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SOME IDENTITIES ON MODULAR EQUATIONS OF DEGREE 5

Devadas Anu Radha and
Belakavadi Radhakrishna Srivatsa Kumar*

Abstract. S. Ramanujan devoted more space for modular equations than any other topic and documented several identities. Motivated by his work, in this paper we establish five modular equations of composite degrees.

1. Introduction

Throughout the paper, we adopt the standard q-series representation and f_k is defined as

$$f_k := (q^k; q^k)_\infty = \prod_{m=1}^{\infty} (1 - q^{mk}), \quad |q| < 1.$$

Recollecting Ramanujan's theta-function $f(a,b)$ as defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Jacobi's triple product identity [**9**, p. 35], can be restated in Ramanujan's script as $f(a,b) = (-a, -b, ab; ab)_\infty$, The most important unique cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

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*Corresponding author

Also after Ramanujan, define

$$\chi(q) := (-q; q^2)_\infty.$$

Ramanujan documented distinct identities which involve $f(-q)$, $f(-q^2)$, $f(-q^n)$ and $f(-q^{2n})$ in his second notebook [**18**] and 'Lost' notebook [**19**]. For example [**10**, p. 206], if

$$P := \frac{f(-q)}{q^{1/6} f(-q^5)} \quad \text{and} \quad Q := \frac{f(-q^2)}{q^{1/3} f(-q^{10})}$$

then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^3 + \left(\frac{P}{Q}\right)^3.$$

In Ramanujan's notebooks there are possibly more modular equations that are found in the combined work of all his forerunners. In fact Ramanujan devoted more space in his notebooks to modular equations than any other topic. He generally established simpler modular equations than his forerunners without proof. He used theta function identities to develop them. In his notebooks [**10**, **18**], Ramanujan listed 23 classic identities so called P-Q modular equations. These identities associate quotients of the eta-function which are designated by P or Q by Ramanujan. Proofs of eighteen of these P-Q identities employing various modular equations of Ramanujan have been given for the first time by Berndt [**9**] and Berndt and Zhang [**11**]. These modular equations play an extensive role in the evaluation of class invariants, continued fractions, ratio of theta functions and many more. For the wonderful work on hypergeometric functions and modular equations one may refer [**1**–**8**, **12**–**16**, **20**–**26**]. Motivated by the above work in this paper, we establish modular equation of degree 5 of various degrees {1, 2, 4, 5, 10, 20}, {1, 3, 5, 9, 15, 45}, {1, 4, 5, 16, 20, 80}. Before proving our results, first we define a modular equation of degree n. In Section 2, we list some P-Q type modular equations which we need to prove our main results and in Section 3, we prove modular equation of degree 5.

Prior to attempting to prove our main results, we define a modular equation as defined by Ramanujan. An equation of the form

$$n \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\beta\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)},$$

is called a modular equation of degree n where β has degree n over α, and

$${}_2F_1(a, b; c; x) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \quad |x| < 1,$$

denotes an ordinary hyper-geometric series with

$$(x)_n := x(x+1)(x+2)\dots(x+n-1),$$

and the multiplier

$$m := \frac{z_1}{z_n} = \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right)}.$$

2. Preliminaries

LEMMA 2.1. *If*

$$P := \frac{f(-q)}{q^{1/6} f(-q^5)} \quad \textit{and} \quad Q_n := \frac{f(-q^n)}{q^{n/6} f(-q^{5n})}$$

then

$$PQ_2 + \frac{5}{PQ_2} = \Big(\frac{Q_2}{P}\Big)^3 + \Big(\frac{P}{Q_2}\Big)^3. \tag{2.1}$$

$$(PQ_3)^3 + \frac{125}{(PQ_3)^3} + \Big(\frac{P}{Q_3}\Big)^6 - \Big(\frac{Q_3}{P}\Big)^6 + 9\Big(\frac{P}{Q_3}\Big)^3 + 9\Big(\frac{Q_3}{P}\Big)^3 = 0. \tag{2.2}$$

$$\begin{aligned}(PQ_4)^3 + \frac{125}{(PQ_4)^3} = \Big(\frac{P}{Q_4}\Big)^5 + \Big(\frac{Q_4}{P}\Big)^5 - 8\Big[\Big(\frac{P}{Q_4}\Big)^3 + \Big(\frac{Q_4}{P}\Big)^3\Big] \\ + 4\Big(\frac{P}{Q_4} + \frac{Q_4}{R}\Big) + 4/\Big(\frac{P}{Q_4} + \frac{Q_4}{P}\Big).\end{aligned} \tag{2.3}$$

For the proof of (2.1), see [**10**] and for the proof of (2.2) and (2.3), see [**19**].

