beta)^2}\Big\{c_3 - \frac{(1+\beta)}{(1+\beta)(1+2\beta)}c_1 c_2\Big\}$$

$$+\frac{(-4\beta^2+4\beta+1)c_2(c_4-c_2^2)}{16(1+2\beta)^3(1+4\beta)}-\frac{c_2}{8(1+3\beta)^2}\Big\{c_4-\frac{\beta(1-\beta)}{(1+\beta)(1+2\beta)}c_1c_3\Big\}$$
$$+\frac{(-8\beta^4-10\beta^3+13\beta^2+8\beta+1)c_1^2c_2^2}{64(1+\beta)^2(1+2\beta)(1+3\beta)^2(1+4\beta)}$$
$$+\frac{(-36\beta^4+12\beta^3+29\beta^2+10\beta+1)c_2c_4}{16(1+2\beta)^3(1+3\beta)^2(1+4\beta)}\Big].$$

Applying the triangle inequality in (2.5), we obtain

$$|H_{3,1}(f)|\leqslant\Big[\frac{|c_4||c_2-c_1^2|}{16(1+\beta)^2(1+4\beta)}+\frac{|c_3|}{16(1+3\beta)^2}\Big|c_3-\frac{(1+\beta)}{(1+\beta)(1+2\beta)}c_1c_2\Big| \tag{2.6}$$
$$+\frac{(-4\beta^2+4\beta+1)|c_2||c_4-c_2^2|}{16(1+2\beta)^3(1+4\beta)}+\frac{|c_2|}{8(1+3\beta)^2}\Big|c_4-\frac{\beta(1-\beta)}{(1+\beta)(1+2\beta)}c_1c_3\Big|$$
$$+\frac{(-8\beta^4-10\beta^3+13\beta^2+8\beta+1)|c_1|^2|c_2|^2}{64(1+\beta)^2(1+2\beta)(1+3\beta)^2(1+4\beta)}$$
$$+\frac{(-36\beta^4+12\beta^3+29\beta^2+10\beta+1)|c_2||c_4|}{16(1+2\beta)^3(1+3\beta)^2(1+4\beta)}\Big].$$

By using Lemmas 1.1, 1.2 and 1.3 in inequality (2.6), it reduces to

$$|H_{3,1}(f)|\leqslant\Big[\frac{1}{4(1+\beta)^2(1+4\beta)}+\frac{1}{4(1+3\beta)^2}+\frac{(-4\beta^2+4\beta+1)}{4(1+2\beta)^3(1+4\beta)}$$
$$+\frac{1}{2(1+3\beta)^2}+\frac{(-8\beta^4-10\beta^3+13\beta^2+8\beta+1)}{4(1+\beta)^2(1+2\beta)(1+3\beta)^2(1+4\beta)}$$
$$+\frac{(-36\beta^4+12\beta^3+29\beta^2+10\beta+1)}{4(1+2\beta)^3(1+3\beta)^2(1+4\beta)}\Big].$$

Further simplification gives

$$|H_{3,1}(f)|\leqslant\Big[\frac{-8\beta^6+240\beta^5+734\beta^4+782\beta^3+377\beta^2+84\beta+7}{4(1+\beta)^2(1+2\beta)^3(1+3\beta)^2(1+4\beta)}\Big]. \tag{2.7}$$ □

Remark 2.1. Choosing $\beta=0$ in (1.4), we get $S^*\mathcal{K}_s(0)=S_s^*$, for which from (2.7), we obtain $|H_{3,1}(f)|\leqslant\frac{7}{4}$.

Remark 2.2. For $\beta=1$ in (1.4), we obtain $S^*\mathcal{K}_s(1)=\mathcal{K}_s$, in this case from (2.7), we get $|H_{3,1}(f)|\leqslant\frac{277}{4320}$.

These two results are far better than those in Vamshee Krishna et al. [**19**].

