PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle serie, tome 111(125) (2022), 123–127 DOI: https://doi.org/10.2298/PIM2225123S

# ON THE COEFFICIENTS OF A CONTINUED FRACTION OF RAMANUJAN

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*Dedicated to my mother Pushpa Srivastava*

Abstract. On expanding Ramanujan's continued fraction in power series, we observe that the sign of the coefficients is periodic with period 4. We also give a combinatorial interpretation for the coefficients.

## 1. Introduction

Thirty years ago, Szekeres noticed that if we expand the celebrated Rogers–Ramanujan continued fraction,

$$R(q) = 1 + \frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\dots}$$

in a series

$$R(q) = 1 + q - q^3 + q^5 + q^6 - q^7 - 2q^8 + 2q^{10} + 2q^{11} - q^{12} - 3q^{13} - q^{14} + 3q^{15} + \cdots$$

the sign of the coefficients is periodic with period 5, after some terms.

Berndt [**6**] explained that at times the term $q^{1/5}$ appears in the definition of $R(q)$, as $R(q)$ belongs to theta functions and $R(q)$'s modular properties are more symmetric and elegant with the term $q^{1/5}$. However, there are also occasions when the factor $q^{1/5}$ is not helpful and the representation of $R(q)$ defined above is used. [**5**, p. 77 et seq.] and [**1**, **11**] contains details of the further work stemming from the Rogers–Ramanujan continued fraction. Andrews [**4**] gave a combinatorial interpretations of the coefficients in the power series expansion of $R(q)$ and gave a proof of Szekeres's observation. Hirschhorn [**8**] gave another proof of the periodicity of the sign of the coefficients and the combinatorial interpretations for the coefficients. In this paper, we consider another continued fraction of Ramanujan, $C(q)$.

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2010 *Mathematics Subject Classification:* 33D15; 11A55.

*Key words and phrases:* $q$-hypergeometric series, continued fraction, quintuple product identity.

Communicated by H. M Srivastava.

The $C(q)$ continued fraction of Ramanujan is defined as:

$$C(q) = \frac{1}{1+}\frac{(1+q)}{1+}\frac{q^2}{1+}\frac{(q+q^3)}{1+}\frac{q^4}{1+\cdots} = \frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty(q^3;q^4)_\infty}$$

We have studied $C(q)$ in detail in my earlier papers [**9**, **10**], where we found that $C(q)$ has properties analogous to $R(q)$. In this paper, we expand $C(q)$ as a power series and show that the sign of the coefficients is periodic with period 4. In addition, we also give the combinatorial interpretation for the coefficients. Recent papers [**11**, **12**] also give interesting combinatorial interpretations, which can be used for further research.

We shall be using the following standard notations: For $|q| < 1$,

$$(a;q^k)_0 = 1, \quad (a;q^k)_n = \prod_{j=1}^{n}(1-aq^{k(j-1)}), \quad (a;q^k)_\infty = \prod_{j=1}^{\infty}(1-aq^{k(j-1)}),$$

$$(a_1,a_2,\ldots,a_r;q^k)_n = (a_1;q^k)_n(a_2;q^k)_n\ldots(a_r;q^k)_n,$$

$$\left({q^{a_1},q^{a_2},\ldots,q^{a_k} \atop q^{b_1},q^{b_2},\ldots,q^{b_k}};q^b\right)_\infty = \frac{(q^{a_1},q^{a_2},\ldots,q^{a_k};q^b)_\infty}{(q^{b_1},q^{b_2},\ldots,q^{b_k};q^b)_\infty}.$$

We shall be using the quintuple product identity in the form [**8**, p. 525]

$$\left(\begin{matrix} a^2q, & a^{-2}, & q \\ aq, & a^{-1} & \end{matrix};q\right)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n a^{3n} q^{\frac{3n^2+n}{2}} + \sum_{n=-\infty}^{\infty}(-1)^n a^{3n-1} q^{\frac{3n^2-n}{2}}. \tag{1.1}$$

Making $q \to q^2$ and writing $-a$ for $a$, we have

$$\left(\begin{matrix} a^2q^2, & a^{-2}, & q^2 \\ -aq^2, & -a^{-1} & \end{matrix};q^2\right)_\infty = \sum_{n=-\infty}^{\infty} a^{3n} q^{3n^2+n} - \sum_{n=-\infty}^{\infty} a^{3n-1} q^{3n^2-n}. \tag{1.2}$$

We prove the following lemma which will be used later

Lemma 1.1. *We have* $C(q) = \dfrac{1}{(q^4;q^4)_\infty}\left[\displaystyle\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+n}(1+q^{4n+1})\right]$.