LEMMA 2.2. *If*

$$P := \frac{\psi(-q)}{q^{1/2}\psi(-q^5)} \quad \textit{and} \quad Q := \frac{\psi(-q^3)}{q^{3/2}\psi(-q^{15})}$$

then

$$(PQ) + \frac{5}{(PQ)} = \Big(\frac{Q}{P}\Big)^2 - \Big(\frac{P}{Q}\Big)^2 + 3\Big[\Big(\frac{P}{Q}\Big) + \Big(\frac{Q}{P}\Big)\Big]. \tag{2.4}$$

For the proof, see [**8**].

LEMMA 2.3. *If*

$$P := \frac{\varphi(-q)}{\varphi(-q^5)} \quad \textit{and} \quad Q := \frac{\varphi(-q^2)}{\varphi(-q^{10})}$$

then

$$\Big(\frac{P}{Q}\Big)^2 + \Big(\frac{Q}{P}\Big)^2 + 4 = Q^2 + \frac{5}{Q^2}. \tag{2.5}$$

For the proof, see [**17**].

3. Modular equations of degree 5

THEOREM 3.1. *If*

$$u = \sqrt{\frac{z_1 z_2}{z_5 z_{10}}}\Big(\frac{\alpha\beta(1-\alpha)^4(1-\beta)^4}{\delta\eta(1-\delta)^4(1-\eta)^4}\Big)^{\frac{1}{24}} \quad \textit{and} \quad v = \sqrt{\frac{z_2 z_4}{z_{10} z_{20}}}\Big(\frac{\beta\gamma(1-\beta)^4(1-\gamma)^4}{\eta\zeta(1-\eta)^4(1-\zeta)^4}\Big)^{\frac{1}{24}}$$

then, we have

$$(u^6v^2(v^2+5) - v^6u^2(u^2+5))(u^2(u^2+5) - v^2(v^2+5)) = (u^6 - v^6)^2,$$

where $\beta, \gamma, \delta, \eta$ and ζ have degrees two, four, five, ten and twenty over α respectively.

PROOF. Let

$$A = \frac{f(-q)}{q^{1/6}f(-q^5)}, \quad B = \frac{f(-q^2)}{q^{1/3}f(-q^{10})} \quad \text{and} \quad C = \frac{f(-q^4)}{q^{2/3}f(-q^{20})}. \tag{3.1}$$

On employing (3.1) in (2.1), it is observed that

$$AB + \frac{5}{AB} = \Big(\frac{A}{B}\Big)^3 + \Big(\frac{B}{A}\Big)^3, \tag{3.2}$$

$$BC + \frac{5}{BC} = \Big(\frac{B}{C}\Big)^3 + \Big(\frac{C}{B}\Big)^3. \tag{3.3}$$

From Entry 12(ii) [**9**, p. 124], set $q = e^{-y}$ for $z = {}_2F_1\big(\frac{1}{2}, \frac{1}{2}; 1; x\big)$ and $y = \pi{}_2F_1\big(\frac{1}{2}, \frac{1}{2}; 1; 1-x\big)/{}_2F_1\big(\frac{1}{2}, \frac{1}{2}; 1; x\big)$, we have

$$f(-q) = \sqrt{z}\; 2^{-1/6}\Big(\frac{x(1-x)^4}{q}\Big)^{1/24}. \tag{3.4}$$

Employing (3.4) in (3.1), we observe that

$$AB = \sqrt{\frac{z_1z_2}{z_5z_{10}}}\Big(\frac{\alpha\beta(1-\alpha)^4(1-\beta)^4}{\delta\eta(1-\delta)^4(1-\eta)^4}\Big)^{1/24} = u, \tag{3.5}$$

$$BC = \sqrt{\frac{z_2z_4}{z_{10}z_{20}}}\Big(\frac{\beta\gamma(1-\beta)^4(1-\gamma)^4}{\eta\zeta(1-\eta)^4(1-\zeta)^4}\Big)^{1/24} = v. \tag{3.6}$$