Theorem 2.2. *If* $f\in S^*\mathcal{K}_s(\beta)$ $(0\leqslant\beta\leqslant1)$, *then*

$$|H_{2,3}(f)|=|a_3a_5-a_4^2|\leqslant\Big[\frac{1}{(1+2\beta)(1+4\beta)}\Big].$$

Proof. Putting the values of a_3, a_4 and a_5 from (2.3) into $H_{2,3}(f)$, we get

$$H_{2,3}(f)=\frac{1}{16}\Big[\frac{2}{(1+2\beta)(1+4\beta)}c_2c_4+\frac{1}{(1+2\beta)(1+4\beta)}c_2^3-\frac{1}{(1+3\beta)^2}c_3^2$$

$$- \frac{1}{4(1+3\beta)^2}c_1^2c_2^2 - \frac{1}{(1+3\beta)^2}c_1c_2c_3\Big],$$

which is equivalent to

$$H_{2,3}(f) = \frac{1}{16}\Big[\frac{(c_2c_4 - c_3^2)}{(1+3\beta)^2} + \frac{c_2^2}{(1+2\beta)(1+4\beta)}\Big\{c_2 - \frac{(1+2\beta)(1+4\beta)}{4(1+3\beta)^2}c_1^2\Big\}$$
$$+ \frac{c_2}{(1+2\beta)(1+4\beta)}\Big\{c_4 - \frac{(1+2\beta)(1+4\beta)}{(1+3\beta)^2}c_1c_3\Big\}$$
$$+ \Big\{\frac{1}{(1+2\beta)(1+4\beta)} - \frac{1}{(1+3\beta)^2}\Big\}c_2c_4\Big],$$

Further, we have

$$H_{2,3}(f) = \frac{1}{16}\Big[\frac{(c_2c_4 - c_3^2)}{(1+3\beta)^2} + \frac{c_2^2}{(1+2\beta)(1+4\beta)}\Big\{c_2 - \frac{(1+2\beta)(1+4\beta)}{4(1+3\beta)^2}c_1^2\Big\}$$
$$+ \frac{c_2}{(1+2\beta)(1+4\beta)}\Big\{c_4 - \frac{(1+2\beta)(1+4\beta)}{(1+3\beta)^2}c_1c_3\Big\}$$
$$+ \left\{\frac{4\beta^2}{(1+2\beta)(1+3\beta)^2(1+4\beta)}\right\}c_2c_4\Big].$$

Applying the same method as we carried in Theorem 2.1 and then using Lemmas 1.2, 1.3, and 1.4, we obtain the result of Theorem 2.2. □

Remark 2.3. Choosing $\beta = 0$ in Theorem 2.2, we obtain

$$|H_{2,3}(f)| = |a_3a_5 - a_4^2| \leqslant 1,$$

it coincides with Zaprawa [**21**]. From this result, we conclude that the UB is the same for the classes $S^*\mathcal{K}_s(0) = S_s^*$ and with $a_2 = 0$ for the class S^*. The extremal function at this stage is

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + \cdots .$$

Remark 2.4. For $\beta = 1$ in Theorem 2.2, we get $|H_{2,3}(f)| = |a_3a_5 - a_4^2| \leqslant \frac{1}{15}$, it coincides with that of Zaprawa [**21**]. From this result, we observe that the UB is same for the classes $S^*\mathcal{K}_s(1) = \mathcal{K}_s$ and with $a_2 = 0$ for the class $\mathcal{K}$. The extremal function in this context is

$$f(z) = \log\left(\sqrt{\frac{1+z}{1-z}}\right) = z + \frac{1}{3}z^3 + \frac{1}{5}z^5 + \cdots .$$

Acknowledgement. The authors are highly grateful to the esteemed referee(s) for a comprehensive reading of the manuscript and making valuable suggestions, leading to a better paper presentation.

