PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 110(124) (2021), 91–102 DOI: https://doi.org/10.2298/PIM2124091B

ON A TYPE OF QUARTER-SYMMETRIC NON-RECURRENT METRIC CONNECTION ON A P-SASAKIAN MANIFOLD

Ajit Barman

ABSTRACT. We study a Para-Sasakian manifold admitting a type of quarter-symmetric non-recurrent-metric connection whose concircular curvature tensor satisfies certain curvature conditions.

1. Introduction

In 1977, Adati and Matsumoto [**1**] defined Para-Sasakian and Special Para-Sasakian manifolds which are special classes of an almost paracontact manifold introduced by Sato [**10**]. Para-Sasakian manifolds have been studied by De and Pathak [**5**], Matsumoto, Ianus and Mihai [**9**], Barman [**2**, **3**] and many others.

In 1924, Friedmann and Schouten [**6**] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\bar{\nabla}$ on a Riemannian manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\bar{\nabla}$ satisfies $T(X,Y) = u(Y)X - u(X)Y$, where u is a 1-form and ρ is a vector field defined by $u(X) = g(X,\rho)$, for all vector fields $X \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

In 1975, Golab [**7**] defined and studied quarter-symmetric connection in differentiable manifolds with affine connections. A liner connection ∇ on an n-dimensional Riemannian manifold (M,g) is called a quarter-symmetric connection [**7**] if its torsion tensor T satisfies $T(X,Y) = u(Y)\phi X - u(X)\phi Y$, where ϕ is a (1,1) tensor field.

In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [**6**]. Thus the notion of the quarter-symmetric connection generalizes the notion of the semi-symmetric connection.

If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition $(\bar{\nabla}_X g)(Y,Z) = 0$, for all $X,Y,Z \in \chi(M)$, then $\bar{\nabla}$ is said to be a quarter-symmetric

2010 *Mathematics Subject Classification:* 53C15; 53C25.

Key words and phrases: para-Sasakian manifold, quarter-symmetric non-recurrent metric connection, concircular curvature tensor, ξ-concircularly flat, ϕ-concircularly flat, η-Einstein manifold.

Communicated by Zoran Rakić.

metric connection and otherwise, $\bar{\nabla}$ is said to be a quarter-symmetric non-metric connection. A quarter-symmetric non-metric connection $\bar{\nabla}$ whose torsion tensor T satisfies $T(X,Y) = u(Y)\phi X - u(X)\phi Y$ and $(\bar{\nabla}_X g)(Y,Z) = 2u(X)g(\phi Y, Z) \neq 0$, is said to be the quarter-symmetric non-recurrent-metric connection.

A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [**8**, **11**]. A concircular transformation is always a conformal transformation [**8**]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism (see also [**4**]). An interesting invariant of a concircular transformation is the concircular curvature tensor $\overline{\mathbb{W}}$. It is defined in [**11**, **12**]

$$\overline{\mathbb{W}}(X,Y)Z = \bar{R}(X,Y)Z - \frac{\bar{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \tag{1.1}$$

From (1.1), it follows that

(1.2)

$$\widetilde{\overline{\mathbb{W}}}(X,Y,Z,U) = \tilde{\bar{R}}(X,Y,Z,U) - \frac{\bar{r}}{n(n-1)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)],$$

$$\widetilde{\overline{\mathbb{W}}}(X,Y,Z,U) = g(\overline{\mathbb{W}}(X,Y)Z,U)\tilde{\bar{R}}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U),$$

where $X,Y,Z,U \in \chi(M)$ and $\bar{R}$ is the curvature tensor and $\bar{r}$ is the scalar curvature with respect to the quarter-symmetric non-recurrent-metric connection respectively. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

A Riemannian manifold M is locally symmetric if its curvature tensor R satisfies $\nabla R = 0$. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and in turn their generalizations. A Riemannian manifold M is said to be semi-symmetric if its curvature tensor R satisfies

$$R(X,Y).R = 0,$$

where $R(X,Y)$ acts on R as a derivation.

The present paper is organized as follows: Section 2 contains some prerequisites of P-Sasakian manifolds. In Section 3 we discuss a type of quarter-symmetric non-recurrent metric connection. In Section 4, we establish the relation of the curvature tensor between the Levi-Civita connection and the quarter-symmetric non-recurrent metric connection of a P-Sasakian manifold. In Section 5, we study ξ-concircularly flat P-Sasakian manifolds with respect to the quarter-symmetric non-recurrent metric connection. In Section 6, we study ϕ-concircularly flat P-Sasakian manifolds with respect to the quarter-symmetric non-recurrent metric connection. Now, section-wise we investigate the curvature conditions $\overline{\mathbb{W}}.\overline{\mathbb{W}} = 0$ and $\bar{R}.\overline{\mathbb{W}} = 0$ in a P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ respectively. Finally, we construct an example of a

5-dimensional P-Sasakian manifold admitting the quarter-symmetric non-recurrent metric connection which verifies the results of Section 4 Section 5 and Section 6.

