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CERTAIN SUBCLASS OF *p*-VALENT FUNCTIONS ASSOCIATED WITH BESSEL FUNCTIONS

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Abstract. We introduce the subclass $\mathcal{S}_p^j(\alpha; c, k; \phi)$, of p-valent functions associated with Bessel functions. Such results as inclusion relationships, convolution properties for this class are proved, coefficient estimates and certain integral preserving properties are also established with this class.

1. Introduction

Let $\mathcal{A}_p$ denote the class of functions of the following form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n}, \tag{1.1}$$

which are analytic in the open unite disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. For simplicity, we write $\mathcal{A}_1 =: \mathcal{A}$. Also let $\mathcal{P}$ be the class of functions of the form

$$\mathcal{P}(z) = 1 + \sum_{n=1}^{\infty} \mathcal{P}_n z^n \quad (z \in \Delta),$$

which are analytic and convex in Δ and satisfy $\text{Re}(\mathcal{P}(z)) > 0$. Let $f, g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by $g(z) = z^p + \sum_{n=1}^{\infty} b_{p+n} z^{p+n}$. Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{n=1}^{\infty} a_{n+p} b_{n+p} z^{n+p} =: (g * f)(z).$$

The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a function w analytic in Δ, with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \Delta)$, such that $f(z) = g(w(z))$ $(z \in \Delta)$. In such a case, we write

$$f \prec g \quad (z \in \Delta) \qquad \text{or} \qquad f(z) \prec g(z) \quad (z \in \Delta).$$

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If the function g is univalent in Δ, then we have (cf. [**13**]),

$$f \prec g \ (z \in \Delta) \Leftrightarrow f(0) = g(0) \ \text{ and } \ f(\Delta) \subset g(\Delta).$$

Let us consider the following second-order linear homogeneous differential equation (see for details [**1**–**3**, **5**–**7**, **14**]):

$$z^2 w''(z) + bzw'(z) + [cz^2 - \gamma^2 + (1-b)\gamma]w(z) = 0, \quad z \in \mathbb{C}. \tag{1.2}$$

The function $w_{\gamma,b,c}(z)$, which is called the generalized Bessel function of the first kind of order γ [**8**], it is defined as a particular solution of (1.2). The function $w_{\gamma,b,c}(z)$ has the familiar representation as follows

$$w_{\gamma,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c)^n}{n!\Gamma(\gamma + n + \frac{b+1}{2})} \Big(\frac{z}{2}\Big)^{2n+\gamma} \qquad (b, c, \gamma; z \in \mathbb{C}),$$

where Γ stands for Euler's gamma function.

Now, we consider the function $\delta_{\gamma,b,c}(z) : \Delta \to \mathbb{C}$, defined in terms of the generalized Bessel function $w_{\gamma,b,c}$ by the transformation

$$\delta_{\gamma,b,c}(z) = 2^{\gamma}\Gamma\Big(\gamma + \frac{b+1}{2}\Big) z^{p-\frac{\gamma}{2}} w_{\gamma,b,c}(\sqrt{z}). \tag{1.3}$$

By using the well-known Pochhammer symbol (or the shifted factorial) $(\lambda)_\mu$, defined, for $\lambda, \mu \in \mathbb{C}$ and in terms of the Euler Γ function, by

$$(\lambda)_\mu := \frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\mu = 0; \ \ \lambda \in \mathbb{C} \setminus \{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\mu, n \in \mathbb{N}; \ \ \lambda \in \mathbb{C}), \end{cases}$$

it being understood conventionally that $(0)_0 = 1$. Therefore, we obtain the following series representation for the function $\delta_{\gamma,b,c}$ given by (1.3):

$$\delta_{\gamma,b,c}(z) = z^p + \sum_{n=1}^{\infty} \frac{(-c)^n z^{n+p}}{4^n (k)_n n!},$$

where and $k = \gamma + \frac{b+1}{2} \notin \mathbb{Z}_0^- = \{0, -1, -2, \dots\}$. For convenience, we write $\delta_{k,c} = \delta_{\gamma,b,c}(z)$.