Proof. Making $q \to q^4$ and putting $a = 1/q$ in (1.1), we have

$$\begin{aligned}\frac{(q^2;q^4)_\infty^2}{(q;q^4)_\infty(q^3;q^4)_\infty} &= \frac{1}{(q^4;q^4)_\infty}\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2-n} + \sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2-5n+1}\right] \\ &= \frac{1}{(q^4;q^4)_\infty}\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+n} + \sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+5n+1}\right] \\ &= \frac{1}{(q^4;q^4)_\infty}\left[\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+n}(1+q^{4n+1})\right],\end{aligned}$$

which is the lemma. $\square$

## 2. Sign of the coefficients

We shall prove the following theorem and show that the sign of the coefficients is periodic of period 4.

THEOREM 2.1. *We have*

$$\begin{aligned}C(q) = \frac{1}{(q^4;q^4)_\infty}&\left[\sum_{n=-\infty}^{\infty} q^{96n^2-4n} - \sum_{n=-\infty}^{\infty} q^{96n^2-68n+12}\right]\\ + \frac{q}{(q^4;q^4)_\infty}&\left[\sum_{n=-\infty}^{\infty} q^{96n^2+20n} - \sum_{n=-\infty}^{\infty} q^{96n^2-44n+4}\right]\\ - \frac{q^2}{(q^4;q^4)_\infty}&\left[\sum_{n=-\infty}^{\infty} q^{96n^2-28n} - \sum_{n=-\infty}^{\infty} q^{96n^2-92n+20}\right]\\ - \frac{q^7}{(q^4;q^4)_\infty}&\left[\sum_{n=-\infty}^{\infty} q^{96n^2-52n} - \sum_{n=-\infty}^{\infty} q^{96n^2-116n+28}\right].\end{aligned}$$

PROOF. In Lemma 1.1, we break the summation into four parts according to the residue modulo 4 of $n$, by replacing $n$ by $4n$, $4n+1$, $4n-2$, $4n-1$, to get

$$\begin{aligned}C(q) = \frac{1}{(q^4;q^4)_\infty}&\left[\sum_{n=-\infty}^{\infty} q^{96n^2-4n} + \sum_{n=-\infty}^{\infty} q^{96n^2-20n+1}\right]\\ - \frac{1}{(q^4;q^4)_\infty}&\left[\sum_{n=-\infty}^{\infty} q^{96n^2+52n+7} + \sum_{n=-\infty}^{\infty} q^{96n^2-68n+12}\right]\\ + \frac{1}{(q^4;q^4)_\infty}&\left[\sum_{n=-\infty}^{\infty} q^{96n^2-92n+22} + \sum_{n=-\infty}^{\infty} q^{96n^2-116n+35}\right]\\ - \frac{1}{(q^4;q^4)_\infty}&\left[\sum_{n=-\infty}^{\infty} q^{96n^2+44n+5} + \sum_{n=-\infty}^{\infty} q^{96n^2-28n+2}\right].\end{aligned}$$

In some sums we have shifted the index of summation. We regroup these eight sums into pairs according to the residue modulo 4 of the powers to get

$$C(q) = \frac{1}{(q^4;q^4)_\infty}\left[\sum_{n=-\infty}^{\infty} q^{96n^2-4n} - \sum_{n=-\infty}^{\infty} q^{96n^2-68n+12}\right] \tag{2.1}$$

$$+ \frac{q}{(q^4;q^4)_\infty}\left[\sum_{n=-\infty}^{\infty} q^{96n^2+20n} - \sum_{n=-\infty}^{\infty} q^{96n^2-44n+4}\right] \tag{2.2}$$

$$- \frac{q^2}{(q^4;q^4)_\infty}\left[\sum_{n=-\infty}^{\infty} q^{96n^2-28n} - \sum_{n=-\infty}^{\infty} q^{96n^2-92n+20}\right] \tag{2.3}$$

$$- \frac{q^7}{(q^4;q^4)_\infty}\left[\sum_{n=-\infty}^{\infty} q^{96n^2-52n} - \sum_{n=-\infty}^{\infty} q^{96n^2-116n+28}\right]. \tag{2.4}$$

The theorem follows from (2.1)–(2.4). $\square$

We now sum each pair of summation in (2.1)–(2.4) by quintuple product identity (1.2) by making $q \to q^{32}$ and taking $a = q^{-12}$ for (2.1), $q \to q^{32}$, $a = q^{-4}$ for (2.2), $q \to q^{32}$, $a = q^{-20}$ for (2.3), $q \to q^{32}$, $a = q^{-28}$ for (2.4), to get

$$C(q) = \frac{(q^{64};q^{64})_\infty}{(q^4;q^4)_\infty} \left[ \begin{matrix} \begin{pmatrix} q^{40}, & q^{24} \\ -q^{52}, & -q^{12} \end{pmatrix}; q^{64} \Bigg)_\infty + q \begin{pmatrix} q^{56}, & q^{8} \\ -q^{60}, & -q^{4} \end{pmatrix}; q^{64} \Bigg)_\infty \\ -q^2 \begin{pmatrix} q^{24}, & q^{40} \\ -q^{44}, & -q^{20} \end{pmatrix}; q^{64} \Bigg)_\infty - q^7 \begin{pmatrix} q^{8}, & q^{56} \\ -q^{36}, & -q^{28} \end{pmatrix}; q^{64} \Bigg)_\infty \end{matrix} \right].$$