2. P-Sasakian manifolds

An n-dimensional differentiable manifold M is said to be an almost para-contact manifold (ϕ, ξ, η, g), if there exists ϕ is a $(1, 1)$ tensor field, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M which satisfy the conditions

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1, \quad g(X, \xi) = \eta(X), \tag{2.1}$$

$$\phi^2(X) = X - \eta(X)\xi,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X, \tag{2.3}$$

for any vector fields X, Y on M.

If moreover, (ϕ, ξ, η, g) satisfy the conditions $d\eta = 0$, $\nabla_X \xi = \phi X$, and

$$(\nabla_X \phi)Y = -g(X, Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.4}$$

then M is called a para-Sasakian manifold or briefly a P-Sasakian manifold.

In a P-Sasakian manifold, the following relations hold [**1**, **10**]:

$$\eta(R(X, Y)Z) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X),$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.5}$$

$$R(\xi, X)\xi = X - \eta(X)\xi,$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.6}$$

$$S(X, \xi) = -(n - 1)\eta(X),$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \tag{2.7}$$

where R and S are the curvature tensor and the Ricci tensor of the Levi-Civita connection respectively.

3. Quarter-symmetric non-recurrent metric connection

Let M be an n-dimensional P-Sasakian manifold with Riemannian metric g. If $\bar{\nabla}$ is the quarter-symmetric non-metric connection of a P-Sasakian manifold M, a linear connection $\bar{\nabla}$ is given by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y. \tag{3.1}$$

Using (3.1), the torsion tensor T of M with respect to the connection $\bar{\nabla}$ is

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] = \eta(Y)\phi X - \eta(X)\phi Y. \tag{3.2}$$

The connection $\bar{\nabla}$ is called a quarter-symmetric connection.

Further using (3.1), we have

$$(\bar{\nabla}_X g)(Y, Z) = \nabla_X g(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = 2\eta(X)g(\phi Y, Z) \neq 0. \tag{3.3}$$

A relation satisfying (3.3) is said to be non-recurrent metric connection. Therefore, $\bar{\nabla}$ defined by (3.1) satisfying (3.2) and (3.3) is a type of quarter-symmetric non-recurrent metric connection.

Conversely, we show that a linear connection $\bar{\nabla}$ defined on M satisfying (3.2) and (3.3) is given by (3.1). Let H be a tensor field of type $(1,2)$ and

$$\bar{\nabla}_X Y = \nabla_X Y + H(X,Y). \tag{3.4}$$

Then we conclude that

$$T(X,Y) = H(X,Y) - H(Y,X). \tag{3.5}$$

Further using (3.4), it follows that

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) &= \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y,Z) - g(Y,\bar{\nabla}_X Z)\\ &= -g(H(X,Y),Z) - g(Y,H(X,Z)).\end{aligned} \tag{3.6}$$

In view of (3.3) and (3.6), it yields

$$g(H(X,Y),Z) + g(Y,H(X,Z)) = -2\eta(X)g(\phi Y,Z). \tag{3.7}$$

Also using (3.7) and (3.5), we derive that

$$\begin{aligned}&g(T(X,Y),Z) + g(T(Z,X),Y) + g(T(Z,Y),X)\\ &\quad = 2g(H(X,Y),Z) - 2\eta(Z)g(X,\phi Y) + 2\eta(Y)g(\phi X,Z) + 2\eta(X)g(\phi Y,Z).\end{aligned}$$

From the above equation, we have

$$\begin{aligned}g(H(X,Y),Z) = {}&\frac{1}{2}[g(T(X,Y),Z) + g(T(Z,X),Y) + g(T(Z,Y),X)]\\ &+ \eta(Z)g(X,\phi Y) - \eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z).\end{aligned} \tag{3.8}$$

Let T' be a tensor field of type $(1,2)$ given by

$$g(T'(X,Y),Z) = g(T(Z,X),Y). \tag{3.9}$$

Adding (2.1), (3.2) and (3.9), we obtain

$$T'(X,Y) = \eta(X)\phi Y - g(\phi X,Y)\xi. \tag{3.10}$$

From (3.8), we have by using (3.9) and (3.10)

$$\begin{aligned}&g(H(X,Y),Z) = \tfrac{1}{2}[g(T(X,Y),Z) + g(T'(X,Y),Z) + g(T'(Y,X),Z)]\\ &\eta(Z)g(X,\phi Y) - \eta(Y)g(\phi X,Z) - \eta(X)g(\phi Y,Z) = -\eta(X)g(\phi Y,Z).\end{aligned} \tag{3.11}$$

Now contracting Z in (3.11) and using (2.1), it implies that

$$H(X,Y) = -\eta(X)\phi Y. \tag{3.12}$$

Combining (3.4) and (3.12), it follows that $\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y$.