Using (3.5) and (3.6) in (3.2) and (3.3), we obtain

$$B^{12} - u^2(u^2+5)B^6 + u^6 = 0, \tag{3.7}$$

$$B^{12} - v^2(v^2+5)B^6 + v^6 = 0, \tag{3.8}$$

respectively. From (3.7) and (3.8), we deduce that

$$\frac{B^{12}}{u^6v^2(v^2+5) - v^6u^2(u^2+5)} = \frac{B^6}{u^6 - v^6} = \frac{1}{u^2(u^2+5) - v^2(v^2+5)},$$

which signifies

$$B^{12} = \frac{u^6v^2(v^2+5) - v^6u^2(u^2+5)}{u^2(u^2+5) - v^2(v^2+5)}, \tag{3.9}$$

$$B^6 = \frac{u^6 - v^6}{u^2(u^2+5) - v^2(v^2+5)}. \tag{3.10}$$

On combining (3.9) and (3.10), we obtain the desired outcome. □

THEOREM 3.2. *If $\beta, \gamma, \delta, \eta$ and ζ have degrees three, five, nine, fifteen, forty five over α respectively and if*

$$u = \sqrt{\frac{z_1z_{15}}{z_3z_5}}\Big(\frac{\alpha\eta(1-\alpha)^4(1-\eta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\Big)^{\frac{1}{24}} \quad \text{and} \quad v = \sqrt{\frac{z_3z_{45}}{z_9z_{15}}}\Big(\frac{\beta\zeta(1-\beta)^4(1-\zeta)^4}{\delta\eta(1-\delta)^4(1-\eta)^4}\Big)^{\frac{1}{24}}$$

then, we have

$$(sv^9 - tu^3)(tu^9 - sv^3) = 125u^6v^6(1-u^6v^6)^2,$$

where

$$s = u^{12} + 9u^9 + 9u^3 - 1 \quad \text{and} \quad t = v^{12} + 9v^9 + 9v^3 - 1.$$

PROOF. Let

$$A = \frac{f(-q)}{q^{1/6}f(-q^5)}, \quad B = \frac{f(-q^3)}{q^{1/2}f(-q^{15})} \quad \text{and} \quad C = \frac{f(-q^9)}{q^{3/2}f(-q^{45})}. \tag{3.11}$$

On employing (3.11) in (2.2), it is observed that

$$(AB)^3 + \frac{125}{(AB)^3} = \Big(\frac{B}{A}\Big)^6 - \Big(\frac{A}{B}\Big)^6 - 9\Big(\frac{A}{B}\Big)^3 - 9\Big(\frac{B}{A}\Big)^3, \tag{3.12}$$

$$(BC)^3 + \frac{125}{(BC)^3} = \Big(\frac{C}{B}\Big)^6 - \Big(\frac{B}{C}\Big)^6 - 9\Big(\frac{B}{C}\Big)^3 - 9\Big(\frac{C}{B}\Big)^3. \tag{3.13}$$

On using (3.4) in (3.11), it is observed that

$$\frac{A}{B} = \sqrt{\frac{z_1 z_{15}}{z_3 z_5}}\Big(\frac{\alpha\eta(1-\alpha)^4(1-\eta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\Big)^{1/24} = u, \tag{3.14}$$

$$\frac{B}{C} = \sqrt{\frac{z_3 z_{45}}{z_9 z_{15}}}\Big(\frac{\beta\zeta(1-\beta)^4(1-\zeta)^4}{\delta\eta(1-\delta)^4(1-\eta)^4}\Big)^{1/24} = v. \tag{3.15}$$