For $f(z) \in \mathcal{A}_p$, we consider the B_k^c-operator, which is defined by

$$B_k^c f(z) = \delta_{k,c}(z) * f(z) = z^p + \sum_{n=1}^{\infty} \frac{(-c)^n a_{p+n} z^{p+n}}{4^n (k)_n n!}. \tag{1.4}$$

We note that by using definition (1.4) we obtain that

$$z[B_{k+1}^c f(z)]^{(j+1)} = (p-j-k)(B_{k+1}^c f)^j(z) + k(B_k^c f)^j(z), \quad k = \gamma + \frac{b+1}{2} \notin \mathbb{Z}_0^-, \tag{1.5}$$

where

$$f^{(j)}(z) = \delta(p,j) z^{p-j} + \sum_{n=1}^{\infty} \delta(n+p, j) a_{p+n} z^{n+p-j}.$$

$$\delta(p,j) = \frac{p!}{(p-j)!} = \begin{cases} 1 & (j = 0) \\ p(p-1)\dots(p-j+1) & (j \neq 0). \end{cases}$$

DEFINITION 1.1. A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_p^j(\alpha; c, k; \phi)$ if it satisfies the subordination condition

$$\frac{z[(1-\alpha)(B_{k+1}^c f)^{(j+1)}(z) + \alpha(B_k^c f)^{(j+1)}(z)(z)]}{(1-\alpha)(B_{k+1}^c f)^j(z) + \alpha(B_k^c f)^j(z)} \prec (p-j)\phi(z) \quad (z \in \Delta) \tag{1.6}$$

for some $\alpha \geqslant 0$ and $j \in \{0, 1, \dots, p-1\}$, where $\phi \in \mathcal{P}$.

For simplicity, we write $\mathcal{S}_p^j(0; c, k; \phi) = \mathcal{S}_p^j(c, k; \phi)$. We also note that for $\phi(z) = \frac{1+(1-2\upsilon)z}{1-z}$, $(0 \leqslant \upsilon < 1)$, we have $\mathcal{S}_p^j(c; k; \phi) = \mathcal{S}_p^j(c, k; \upsilon)$.

In order to establish our main results, we shall also make use of the following lemma.

LEMMA 1.1. [**9**] *Let $\beta, \eta \in \mathbb{C}$ and let $\phi(z)$ is convex and univalent in Δ with $\phi(0) = 1$ and $\mathrm{Re}(\beta\phi(z) + \eta) > 0$ $(z \in \Delta)$. If $q(z)$ is analytic in Δ with $q(0) = 1$, then the subordination*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \eta} \prec \phi(z) \quad (z \in \Delta)$$

implies that $q(z) \prec \phi(z)$ $(z \in \Delta)$.

We prove such results as inclusion relationships, convolution properties, coefficient estimates and certain integral preserving properties for the class $\mathcal{S}_p^j(\alpha; c, k; \phi)$. The results extend those given in earlier works.

2. A set of inclusion relationships

At first, we prove some inclusion relationships for the class $\mathcal{S}_p^j(\alpha; c, k; \phi)$, which was defined in the preceding section.

THEOREM 2.1. *Let $\phi \in \mathcal{P}$ with*

$$\mathrm{Re}\left\{(p-j)\phi(z) + \frac{k}{\alpha} + j - p\right\} > 0 \quad (\alpha > 0;\ j \in \{0, 1, \dots, p-1\};\ z \in \Delta).$$

Then $\mathcal{S}_p^j(\alpha; c, k; \phi) \subset \mathcal{S}_p^j(c, k; \phi)$.

PROOF. Let $f \in \mathcal{S}_p^j(\alpha; c, k; \phi)$ and suppose that

$$\psi(z) = \frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c f)^j(z)} \quad (z \in \Delta). \tag{2.1}$$

Then ψ is analytic in Δ and $\psi(0) = 1$. It follows from (1.5) and (2.1) that

$$k + j - p + (p-j)\psi(z) = \frac{k(B_k^c f)^j(z)}{(B_{k+1}^c f)^j(z)}. \tag{2.2}$$

We can easily find from (2.1) and (2.2) that

$$z(B_k^c f)^{(j+1)}(z) = \frac{(p-j)}{k}\{z\psi'(z) + [k+j-p+(p-j)\psi(z)]\psi(z)\}(B_{k+1}^c f)^j(z). \tag{2.3}$$