Now, we are in a strong position to state the following theorem:

THEOREM 3.1. *The linear connection $\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y$ is a quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ on P-Sasakian manifolds.*

4. Curvature tensor of a P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection

THEOREM 4.1. *For a P-Sasakian manifold M with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$*

(i) *The curvature tensor $\bar{R}$ is given by $\bar{R}(X,Y)Z = R(X,Y)Z - \eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X$,*

(ii) *The Ricci tensor $\bar{S}$ is given by $\bar{S}(Y,Z) = S(Y,Z) - g(Y,Z) + n\eta(Y)\eta(Z)$,*

(iii) *The scalar curvatures with respect to the Levi-Civita connection ∇ and the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ are equal,*

(iv) $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$,

(v) *The Ricci tensor with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ is symmetric.*

PROOF. Let M be a P-Sasakian manifold. A relation between the curvature tensor $\bar{R}$ of the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ and the curvature tensor R of the Levi-Civita connection ∇ is given by

$$\bar{R}(X,Y)Z = R(X,Y)Z - (\nabla_X\eta)(Y)\phi Z + (\nabla_Y\eta)(X)\phi Z + \eta(X)(\nabla_Y\phi)(Z) - \eta(Y)(\nabla_X\phi)(Z). \tag{4.1}$$

Then by making use of (2.3), (2.4) and (4.1), we get

$$\bar{R}(X,Y)Z = R(X,Y)Z - \eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi - \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X. \tag{4.2}$$

From (4.2), we obtain that the curvature tensor $\bar{R}$ satisfies

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z.$$

Taking the inner product of (4.2) with U, it follows that

$$\tilde{\bar{R}}(X,Y,Z,U) = \tilde{R}(X,Y,Z,U) - \eta(X)\eta(U)g(Y,Z) + \eta(Y)\eta(U)g(X,Z) - \eta(X)\eta(Z)g(Y,U) + \eta(Y)\eta(Z)g(X,U), \tag{4.3}$$

where $\tilde{\bar{R}}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U)$.

Now putting $X = \xi$ in (4.2) and using (2.1) and (2.5), we have

$$\bar{R}(\xi,Y)Z = 2\eta(Y)\eta(Z)\xi - 2g(Y,Z)\xi. \tag{4.4}$$

Again putting $Z = \xi$ in (4.2) and using (2.1) and (2.6), we get

$$\bar{R}(X,Y)\xi = 0. \tag{4.5}$$

Let $\{e_1,\ldots,e_n\}$ be a local orthonormal basis of vector fields in M. Then by putting $X = U = e_i$ in (4.3), summing over i, $1 \leqslant i \leqslant n$ and using (2.1), we get

$$\bar{S}(Y,Z) = S(Y,Z) - g(Y,Z) + n\eta(Y)\eta(Z), \tag{4.6}$$

where $\bar{S}$ is the Ricci tensor of the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$.

Putting $Y = Z = e_i$ in (4.6), summing over i, $1 \leqslant i \leqslant n$ and using (2.1), we have

$$\bar{r} = r, \tag{4.7}$$

where r and $\bar{r}$ are the scalar curvatures with respect to the Levi-Civita connection ∇ and the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ respectively. □

5. ξ-concircularly flat in P-Sasakian manifolds with respect to the quarter-symmetric non-recurrent-metric connection

DEFINITION 5.1. A P-Sasakian manifold is said to be ξ-concircularly flat with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ if $\overline{W}(X,Y)\xi = 0$, where $X, Y \in \chi(M)$.

THEOREM 5.1. *An n-dimensional P-Sasakian manifold is ξ-concircularly flat with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ if the scalar curvature vanishes with respect to the Levi-Civita connection.*

PROOF. Putting (4.2) and (4.7) in (1.1), we get

$$\begin{aligned} \overline{\mathbb{W}}(X,Y)Z = R(X,Y)Z &- \eta(X)g(Y,Z)\xi + \eta(Y)g(X,Z)\xi \\ &- \eta(X)\eta(Z)Y + \eta(Y)\eta(Z)X \\ &- \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{5.1}$$

From (5.1), it yields

$$\overline{\mathbb{W}}(X,Y)Z = -\overline{\mathbb{W}}(Y,X)Z.$$

Also putting $X = \xi$ in (5.1) and using (2.1) and (2.5), it follows that

$$\overline{\mathbb{W}}(\xi,Y)Z = \frac{r}{n(n-1)}\eta(Z)Y - \left[2 + \frac{r}{n(n-1)}\right]g(Y,Z)\xi + 2\eta(Y)\eta(Z)\xi. \tag{5.2}$$