Employing (3.14) and (3.15) in (3.12) and (3.13), we have

$$u^9 B^{12} + sB^6 + 125u^3 = 0, \tag{3.16}$$

$$v^3 B^{12} + tB^6 + 125v^9 = 0, \tag{3.17}$$

respectively, where $s = u^{12} + 9u^9 + 9u^3 - 1$ and $t = v^{12} + 9v^9 + 9v^3 - 1$. From (3.16) and (3.17), we have

$$\frac{B^{12}}{125(sv^9 - tu^3)} = \frac{B^6}{125u^3v^3(1 - u^6v^6)} = \frac{1}{tu^9 - sv^3},$$

which signifies

$$B^{12} = \frac{125(sv^9 - tu^3)}{tu^9 - sv^3} \tag{3.18}$$

and

$$B^6 = \frac{125u^3v^3(1 - u^6v^6)}{tu^9 - sv^3}. \tag{3.19}$$

On combining (3.18) and (3.19), we obtain the desired outcome. □

THEOREM 3.3. *If*

$$u = \sqrt{\frac{z_1 z_{20}}{z_4 z_5}}\Big(\frac{\alpha\eta(1-\alpha)^4(1-\eta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\Big)^{\frac{1}{24}} \quad \text{and} \quad v = \sqrt{\frac{z_4 z_{80}}{z_{16} z_{20}}}\Big(\frac{\beta\zeta(1-\beta)^4(1-\zeta)^4}{\delta\eta(1-\delta)^4(1-\eta)^4}\Big)^{\frac{1}{24}}$$

then, we have

$$u^3v^3(n - mu^3v^3)(m - nu^3v^3) = 125(1 - u^6v^6)^2,$$

where

$$m = s^5 - 13s^3 - 49s + 4\Big(s + \frac{1}{s}\Big), \quad n = t^5 - 13t^3 - 49t + 4\Big(t + \frac{1}{t}\Big),$$

$$s = u + \frac{1}{u}, \quad \textit{and} \quad t = v + \frac{1}{v}.$$

and $\beta, \gamma, \delta, \eta$ *and* ζ *have degrees four, five, sixteen, twenty and eighty respectively over* α.

Proof. Let

$$A = \frac{f(-q)}{q^{1/6}f(-q^5)}, \quad B = \frac{f(-q^4)}{q^{2/3}f(-q^{20})} \quad \text{and} \quad C = \frac{f(-q^{16})}{q^{8/3}f(-q^{80})}. \tag{3.20}$$

On employing (3.20) in (2.3), we have

$$(AB)^3 + \frac{125}{(AB)^3} = \Big(\frac{A}{B}\Big)^5 + \Big(\frac{B}{A}\Big)^5 - 8\Big[\Big(\frac{A}{B}\Big)^3 + \Big(\frac{B}{A}\Big)^3\Big] + 4\Big[\frac{A}{B} + \frac{B}{A}\Big] + 4/\Big[\frac{A}{B} + \frac{B}{A}\Big], \tag{3.21}$$

$$(BC)^3 + \frac{125}{(BC)^3} = \Big(\frac{B}{C}\Big)^5 + \Big(\frac{C}{B}\Big)^5 - 8\Big[\Big(\frac{B}{C}\Big)^3 + \Big(\frac{C}{B}\Big)^3\Big] + 4\Big[\frac{B}{C} + \frac{C}{B}\Big] + 4/\Big[\frac{B}{C} + \frac{C}{B}\Big]. \tag{3.22}$$

On using (3.4) in (3.20), it is observed that

$$\frac{A}{B} = \sqrt{\frac{z_1 z_{20}}{z_4 z_5}}\Big(\frac{\alpha\eta(1-\alpha)^4(1-\eta)^4}{\beta\gamma(1-\beta)^4(1-\gamma)^4}\Big)^{1/24} = u, \tag{3.23}$$

$$\frac{B}{C} = \sqrt{\frac{z_4 z_{80}}{z_{16} z_{20}}}\Big(\frac{\beta\zeta(1-\beta)^4(1-\zeta)^4}{\delta\eta(1-\delta)^4(1-\eta)^4}\Big)^{1/24} = v. \tag{3.24}$$