It now follows from (1.5), (2.1), (2.2) and (2.3) that

$$\begin{aligned}(2.4)\quad &\frac{z[(1-\alpha)(B_{k+1}^c f)^{(j+1)}(z)+\alpha(B_k^c f)^{(j+1)}(z)]}{(p-j)(1-\alpha)(B_{k+1}^c f)^j(z)+\alpha(B_k^c f)^j(z)}\\ &=\frac{(1-\alpha)\psi(z)+\frac{\alpha}{k}\{z\psi'(z)+[(k+j-p)+(p-j)\psi(z)]\psi(z)\}}{(1-\alpha)+\frac{\alpha}{k}[(k+j-p)+(p-j)\psi(z)]}\\ &=\frac{\frac{\alpha}{k}z\psi'(z)+\left\{(1-\alpha)+\frac{\alpha}{k}[(k+j-p)+(p-j)\psi(z)]\right\}\psi(z)}{(1-\alpha)+\frac{\alpha}{k}[(k+j-p)+(p-j)\psi(z)]}\\ &=\psi(z)+\frac{z\psi'(z)}{(\frac{k}{\alpha}+j-p)+(p-j)\psi(z)}\prec\phi(z)\quad(z\in\Delta).\end{aligned}$$

Moreover, since $\operatorname{Re}\left\{(p-j)\phi(z)+\left(\frac{k}{\alpha}+j-p\right)\right\}>0$ $(\alpha>0;\ z\in\Delta)$, by Lemma 1.1 and (2.4), we know that

$$\psi(z)=\frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c f)^j(z)}\prec\phi(z)\qquad(z\in\Delta),$$

that is, that $f\in\mathcal{S}_p^j(c,k;\phi)$. This implies that $\mathcal{S}_p^j(\alpha;c,k;\phi)\subset\mathcal{S}^j(c,k;\phi)$. □

THEOREM 2.2. *Let* $\phi\in\mathcal{P}$ *with*

$$\operatorname{Re}\{(p-j)\phi(z)+(k+j+1-p)\}>0\quad(j\in\{0,1,\dots,p-1\};\ z\in\Delta).$$

Then $\mathcal{S}_p^j(c,k;\phi)\subset\mathcal{S}_p^j(c,k+1;\phi)$.

PROOF. Suppose that $f\in\mathcal{S}_p^j(c,k;\phi)$. Then we have

$$\frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c f)^j(z)}\prec\phi(z)\quad(z\in\Delta).\tag{2.5}$$

Suppose that

$$\Psi(z)=\frac{z(B_{k+2}^c f)^{(j+1)}(z)}{(p-j)(B_{k+2}^c f)^j(z)}\prec\phi(z)\quad(z\in\Delta).\tag{2.6}$$

Then Ψ is analytic in Δ and $\Psi(0)=1$. Using (1.5) and (2.6), we have

$$\Psi(z)+\frac{z\Psi'(z)}{(k+1+j-p)+(p-j)\Psi(z)}=\frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c f)^j(z)}\quad(z\in\Delta).\tag{2.7}$$

It now follows from (2.5) and (2.7) that

$$\Psi(z)+\frac{z\Psi'(z)}{(k+1+j-p)+(p-j)\Psi(z)}\prec\phi(z)\quad(z\in\Delta).\tag{2.8}$$

Moreover, since $\operatorname{Re}\{(p-j)\phi(z)+(k+j+1-p))\}>0$ $(z\in\Delta)$, by (2.8) and Lemma 1.1, we know that

$$\Psi(z)=\frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c f)^j(z)}\prec\phi(z)\qquad(z\in\Delta),$$

that is, that $f\in\mathcal{S}_p^j(c,k+1;\phi)$. This implies that $\mathcal{S}_p^j(c,k;\phi)\subset\mathcal{S}_p^j(c,k+1;\phi)$. □

3. Convolution properties

In this section, we provide some convolution properties for the class $\mathcal{S}_p^j(c,k;\phi)$.

THEOREM 3.1. *Let* $f \in \mathcal{S}_p^j(c,k;\phi)$. *Then*

$$f^{(j)}(z) = \left[z^{p-j}\exp\left((p-j)\int_0^z \frac{\phi(w(\zeta))-1}{\zeta}d\zeta\right)\right] * \left(\sum_{n=0}^{\infty}\frac{4^n(k+1)_n n!}{(-c)^n}z^{n-j+p}\right), \tag{3.1}$$

$(j \in \{0,1,\dots,p-1\})$ *where* w *is analytic in* Δ *with* $w(0)=0$ *and* $|w(z)|<1$ $(z\in\Delta)$.