Again putting $Z = \xi$ in (5.1) and using (2.1) and (2.6), we obtain

$$\overline{\mathbb{W}}(X,Y)\xi = \frac{r}{n(n-1)}[\eta(X)Y - \eta(Y)X]. \tag{5.3}$$

If $r = 0$, then $\overline{\mathbb{W}}(X,Y)\xi = 0$. □

6. ϕ-concircularly flat P-Sasakian manifolds with respect to the quarter-symmetric non-recurrent metric connection

DEFINITION 6.1. A P-Sasakian manifold is said to be ϕ-concircularly flat with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ if

$$\widetilde{\overline{\mathbb{W}}}(\phi X, \phi Y, \phi Z, \phi U) = 0, \tag{6.1}$$

where $X, Y, Z, U \in \chi(M)$.

DEFINITION 6.2. A P-Sasakian manifold is said to be an η-Einstein manifold if its Ricci tensor S of the Levi-Civita connection is of the form $S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y)$, where a and b are smooth functions on the manifold.

THEOREM 6.1. *If a P-Sasakian manifold is ϕ-concircularly flat with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$, then the manifold is an η-Einstein manifold.*

PROOF. Putting $X = \phi X, Y = \phi Y, Z = \phi Z$ and $U = \phi U$ in (1.2), we obtain

$$\widetilde{\mathbb{W}}(\phi X, \phi Y, \phi Z, \phi U) = \tilde{\tilde{R}}(\phi X, \phi Y, \phi Z, \phi U) - \frac{\bar{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)]. \tag{6.2}$$

Using (2.1) and (4.3) in (6.2), we have

$$\widetilde{\mathbb{W}}(\phi X, \phi Y, \phi Z, \phi U) = \widetilde{R}(\phi X, \phi Y, \phi Z, \phi U) - \frac{\bar{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)]. \tag{6.3}$$

Since the P-Sasakian manifold is ϕ-concircularly flat with respect to the quarter-symmetric non-recurrent metric connection. Now Combining (4.7), (6.1) and (6.3), it follows that

$$\widetilde{R}(\phi X, \phi Y, \phi Z, \phi U) = \frac{r}{n(n-1)}\big[g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)\big]. \tag{6.4}$$

Let $\{e_1, \ldots, e_{n-1}, \xi\}$ be a local orthonormal basis of vector fields in M; then $\{\phi e_1, \ldots, \phi e_{n-1}, \xi\}$ is also a local orthonormal basis. Putting $X = U = e_i$ in (6.4) and summing over $i = 1$ to $n-1$, we get

$$S(\phi Y, \phi Z) = \frac{(n-2)r}{n(n-1)}g(\phi Y, \phi Z). \tag{6.5}$$

By making use of (2.7) and (2.2) in (6.5), we have

$$S(Y, Z) = \frac{(n-2)r}{n(n-1)}g(Y, Z) - \frac{n(n-1)^2 + (n-2)r}{n(n-1)}\eta(Y)\eta(Z).$$

This result shows that the manifold is an η-Einstein manifold, where $a = \frac{(n-2)r}{n(n-1)}$, $b = -\frac{n(n-1)^2+(n-2)r}{n(n-1)}$. □

7. P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ satisfying $\overline{\mathbb{W}}.\overline{\mathbb{W}} = 0$

THEOREM 7.1. *If a P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ satisfies $\overline{\mathbb{W}}.\overline{\mathbb{W}} = 0$, then the manifold is an η-Einstein manifold.*

PROOF. We suppose that the manifold under consideration is concircular semi-symmetric with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$, that is, $(\overline{\mathbb{W}}(X, Y).\overline{\mathbb{W}})(U, V)Z = 0$. Then we have

$$(\overline{\mathbb{W}}(X, Y))\overline{\mathbb{W}}(U, V)Z - \overline{\mathbb{W}}(\overline{\mathbb{W}}(X, Y)U, V)Z - \overline{\mathbb{W}}(U, \overline{\mathbb{W}}(X, Y)V)Z - \overline{\mathbb{W}}(U, V)\overline{\mathbb{W}}(X, Y)Z = 0. \tag{7.1}$$

Putting $X = \xi$ in (7.1), it follows that

$$(\overline{\mathbb{W}}(\xi, Y))\overline{\mathbb{W}}(U, V)Z - \overline{\mathbb{W}}(\overline{\mathbb{W}}(\xi, Y)U, V)Z - \overline{\mathbb{W}}(U, \overline{\mathbb{W}}(\xi, Y)V)Z - \overline{\mathbb{W}}(U, V)\overline{\mathbb{W}}(\xi, Y)Z = 0. \tag{7.2}$$