Using (3.23) and (3.24) in (3.21) and (3.22), we obtain

$$u^6B^{12} - mu^3B^6 + 125 = 0, \tag{3.25}$$

$$B^{12} - nv^3B^6 + 125v^6 = 0, \tag{3.26}$$

respectively and m, n are as defined as in Theorem 3.3. From (3.25) and (3.26), we deduce

$$\frac{B^{12}}{125(nv^3 - mu^3v^6)} = \frac{B^6}{125(1 - u^6v^6)} = \frac{1}{mu^3 - nu^6v^3},$$

which signifies

$$B^{12} = \frac{125v^3(n - mu^3v^3)}{u^3(m - nu^3v^3)} \tag{3.27}$$

and

$$B^6 = \frac{125(1 - u^6v^6)}{u^3(m - nu^3v^3)}. \tag{3.28}$$

On combining (3.27) and (3.28), we obtain the required result. □

THEOREM 3.4. *If*

$$u=\sqrt{\frac{z_1z_{15}}{z_3z_5}}\Big(\frac{\alpha\eta(1-\alpha)(1-\eta)}{\beta\gamma(1-\beta)(1-\gamma)}\Big)^{1/8}\quad and\quad v=\sqrt{\frac{z_3z_{45}}{z_9z_{15}}}\Big(\frac{\beta\zeta(1-\beta)(1-\zeta)}{\delta\eta(1-\delta)(1-\eta)}\Big)^{1/8}$$

then, we have

$$(ut-v^3s)(vs-u^3t)=5u^2v^2(1-u^2v^2),$$

where $s=1+3u+3u^3-u^4$, $t=1+3v+3v^3-v^4$, *and* β,γ,δ,η *and* ζ *have degrees three, five, nine, fifteen and forty five respectively over* α.

PROOF. Let

$$A=\frac{\psi(-q)}{q^{1/2}\psi(-q^5)},\quad B=\frac{\psi(-q^3)}{q^{3/2}\psi(-q^{15})}\quad \text{and}\quad C=\frac{\psi(-q^9)}{q^{9/2}\psi(-q^{45})} \tag{3.29}$$

On employing (3.29) in (2.4), it is observed that

$$AB+\frac{5}{AB}=\Big(\frac{B}{A}\Big)^2-\Big(\frac{A}{B}\Big)^2+3\Big(\frac{A}{B}+\frac{B}{A}\Big), \tag{3.30}$$

$$BC+\frac{5}{BC}=\Big(\frac{C}{B}\Big)^2-\Big(\frac{B}{C}\Big)^2+3\Big(\frac{B}{C}+\frac{C}{B}\Big). \tag{3.31}$$

From [**9**, p. 123], we have

$$\psi(-q)=\sqrt{\frac{z}{2}}\Big(\frac{x(1-x)}{q}\Big)^{1/8}. \tag{3.32}$$

Using (3.32) in (3.29), it is observed that

$$\frac{A}{B}=\sqrt{\frac{z_1z_{15}}{z_3z_5}}\Big(\frac{\alpha\eta(1-\alpha)(1-\eta)}{\beta\gamma(1-\beta)(1-\gamma)}\Big)^{1/8}=u, \tag{3.33}$$

$$\frac{B}{C}=\sqrt{\frac{z_3z_{45}}{z_9z_{15}}}\Big(\frac{\beta\zeta(1-\beta)(1-\zeta)}{\delta\eta(1-\delta)(1-\eta)}\Big)^{1/8}=v. \tag{3.34}$$

Employing (3.33) and (3.34) in (3.30) and (3.31), we have

$$u^3B^4-sB^2+5u=0, \tag{3.35}$$

$$vB^4-tB^2+5v^3=0, \tag{3.36}$$

here s and t are as given in Theorem 3.4. From (3.35) and (3.36), we deduce that

$$\frac{B^4}{5ut-5v^3s}=\frac{B^2}{5uv(1-u^2v^2)}=\frac{1}{vs-u^3t},$$

which signifies

$$B^4=\frac{5(ut-v^3s)}{sv-u^3t}, \tag{3.37}$$

$$B^2=\frac{5uv(1-u^2v^2)}{sv-u^3t}. \tag{3.38}$$

On combining (3.37) and (3.38) and streamlining, we obtain the required result. □