PROOF. Suppose that $f \in \mathcal{S}_p^j(c,k;\phi)$. We know from (1.6) (with $\alpha = 0$) that

$$\frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c f)^j(z)} = \phi(w(z)) \qquad (z\in\Delta). \tag{3.2}$$

where w is analytic in Δ with $w(0)=0$ and $|w(z)|<1$ $(z\in\Delta)$. We next find from (3.2) that

$$\frac{(B_{k+1}^c f)^{(j+1)}(z)}{(B_{k+1}^c f)^j(z)} - \frac{p-j}{z} = (p-j)\frac{[\phi(w(z))-1]}{z} \quad (z\in\Delta). \tag{3.3}$$

Upon integrating (3.3), we have

$$\log\left(\frac{(B_{k+1}^c f)^j(z)}{z^{p-j}}\right) = (p-j)\int_0^z \frac{\phi(w(\zeta))-1}{\zeta}d\zeta,$$

or equivalently,

$$(B_{k+1}^c f)^j(z) = z^{p-j}.\exp\left((p-j)\int_0^z \frac{\phi(w(\zeta))-1}{\zeta}d\zeta\right). \tag{3.4}$$

On the other hand, we know from (1.4) that

$$(B_{k+1}^c f)^j(z) = \left(\sum_{n=0}^{\infty}\frac{(-c)^n}{4^n(k+1)_n n!}z^{n-j+p}\right) * f^{(j)}(z). \tag{3.5}$$

Assertion (3.1) of Theorem 3.1 can now easily be derived from (3.4) and (3.5). □

THEOREM 3.2. *Let* $f\in\mathcal{A}_p$ *and* $\phi\in\mathcal{P}$. *Then* $f\in\mathcal{S}_p^j(c,k;\phi)$ *if and only if*

$$\frac{1}{z^{p-j}}\left[f^{(j)}(z) * \left(\sum_{n=1}^{\infty}\frac{(-c)^n}{4^n(k+1)_n n!}[n+p-j-(p-j)\phi(e^{i\theta})]z^{n+p-j}\right)\right] \neq 0 \tag{3.6}$$

$$(z\in\Delta;\ 0\leqslant\theta<2\pi).$$

PROOF. Suppose that $f\in\mathcal{S}_p^j(c,k;\phi)$. Since the subordination condition

$$\frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c f)^j(z)} \prec \phi(z)$$

is equivalent to

$$\frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c f)^j(z)} \neq \phi(e^{i\theta}) \quad (z \in \Delta;\ 0 \leqslant \theta < 2\pi). \tag{3.7}$$

It is easy to see that (3.7) can be written as

$$\frac{1}{z^{p-j}}[z(B_{k+1}^c f)^{(j+1)}(z) - (p-j)(B_{k+1}^c f)^j(z)\phi(e^{i\theta})] \neq 0 \quad (z \in \Delta;\ 0 \leqslant \theta < 2\pi). \tag{3.8}$$

On the other hand, we know from (1.4) that

$$z(B_{k+1}^c f)^{(j+1)}(z) = \left(\sum_{n=1}^{\infty} \frac{(-c)^n}{4^n(k+1)_n n!}(n+p-j)z^{n+p-j}\right) * f^{(j)}(z). \tag{3.9}$$

Upon substituting (3.5) and (3.9) in (3.8), we can easily get the convolution property (3.6) asserted by Theorem 3.2. □

4. Coefficient estimates

In this section, we give the coefficient estimates of functions belonging to the class $\mathcal{S}_p^j(\alpha; c, k; \phi)$.

THEOREM 4.1. *If the function f of the form* (1.1) *satisfies the inequality*

$$\sum_{n=1}^{\infty} \frac{(+c)^n}{4^n(k)_n n!}\Big(\frac{k+\alpha n}{k+n}\Big)(p+n-j)\delta(p+n,j)|a_{p+n}| \leqslant (p-j)\delta(p,j)$$

($j \in \{0,1,\dots,p-1\}$, $n, p \in \mathbb{N}$), *then f belongs to the class $\mathcal{S}_p^j(\alpha; c, k; \phi)$.*