In view of (5.2), (5.3) and (7.2), it yields

$$\begin{aligned}(\overline{\mathbb{W}}(\xi, Y)\overline{\mathbb{W}})(U, V)Z &- \frac{r}{n(n-1)}\eta(U)\overline{\mathbb{W}}(Y, V)Z \\ &+ \Big[2 + \frac{r}{n(n-1)}\Big]g(Y, U)\overline{\mathbb{W}}(\xi, V)Z - 2\eta(Y)\eta(U)\overline{\mathbb{W}}(\xi, V)Z \\ &- \frac{r}{n(n-1)}\eta(V)\overline{\mathbb{W}}(Y, U)Z + \Big[2 + \frac{r}{n(n-1)}\Big]g(Y, V)\overline{\mathbb{W}}(\xi, U)Z \\ &- 2\eta(Y)\eta(V)\overline{\mathbb{W}}(\xi, U)Z - \frac{r}{n(n-1)}\eta(Z)\overline{\mathbb{W}}(U, V)Y \\ &+ \Big[\Big\{2 + \frac{r}{n(n-1)}\Big\}g(Y, Z) - 2\eta(Y)\eta(Z)\Big]\Big[\frac{r}{n(n-1)}\{\eta(U)V - \eta(V)U\}\Big] = 0.\end{aligned} \tag{7.3}$$

And also putting $X = \xi$ in (5.3) and using (2.1), it implies that

$$\overline{\mathbb{W}}(\xi, Y)\xi = \frac{r}{n(n-1)}[Y - \eta(Y)\xi]. \tag{7.4}$$

Again putting $U = \xi$ in (7.3) and using (5.1), (5.2) and (7.4), we get

$$\begin{aligned}\frac{r}{n(n-1)}\eta(Z)\overline{\mathbb{W}}(\xi, Y)V &- \Big[2 + \frac{r}{n(n-1)}\Big]g(V, Z)\overline{\mathbb{W}}(\xi, Y)\xi \\ &+ 2\eta(V)\eta(Z)\overline{\mathbb{W}}(\xi, Y)\xi - \frac{r}{n(n-1)}\Big[R(Y, V)Z \\ &- \eta(Y)g(V, Z)\xi + \eta(V)g(Y, Z)\xi - \eta(Y)\eta(Z)V + \eta(V)\eta(Z)Y \\ &- \frac{r}{n(n-1)}\{g(V, Z)Y - g(Y, Z)V\}\Big] + \Big[\frac{r}{n(n-1)}\Big]^2\eta(Y)\eta(Z)V \\ &- \Big[2 + \frac{r}{n(n-1)}\Big]\Big[\frac{r}{n(n-1)}\Big]\eta(Y)g(V, Z)\xi - \Big[\frac{r}{n(n-1)}\Big]^2\eta(Y)\eta(Z)V \\ &- \Big[\frac{2r}{n(n-1)}\Big]\eta(V)\eta(Z)Y + \Big[2 + \frac{r}{n(n-1)}\Big]\Big[\frac{r}{n(n-1)}\Big]g(Y, V)\eta(Z)\xi \\ &- \Big[\frac{r}{n(n-1)}\Big]^2\eta(Z)\eta(V)Y + \Big[2 + \frac{r}{n(n-1)}\Big]\Big[\frac{r}{n(n-1)}\Big]g(Y, Z)V = 0.\end{aligned} \tag{7.5}$$

Now contracting Y in (7.5) and (2.1), we have

$$S(V, Z) = (n-2)\eta(Z)\eta(V) + (3-2n)g(Z, V). \tag{7.6}$$

Form (7.6), we can write $S(V, Z) = ag(V, Z) + b\eta(V)\eta(Z)$, where $a = (3-2n)$, $b = (n-2)$. This result shows that the manifold is an η-Einstein manifold. □

8. P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ satisfying $\bar{R}(\xi, Y).\overline{\mathbb{W}} = 0$

THEOREM 8.1. *An n-dimensional P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ satisfies $\bar{R}(\xi, Y).\overline{\mathbb{W}} = 0$ if the scalar curvature tensor of the manifold with respect to the Levi-Civita connection vanishes.*

PROOF. We obtain a necessary condition for a P-Sasakian manifold to satisfy the derivation condition $(\bar{R}(\xi, Y).\overline{\mathbb{W}})(U, V)Z = 0$, where $\bar{R}(\xi, Y).\bar{\mathbb{W}}$ denotes $\bar{R}(\xi, Y)$ acting on $\overline{\mathbb{W}}$ as a derivation. Then we have

$$(\bar{R}(\xi, Y))\overline{\mathbb{W}}(U, V)Z - \overline{\mathbb{W}}(\bar{R}(\xi, Y)U, V)Z - \overline{\mathbb{W}}(U, \bar{R}(\xi, Y)V)Z - \overline{\mathbb{W}}(U, V)\bar{R}(\xi, Y)Z = 0. \tag{8.1}$$