If we define $C(q) = \sum_{n=0}^{\infty} c(n)q^n$, then

$$\sum_{n=0}^{\infty} c(4n)q^n = 1/(q^1,q^2,q^4,q^5,q^7,q^8,q^9,q^{11},q^{12},q^{14},q^{15};q^{16})_\infty (q^6,q^{26};q^{32})_\infty, \tag{2.5}$$

$$\sum_{n=0}^{\infty} c(4n+1)q^n = 1/(q^3,q^4,q^5,q^6,q^7,q^8,q^9,q^{10},q^{11},q^{12},q^{13};q^{16})_\infty (q^2,q^{30};q^{32})_\infty, \tag{2.6}$$

$$\sum_{n=0}^{\infty} c(4n+2)q^n = -1/(q^1,q^2,q^3,q^4,q^7,q^8,q^9,q^{12},q^{13},q^{14},q^{15};q^{16})_\infty (q^{10},q^{22};q^{32})_\infty, \tag{2.7}$$

$$\sum_{n=0}^{\infty} c(4n+3)q^n = -q/(q^1,q^3,q^4,q^5,q^6,q^8,q^{10},q^{11},q^{12},q^{13},q^{15};q^{16})_\infty (q^{14},q^{18};q^{32})_\infty. \tag{2.8}$$

From (2.5)–(2.8), it follows that $c(4n) > 0$, $c(4n+1) > 0$, $c(4n+2) < 0$ and $c(4n+3) < 0$ which shows that, apart from some coefficients in the beginning, the sign of the coefficients in the expansion of $C(q)$ is periodic with period 4.

## 3. Combinatorial Interpretation

We give the following combinatorial interpretations for the coefficients.

$c(4n)$ is the number of partitions of $n$ into parts which are $\pm1, \pm2, \pm4, \pm5, \pm7, \pm8 \pmod{16}$ and $\pm6 \pmod{32}$.

$c(4n+1)$ is the number of partitions of $n$ into parts which are $\pm3, \pm4, \pm5, \pm6, \pm7, \pm8 \pmod{16}$ and $\pm2 \pmod{32}$.

$c(4n+2)$ is the number of partitions of $n$ into parts which are $\pm1, \pm2, \pm3, \pm4, \pm7, \pm8 \pmod{16}$ and $\pm10 \pmod{32}$.

$c(4n+3)$ is the number of partitions of $n-1$ into parts which are $\pm1, \pm3, \pm4, \pm5, \pm6, \pm8 \pmod{16}$ and $\pm14 \pmod{32}$.

**Acknowledgments.** I am thankful to the referee for his valuable comments and suggestions.

## References

1. C. Adiga, N. A. S. Bulkhali, Y. Simsek, H. M. Srivastava, *A continued fraction of Ramanujan and some Ramanujan–Weber class invariants*, Filomat **31** (2017), 3975–3997.
2. G. E. Andrews, *Applications of basic hypergeometric functions*, SIAM Rev. **16** (1974), 441–484.
3. ______, *An introduction to Ramanujan's "lost" notebook*, Am. Math. Monthly **86** (1979), 89–108.
4. ______, *Ramanujan's "lost" notebook III: The Rogers-Ramanujan continued fraction*, Adv. Math. **41** (1981), 186–208.
5. B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer-Verlag, New York, 1991.
6. ______, *Number Theory in the Spirit of Ramanujan*, Am. Math. Soc., Providence, Rhode Island, 2006.
7. G. Gasper, M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 1990.
8. M. D. Hirschhorn, *On the expansion of Ramanujan's continued fraction*, Ramanujan J. **2** (1998), 521–527.
9. B. Srivastava, *A note on an analogous continued fraction of Ramanujan*, Kyungpook Math. J. **45** (2005), 603–607.
10. ______, *Some identities connected with a continued fraction of Ramanujan*, Taiwanese J. Math. **16** (2012), 829–838.
11. H. M. Srivastava, M. P. Chaudhary, *Some relationships between q-product identities, combinatorial partition identities and continued-fraction identities*, Adv. Stud. Contemp. Math., Kyungshang **25** (2015), 265–272.
12. H. M. Srivastava, R. Srivastava, M. P. Chaudhary, S. Uddin, *A family of theta-function identities based upon combinatorial partition identities related to Jacobi's triple-product identity*, Mathematics **8** (2020), ID 918, 1–14.

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(Received 26 02 2021)
(Revised 13 05 2021)