References

1. A. Lecko, Y. J Sim, B. Smiarowska, *The sharp bound of the Hankel determinant of the third kind for starlike functions of order* 1/2, Complex Anal. Oper. Theory **13** (2019), 2231–2238.
2. A. Muhammad, L. Rani, M. Raza, P. Zaprawa, *Fourth Hankel determinant for the family of functions with bounded turning*, Bull. Korean Math. Soc. **55** (2018), 1703–1711.

3. B. Kowalczyk, A. Lecko, Y. J. Sim, *The sharp bound for the Hankel determinant of the third kind for convex functions*, Bull. Aust. Math. Soc. **97** (2018), 435–445.
4. K. O. Babalola, *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, In: Y. J. Cho (ed.), *Inequality Theory and Applications*, Nova Science Publishers, New York, 2010, 1–7.
5. D. Bansal, S. Maharana, J. K. Prajapat, *Third order Hankel determinant for certain univalent functions*, J. Korean Math. Soc. **52** (2015), 1139–1148.
6. R. N. Das, P. Singh, *On subclass of schlicht mappings*, Indian J. Pure Appl. Math. **8** (1977), 864–872.
7. P. L. Duren, *Univalent Functions*, Grundlehren Math. Wiss. **259**, Springer, New York, 1983.
8. T. Hayami, S. Owa, *Generalized Hankel determinant for certain classes*, Int. J. Math. Anal., Ruse **4** (2010), 2573–2585.
9. A. Janteng, S. A. Halim. M. Darus, *Hankel Determinant for starlike and convex functions*, Int. J. Math. Anal., Ruse **1** (2007), 619–625.
10. ———, *Coefficient inequality for a function whose derivative has a positive real part*, JIPAM, J. Inequal. Pure Appl. Math. **7** (2006), 1–5.
11. R. J. Libera, E. J. Zlotkiewicz, *Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$*, Proc. Am. Math. Soc. **87** (1983), 251–257.
12. A. E. Livingston, *The coefficients of multivalent close-to-convex functions*, Proc. Am. Math. Soc. **21** (1969), 545–552.
13. H. Orhan, P. Zaprawa, *Third Hankel determinants for starlike and convex funxtions of order alpha*, Bull. Korean Math. Soc. **55** (2018), 165–173.
14. Ch. Pommerenke, *Univalent Functions*, Gottingen: Vandenhoeck and Ruprecht, 1975.
15. ———, *On the coefficients and Hankel determinants of univalent functions*, J. Lond. Math. Soc. **41** (1966), 111–122.
16. M. Raza, S. N. Malik, *Upper bound of third Hankel determinant for a class of analytic functions related with lemniscate of Bernoulli*, J. Inequal. Appl. **2013** (2013), 412, doi: 10.1186/1029-242X-2013-412.
17. K. Sakaguchi, *On a certain univalent mapping*, J. Math. Soc. Japan **11** (1959), 72–75.
18. T. V. Sudharsan, S. P. Vijayalakshmi, B. Adolf Stephen, *Third Hankel determinant for a subclass of analytic functions*, Malaya J. Mat. **2** (2014), 438–444.
19. D. Vamshee Krishna, B. Venkateswarlu, T. RamReddy, *Third Hankel determinant for starlike and convex functions with respect to symmetric points*, Ann. Univ. Mariae Curie-Skłodowska, Sect. A **70** (2016), 37–45.
20. D. Vamshee Krishna, T. RamReddy, *Coefficient inequality for certain p-valent analytic functions*, Rocky Mt. J. Math. **44** (2014), 1941–1959.
21. P. Zaprawa, *On Hankel determinant $H_2(3)$ for univalent functions*, Results Math. **73** (2018), 89.

Department of Mathematics, Gitam School of Science
GITAM University, Visakhapatnam, India
`vamsheekrishna1972@gmail.com`

(Received 01 01 2021)
(Revised 23 08 2021)

Department of Mathematics, Dr. B. R. Ambedkar University
Srikakulam, India
`shaliniraj1005@gmail.com`