Combining (4.4) and (8.1), we get

$$(\bar{R}(\xi, Y))\overline{\mathbb{W}}(U, V)Z - 2[\eta(U)\eta(Y) - g(Y, U)]\overline{\mathbb{W}}(\xi, V)Z - 2[\eta(V)\eta(Y) - g(Y, V)]\overline{\mathbb{W}}(U, \xi)Z - 2[\eta(Z)\eta(Y) - g(Y, Z)]\overline{\mathbb{W}}(U, V)\xi = 0. \tag{8.2}$$

Putting $U = \xi$ in (8.2), we obtain

$$(\bar{R}(\xi, Y))\overline{\mathbb{W}}(\xi, V)Z - 2[\eta(V)\eta(Y) - g(Y, V)]\overline{\mathbb{W}}(\xi, \xi)Z - 2[\eta(Z)\eta(Y) - g(Y, Z)]\overline{\mathbb{W}}(\xi, V)\xi = 0. \tag{8.3}$$

Now putting $Y = \xi$ in (5.2) and using (2.1), we get

$$\overline{\mathbb{W}}(\xi, \xi)Z = 0. \tag{8.4}$$

Using (5.2), (8.4) and (7.4) in (8.3), it follows that

$$\frac{2r}{n(n-1)}\big[2\eta(Y)\eta(Z)\eta(V)\xi - \eta(Z)g(Y, V)\xi - \eta(Y)\eta(Z)V - g(Y, Z)V + \eta(V)g(Y, Z)\xi\big] = 0. \tag{8.5}$$

Again putting $Y = Z = \xi$ in (8.5) and using (2.1), we have

$$\frac{2r}{n(n-1)}[2\eta(V)\xi - V] = 0.$$

Either $r = 0$, or, $2\eta(V)\xi - V = 0$, which is not possible. Therefore, $r = 0$. □

THEOREM 8.2. *An n-dimensional P-Sasakian manifold is semi-symmetric with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ if the scalar curvature tensor of the manifold with respect to the Levi-Civita connection vanishes.*

PROOF. From the definition of concircular curvature tensor, it follows that $\bar{R}(\xi, Y).\overline{\mathbb{W}} = \bar{R}(\xi, Y).\bar{R}$. □

9. Example

In this section, we construct an example on P-Sasakian manifold with respect to the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ which verifies the results of Section 4, Section 5 and Section 6.

We consider the 5-dimensional manifold $\{(x,y,z,u,v)\in\mathbb{R}^5\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. We choose the vector fields

$$e_1=\frac{\partial}{\partial x},\ e_2=\frac{\partial}{\partial y},\ e_3=\frac{\partial}{\partial z},\ e_4=\frac{\partial}{\partial u},\ e_5=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}+u\frac{\partial}{\partial u}+\frac{\partial}{\partial v},$$

which are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_i,e_j)=\begin{cases}1 & \text{if} \quad i=j\\ 0 & \text{if} \quad i\neq j;\ i,j=1,2,3,4,5.\end{cases}$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_5)$, for any $Z\in\chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1)=e_1,\quad \phi(e_2)=e_2,\quad \phi(e_3)=e_3,\quad \phi(e_4)=e_4,\quad \phi(e_5)=0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_5)=1,\quad \phi^2Z=Z-\eta(Z)e_5,\quad g(\phi Z,\phi U)=g(Z,U)-\eta(Z)\eta(U),$$

for any vector fields $Z,U\in\chi(M)$. Thus for $e_5=\xi$, the structure (ϕ,ξ,η,g) defines an almost paracontact metric structure on M. Then we have

$$\begin{aligned}
&[e_1,e_2]=0, && [e_1,e_3]=0, \quad [e_1,e_4]=0, \quad [e_1,e_5]=e_1,\\
&[e_2,e_3]=[e_2,e_4]=0, && [e_2,e_5]=e_2,\\
&[e_3,e_4]=0, && [e_3,e_5]=e_3, \quad [e_4,e_5]=e_4.
\end{aligned}$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula

$$\begin{aligned}2g(\nabla_XY,Z)&=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)\\ &\quad -g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]).\end{aligned}$$

Using Koszul's formula, we get

$$\begin{array}{lllll}
\nabla_{e_1}e_1=-e_5, & \nabla_{e_1}e_2=0, & \nabla_{e_1}e_3=0, & \nabla_{e_1}e_4=0, & \nabla_{e_1}e_5=e_1,\\
\nabla_{e_2}e_1=0, & \nabla_{e_2}e_2=-e_5, & \nabla_{e_2}e_3=0, & \nabla_{e_2}e_4=0, & \nabla_{e_2}e_5=e_2,\\
\nabla_{e_3}e_1=0, & \nabla_{e_3}e_2=0, & \nabla_{e_3}e_3=-e_5, & \nabla_{e_3}e_4=0, & \nabla_{e_3}e_5=e_3,\\
\nabla_{e_4}e_1=0, & \nabla_{e_4}e_2=0, & \nabla_{e_4}e_3=0, & \nabla_{e_4}e_4=-e_5, & \nabla_{e_4}e_5=e_4,\\
\nabla_{e_5}e_1=0, & \nabla_{e_5}e_2=0, & \nabla_{e_5}e_3=0, & \nabla_{e_5}e_4=0, & \nabla_{e_5}e_5=0.
\end{array}$$