PROOF. A function f of the form (1.1) belongs to the class $\mathcal{S}_p^j(\alpha; c, k; \phi)$ if and only if there exists a function w, $|w(z)| \leqslant |z|$ ($z \in \Delta$), such that

$$\frac{z[(1-\alpha)(B_{k+1}^c f)^{(j+1)}(z) + \alpha(B_k^c f)^{(j+1)}(z)]}{(p-j)[(1-\alpha)(B_{k+1}^c f)^j(z) + \alpha(B_k^c f)^j(z)]} = \frac{1+w(z)}{1-w(z)} \qquad (z \in \Delta)$$

or equivalently

$$\left|\frac{(1-\alpha)[z(B_{k+1}^c f)^{(j+1)}(z) - (p-j)(B_{k+1}^c f)^j(z)] + \alpha[z(B_k^c f)^{(j+1)}(z) - (p-j)(B_k^c f)^j(z)]}{(1-\alpha)[z(B_{k+1}^c f)^{(j+1)}(z) + (p-j)(B_{k+1}^c f)^j(z)] + \alpha[z(B_k^c f)^{(j+1)}(z) + (p-j)(B_k^c f)^j(z)]}\right| < 1. \tag{4.1}$$

Thus, it is sufficient to prove that

$$\begin{aligned}&\big|(1-\alpha)[z(B_{k+1}^c f)^{(j+1)}(z) - (p-j)(B_{k+1}^c f)^j(z)] \\ &\qquad + \alpha[z(B_k^c f)^{(j+1)}(z) - (p-j)(B_k^c f)^j(z)]\big| \\ &\quad - \big|(1-\alpha)[z(B_{k+1}^c f)^{(j+1)}(z) + (p-j)(B_{k+1}^c f)^j(z)] \\ &\qquad + \alpha[z(B_k^c f)^{(j+1)}(z) + (p-j)(B_k^c f)^j(z)]\big| < 0 \quad (z \in \Delta \setminus \{0\}).\end{aligned}$$

Indeed, letting $\mu=\sum_{n=1}^{\infty}\frac{(-c)^n}{4^n(k+1)_n n!}$, $\eta=\sum_{n=1}^{\infty}\frac{(-c)^n}{4^n(k)_n n!}$ and $|z|=r$ $(0\leqslant r<1)$ we have

$$\begin{aligned}
&|(1-\alpha)[z(B_{k+1}^c f)^{(j+1)}(z)-(p-j)(B_{k+1}^c f)^j(z)]\\
&\qquad\qquad+\alpha[z(B_k^c f)^{(j+1)}(z)-(p-j)(B_k^c f)^j(z)]|\\
&\qquad-|(1-\alpha)[z(B_{k+1}^c f)^{(j+1)}(z)+(p-j)(B_{k+1}^c f)^j(z)]\\
&\qquad\qquad+\alpha[z(B_k^c f)^{(j+1)}(z)+(p-j)(B_k^c f)^j(z)]|\\
&\qquad=\left|\sum_{n=1}^{\infty}n[(1-\alpha)\mu+\alpha\eta]\delta(p+n,j)a_{p+n}z^n\right|\\
&-\left|2(p-j)\delta(p,j)+\sum_{n=1}^{\infty}[(1-\alpha)\mu+\alpha\eta](2p+n-2j)\delta(p+n,j)a_{p+n}z^n\right|\\
&\leqslant 2\sum_{n=1}^{\infty}|(1-\alpha)\mu+\alpha\eta|(p+n-j)\delta(p+n,j)|a_{p+n}|r^n-2(p-j)\delta(p,j)\leqslant 0\\
&\qquad\Rightarrow\sum_{n=1}^{\infty}|(1-\alpha)\mu+\alpha\eta|(p+n-j)\delta(p+n,j)|a_{p+n}|\leqslant(p-j)\delta(p,j)
\end{aligned}$$

whence $f\in\mathcal{S}_p^j(\alpha;c,k;\phi)$, □

5. A set of integral-preserving properties

In this section, we recall the generalized Bernardi–Libera–Livingston integral operator $J_\delta^p:\mathcal{A}_p\to\mathcal{A}_p$ defiend by

(5.1)
$$J_\delta^p f(z)=\frac{\delta+p}{z^\delta}\int_0^z t^{\delta-1}f(t)dt=z^p+\sum_{n=1}^{\infty}\Big(\frac{\delta+p}{\delta+n}\Big)a_{n+p}z^{n+p}\quad(f\in\mathcal{A}_p;\ \delta>-1).$$

This operator was itroduced by Bernardi [**4**] and studied by Libera [**11**] and Livingston [**12**].