THEOREM 3.5. *If*

$$u=\sqrt{\frac{z_1z_2}{z_5z_{20}}}\Big(\frac{(1-\alpha)(1-\beta)}{(1-\delta)(1-\eta)}\Big)^{1/4} \quad \textit{and} \quad v=\sqrt{\frac{z_2z_4}{z_{10}z_{20}}}\Big(\frac{(1-\beta)(1-\gamma)}{(1-\eta)(1-\zeta)}\Big)^{1/4}$$

then, we have

$$(u^2v^4(4-s)-u^4v^2(4-t))(v^2(4-t)-u^2(4-s))=(u^4-v^4)^2,$$

where

$$s=\frac{z_2}{z_{10}}\Big(\frac{1-\beta}{1-\eta}\Big)^{1/2}+5\frac{z_{10}}{z_2}\Big(\frac{1-\eta}{1-\beta}\Big)^{1/2} \quad \textit{and}\ t=\frac{z_4}{z_{20}}\Big(\frac{1-\gamma}{1-\zeta}\Big)^{1/2}+5\frac{z_{20}}{z_4}\Big(\frac{1-\zeta}{1-\gamma}\Big)^{1/2},$$

and β,γ,δ,η and ζ have degrees two, four, five, ten and twenty over α respectively.

PROOF. Let

$$A=\frac{\varphi(-q)}{\varphi(-q^5)},\quad B=\frac{\varphi(-q^2)}{\varphi(-q^{10})}\quad \text{and}\quad C=\frac{\varphi(-q^4)}{\varphi(-q^{20})}. \tag{3.39}$$

On employing (3.39) in (2.5), we have

$$\Big(\frac{A}{B}\Big)^2+\Big(\frac{B}{A}\Big)^2+4=B^2+\frac{5}{B^2}, \tag{3.40}$$

$$\Big(\frac{B}{C}\Big)^2+\Big(\frac{C}{B}\Big)^2+4=C^2+\frac{5}{C^2}. \tag{3.41}$$

From [**9**, p.122], we have

$$\varphi(-q)=\sqrt{z}(1-x)^{1/4}. \tag{3.42}$$

Using (3.42) in (3.39), we have

$$AB=\sqrt{\frac{z_1z_2}{z_5z_{20}}}\Big(\frac{(1-\alpha)(1-\beta)}{(1-\delta)(1-\eta)}\Big)^{1/4}=u, \tag{3.43}$$

$$BC=\sqrt{\frac{z_2z_4}{z_{10}z_{20}}}\Big(\frac{(1-\beta)(1-\gamma)}{(1-\eta)(1-\zeta)}\Big)^{1/4}=v. \tag{3.44}$$

Employing (3.43) and (3.44) in (3.40) and (3.41), we have

$$B^8+u^2(4-s)B^4+u^4=0, \tag{3.45}$$

$$B^8+v^2(4-t)B^4+v^4=0, \tag{3.46}$$

respectively and s and t are as defined as in Theorem 3.5. From (3.45) and (3.46), we deduce

$$\frac{B^8}{u^2v^4(4-s)-u^4v^2(4-t)}=\frac{B^4}{u^4-v^4}=\frac{1}{v^2(4-t)-u^2(4-s)},$$

which signifies

$$B^8=\frac{u^2v^4(4-s)-u^4v^2(4-t)}{v^2(4-t)-u^2(4-s)}, \tag{3.47}$$

$$B^4=\frac{u^4-v^4}{v^2(4-t)-u^2(4-s)}. \tag{3.48}$$

On combining (3.47) and (3.48) and streamlining, we obtain the required result. □

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Department of Mathematics
Manipal Institute of Technology
Manipal Academy of Higher Education
Manipal
India
anurad13@gmail.com
sri_vatsabr@yahoo.com

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