In view of the above relations, we see that

$$\nabla_X\xi=\phi X,\quad (\nabla_X\phi)Y=-g(X,Y)\xi-\eta(Y)X+2\eta(X)\eta(Y)\xi,\quad \text{for all}\ \ e_5=\xi.$$

Therefore the manifold is a P-Sasakian manifold with the structure (ϕ, ξ, η, g). Using (3.1) in the above equations, we obtain

$$\begin{array}{lllll}\bar{\nabla}_{e_1}e_1 = -e_5, & \bar{\nabla}_{e_1}e_2 = 0, & \bar{\nabla}_{e_1}e_3 = 0, & \bar{\nabla}_{e_1}e_4 = 0, & \bar{\nabla}_{e_1}e_5 = 2e_1,\\ \bar{\nabla}_{e_2}e_1 = 0, & \bar{\nabla}_{e_2}e_2 = -e_5, & \bar{\nabla}_{e_2}e_3 = 0, & \bar{\nabla}_{e_2}e_4 = 0, & \bar{\nabla}_{e_2}e_5 = 2e_2,\\ \bar{\nabla}_{e_3}e_1 = 0, & \bar{\nabla}_{e_3}e_2 = 0, & \bar{\nabla}_{e_3}e_3 = -e_5, & \bar{\nabla}_{e_3}e_4 = 0, & \bar{\nabla}_{e_3}e_5 = 2e_3,\\ \bar{\nabla}_{e_4}e_1 = 0, & \bar{\nabla}_{e_4}e_2 = 0, & \bar{\nabla}_{e_4}e_3 = 0, & \bar{\nabla}_{e_4}e_4 = -e_5, & \bar{\nabla}_{e_4}e_5 = 2e_4,\\ \bar{\nabla}_{e_5}e_1 = -e_1, & \bar{\nabla}_{e_5}e_2 = -e_2, & \bar{\nabla}_{e_5}e_3 = -e_3, & \bar{\nabla}_{e_5}e_4 = -e_4, & \bar{\nabla}_{e_5}e_5 = e_5.\end{array}$$

Now, we can easily obtain the non-zero components of the curvature tensors

$$\begin{array}{llll}R(e_1,e_2)e_1 = e_2, & R(e_1,e_2)e_2 = -e_1, & R(e_1,e_3)e_1 = e_3, & R(e_1,e_3)e_3 = -e_1,\\ R(e_1,e_4)e_1 = e_4, & R(e_1,e_4)e_4 = -e_1, & R(e_1,e_5)e_1 = e_5, & R(e_1,e_5)e_5 = -e_1,\\ R(e_2,e_3)e_2 = e_3, & R(e_2,e_3)e_3 = -e_2, & R(e_2,e_4)e_2 = e_4, & R(e_2,e_4)e_4 = -e_2,\\ R(e_2,e_5)e_2 = e_5, & R(e_2,e_5)e_5 = -e_2, & R(e_3,e_4)e_3 = e_4, & R(e_3,e_4)e_4 = -e_3,\\ R(e_3,e_5)e_3 = e_5, & R(e_3,e_5)e_5 = -e_3, & R(e_4,e_5)e_4 = e_5, & R(e_4,e_5)e_5 = -e_4.\end{array}$$

and

$$\begin{array}{llll}\bar{R}(e_1,e_2)e_1 = e_2, & \bar{R}(e_1,e_2)e_2 = -e_1, & \bar{R}(e_1,e_3)e_1 = e_3, & \bar{R}(e_1,e_3)e_3 = -e_1,\\ \bar{R}(e_1,e_4)e_1 = e_4, & \bar{R}(e_1,e_4)e_4 = -e_1, & \bar{R}(e_1,e_5)e_1 = 2e_5, & \bar{R}(e_1,e_5)e_5 = 0,\\ \bar{R}(e_2,e_3)e_2 = e_3, & \bar{R}(e_2,e_3)e_3 = -e_2, & \bar{R}(e_2,e_4)e_2 = e_4, & \bar{R}(e_2,e_4)e_4 = -e_2,\\ \bar{R}(e_2,e_5)e_2 = 2e_5, & \bar{R}(e_2,e_5)e_5 = 0, & \bar{R}(e_3,e_4)e_3 = e_4, & \bar{R}(e_3,e_4)e_4 = -e_3,\\ \bar{R}(e_3,e_5)e_3 = 2e_5, & \bar{R}(e_3,e_5)e_5 = 0, & \bar{R}(e_4,e_5)e_4 = 2e_5, & \bar{R}(e_4,e_5)e_5 = 0.\end{array}$$