From (5.1) and (1.5), it is clear $J_\delta^p f(z)$ satisfies

$$z[B_{k+1}^c J_\delta^p f(z)]^{(j+1)}=(\delta+p)(B_{k+1}^c f)^j(z)-(j+\delta)(B_{k+1}^c J_\delta^p f)^j(z),\qquad k=\gamma+\frac{b+1}{2}\notin\mathbb{Z}_0^-. \tag{5.2}$$

In order to obtain integral-preserving properties involving the integral operator J_δ^p, we need the following lemma which is known as Jack's Lemma.

LEMMA 5.1. [**10**] *Let $w(z)$ be a nonconstant function analytic in Δ with $w(0)=0$. If $w(z)$ attains its maximum value in the circle $|z|=r<1$ at z_0, then $z_0w'(z_0)=\xi w(z_0)$ where ξ is a real number and $\xi\geqslant 1$.*

THEOREM 5.1. *If $f(z)\in\mathcal{S}_p^j(c,k;\upsilon)$, then $J_\delta^p f(z)\in\mathcal{S}_p^j(c,k;\upsilon)$.*

PROOF. Suppose that $f \in \mathcal{S}_p^j(c,k;\upsilon)$, and let

$$\frac{z(B_{k+1}^c J_\delta^p f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c J_\delta^p f)^j(z)} = \frac{1+(1-2\upsilon)w(z)}{1-w(z)}, \tag{5.3}$$

where $w(0)=0$. Then, by using (5.2) and (5.3), we have

$$\frac{(B_{k+1}^c f)^j(z)}{(p-j)(B_{k+1}^c J_\delta^p f)^j(z)} = \frac{(\delta+p)+[(p-j)(1-2\upsilon)-(j+\delta)]w(z)}{(p-j)(\delta+p)(1-w(z))},$$

which, upon logarithmic differentiation, yields

$$\begin{aligned}\frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c f)^j(z)} &= \frac{1+(1-2\upsilon)w(z)}{1-w(z)}\\ &+ \frac{[(p-j)(1-2\upsilon)-(j+\delta)]zw'(z)}{(p-j)\{(\delta+p)+[(p-j)(1-2\upsilon)-(j+\delta)]w(z)\}} + \frac{zw'(z)}{(p-j)(1-w(z))},\end{aligned}$$

so that

$$\begin{aligned}\frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_k^c f)^j(z)} - \upsilon &= (1-\upsilon)\frac{1+w(z)}{1-w(z)}\\ &+ \frac{[(p-j)(1-2\upsilon)-(j+\delta)]zw'(z)}{(p-j)\{(\delta+p)+[(p-j)(1-2\upsilon)-(j+\delta)]w(z)\}} + \frac{zw'(z)}{(p-j)(1-w(z))}.\end{aligned}$$

Now, assuming that $\max_{|z|\leqslant|z_0|}|w(z)| = |w(z_0)| = 1$ $(z \in \Delta)$ and applying Jack's lemma, we obtain

$$z_0 w'(z_0) = \xi w(z_0) \qquad (\xi \in \mathbb{R},\ \xi \geqslant 1). \tag{5.4}$$

If we set $w(z_0) = e^{i\theta} (\theta \in \mathbb{R})$ in (5.4) and observe that

$$\operatorname{Re}\left\{(1-\upsilon)\frac{1+w(z_0)}{1-w(z_0)}\right\} = 0,$$

then, we obtain

$$\begin{aligned}&\operatorname{Re}\left\{\frac{z(B_{k+1}^c f)^{(j+1)}(z)}{(p-j)(B_{k+1}^c f)^j(z)} - \upsilon\right\}\\ &\quad= \frac{1}{p-j}\operatorname{Re}\left\{\frac{z_0 w'(z_0)}{(1-w(z_0))} + \frac{[(p-j)(1-2\upsilon)-(j+\delta)]z_0 w'(z_0)}{(\delta+p)+[(p-j)(1-2\upsilon)-(j+\delta)]w(z_0)}\right\}\\ &\quad= \frac{1}{p-j}\operatorname{Re}\left\{\frac{\xi e^{i\theta}}{(1-e^{i\theta})} + \frac{[(p-j)(1-2\upsilon)-(j+\delta)]\xi e^{i\theta}}{(\delta+p)+[(p-j)(1-2\upsilon)-(j+\delta)]e^{i\theta}}\right\}\\ &\quad= \frac{-\xi}{2(p-j)}\,\frac{\upsilon(p-j)+(j+\delta)}{(p-j)(1-\upsilon)} < 0,\end{aligned}$$

which obviously contradicts the hypothesis $f \in \mathcal{S}_p^j(c,k;\upsilon)$, □

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