With the help of the above results we find the Ricci tensors

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = S(e_4,e_4) = S(e_5,e_5) = -4,$$
$$\bar{S}(e_1,e_1) = \bar{S}(e_2,e_2) = \bar{S}(e_3,e_3) = \bar{S}(e_4,e_4) = -5, \quad \bar{S}(e_5,e_5) = 0.$$

Also, it follows that the scalar curvature tensors with respect to the Levi-Civita connection and the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$ are $r = -20$ and $\bar{r} = -20$ respectively. Thus the manifold under consideration satisfies equations (4.5), (4.6) and (4.7) of Section 4.

Let X, Y, Z and U be any four vector fields given by

$$X = a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5, \quad Y = b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5,$$
$$Z = c_1e_1 + c_2e_2 + c_3e_3 + c_4e_4 + c_5e_5, \quad W = d_1e_1 + d_2e_2 + d_3e_3 + d_4e_4 + d_5e_5$$

where a_i, b_i, c_i, d_i, for all $i = 1, 2, 3, 4, 5$ are all non-zero real numbers.

Using the above curvature tensors and the scalar curvatures of the quarter-symmetric non-recurrent metric connection $\bar{\nabla}$, we obtain

$$\overline{\mathbb{W}}(X,Y)\xi = [(a_5b_1 - a_1b_5)e_1 + (a_5b_2 - a_2b_5)e_2 + (a_5b_3 - a_3b_5)e_3 + (a_5b_4 - a_4b_5)e_4].$$

Hence P-Sasakian manifolds will be ξ-concircularly flat with respect to the quarter-symmetric non-recurrent metric connections $\bar{\nabla}$ if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{a_4}{b_4} = \frac{a_5}{b_5}$ and $a_1 b_2 c_2 d_1 + a_2 b_1 c_1 d_2 = \frac{4}{3}$ which verifies the result of Section 5.

From the above relations, we see that $\overline{\mathbb{W}}(\phi X, \phi Y, \phi Z, \phi U) = 0$.

The above arguments tell us that the 5-dimensional P-Sasakian manifolds with respect to the quarter-symmetric non-recurrent metric connections $\bar{\nabla}$ under consideration agree with the Section 6.

Acknowledgement. The author is thankful to the referee for his/ her valuable comments towards the improvement of my paper.

References

1. T. Adati, K. Matsumoto, *On conformally recurrent and conformally symmetric P-Sasakian manifolds,* TRU Math. **13** (1977), 25–32.
2. A. Barman, *Semi-symmetric non-metric connection in a P-Sasakian manifold*, Novi Sad J. Math. **43** (2013), 117–124.
3. ______, *On Para-Sasakian manifolds admitting semi-symmetric metric connection*, Publ. Inst. Math., Nouv. Sér. **95**(109) (2014), 239–247.
4. D. E. Blair, *Inversion Theory and Conformal Mapping*, Stud. Math. Libr. **9**, American Mathematical Society, Providence, RI, 2000.
5. U. C. De, G. Pathak, *On P-Sasakian manifolds satisfying certain conditions*, J. Indian Acad. Math. **16** (1994), 72–77.
6. A. Friedmann, J. A. Schouten, *Über die Geometric der halbsymmetrischen Übertragung*, Math. Z. **21** (1924), 211–223.
7. S. Golab, *On semi-symmetric and quarter-symmetric liner connections*, Tensor, New Ser. **29** (1975), 249–254.
8. W. Kuhnel, *Conformal transformations between Einstein spaces, Conformal geometry* (Bonn, 1985/1986), 105–146, Aspects Math. **E12**, Vieweg, Braunschweig, 1988.
9. K. Matsumoto, S. Ianus, I. Mihai, *On a P-Sasakian manifolds which admit certain tensor fields*, Publ. Math. **33** (1986), 61–65.
10. I. Sato, *On a structure similar to the almost contact structure*, Tensor, New Ser. **30** (1976), 219–224.
11. K. Yano, *Concircular geometry I. Concircular transformations*, Proc. Imp. Acad. Tokyo **16** (1940), 195–200.
12. K. Yano, S. Bochner, *Curvature and Betti Numbers*, Ann. Math. Stud. **32**, Princeton University Press, Princeton, NJ, 1953.

Department of Mathematics
Ramthakur College
Agartala, West Tirpura
Tripura
India
ajitbarmanaw@yahoo.in

(Received 03 07 2016)
(Revised 09 05 2018 and 01 02 2019)