

## WEIGHTED GENERALIZATION OF SOME INEQUALITIES FOR DOUBLE INTEGRALS

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ABSTRACT. We give some weighted double integral inequalities of Hermite–Hadamard type for co-ordinated convex functions in a rectangle from  $\mathbb{R}^2$ . The inequalities obtained provide generalizations of some result given in earlier works.

### 1. Introduction

The Hermite–Hadamard inequality discovered by C. Hermite and J. Hadamard (see, e.g., [10], [23, p.137]) is one of the most well established inequalities in the theory of convex functions with a geometrical interpretation and many applications. These inequalities state that if  $f : I \rightarrow \mathbb{R}$  is a convex function on the interval  $I$  of real numbers and  $a, b \in I$  with  $a < b$ , then

$$(1.1) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Both inequalities hold in the reversed direction if  $f$  is concave. We note that the Hermite–Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. The Hermite–Hadamard inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been studied (see, for example, [4, 7, 9, 11, 15, 22, 26, 28, 31, 32, 34]).

The most well-known inequalities related to the integral mean of a convex function are the Hermite–Hadamard inequalities or its weighted versions, the so-called Hermite–Hadamard–Fejér inequalities. In [14], Fejér gave a weighted generalization of the inequalities (1.1) as follows:

THEOREM 1.1. *If  $f : [a, b] \rightarrow \mathbb{R}$ , is a convex function, then the inequality*

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq \frac{f(a)+f(b)}{2} \int_a^b g(x)dx$$

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holds, where  $g : [a, b] \rightarrow \mathbb{R}$  is nonnegative, integrable, and symmetric about  $x = \frac{a+b}{2}$  (i.e.,  $g(x) = g(a+b-x)$ ).

In [28], Wu gives the following interesting result which is a weighted generalization of the Hermite–Hadamard inequality (i.e., this is Fejer’s inequality):

**THEOREM 1.2.** *Let  $f : [a, b] \rightarrow \mathbb{R}$  be a convex function on  $[a, b]$ , let  $g$  be a nonnegative, integrable function on  $[0, 1]$ , and let  $\lambda = \int_0^1 xg(x)dx / \int_0^1 g(x)dx$ . Then,*

$$(1.3) \quad f(\lambda a + (1-\lambda)b) \leq \frac{\int_a^b f(x)g(\frac{b-x}{b-a})dx}{\int_a^b g(\frac{b-x}{b-a})dx} \leq \lambda f(a) + (1-\lambda)f(b).$$

A formal definition for co-ordinated convex function may be stated as follows:

**DEFINITION 1.1.** A function  $f : \Delta \rightarrow \mathbb{R}$  is called co-ordinated convex on  $\Delta$ , for all  $(x, u), (y, v) \in \Delta$  and  $t, s \in [0, 1]$ , if it satisfies the following inequality:

$$(1.4) \quad f(tx + (1-t)y, su + (1-s)v) \leq tsf(x, u) + t(1-s)f(x, v) + s(1-t)f(y, u) + (1-t)(1-s)f(y, v).$$

The mapping  $f$  is a coordinated concave on  $\Delta$  if inequality (1.4) holds in reversed direction for all  $t, s \in [0, 1]$  and  $(x, u), (y, v) \in \Delta$ .

In [8], Dragomir proved the following inequalities which are the Hermite–Hadamard type inequalities for co-ordinated convex functions on the rectangle from  $\mathbb{R}^2$ .

**THEOREM 1.3.** *Suppose that  $f : \Delta \rightarrow \mathbb{R}$  is co-ordinated convex, then we have the following inequalities:*

$$(1.5) \quad \begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c} \int_c^d f\left(\frac{a+b}{2}, y\right) dy \right] \\ &\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\ &\leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\ &\quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\ &\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \end{aligned}$$

The above inequalities are sharp. The inequalities in (1.5) hold in the reverse direction if the mapping  $f$  is a coordinated concave mapping.

Over the years, many papers are dedicated to the generalizations and new versions of the inequalities (1.5) using the different type convex functions. For

the other Hermite–Hadamard type inequalities for co-ordinated convex functions, please refer to ([1, 2, 5, 6, 21, 24, 25, 27, 29, 33]).

Moreover, Farid et al. established a weighted version of inequalities (1.5) in [12]. Please see [3, 13, 16–20, 30] for other papers focused on the Hermite–Hadamard–Fejér inequalities for co-ordinated convex functions.

Here we establish a new weighed generalizations of the Hermite–Hadamard type double integral inequalities (1.5). The results provide extensions of those given in [3, 8, 12].

## 2. Hermite–Hadamard–Fejér type inequalities

Let start with the following Hermite–Hadamard–Fejér type inequalities:

**THEOREM 2.1.** *Let  $\Delta := (a, b) \times (c, d)$  and  $f : \Delta \rightarrow \mathbb{R}$  be a convex function on co-ordinated in  $\Delta$  and let  $g_1 : (a, b) \rightarrow \mathbb{R}^+$  and  $g_2 : (c, d) \rightarrow \mathbb{R}^+$  be two integrable functions. Also let*

$$\lambda = \frac{1}{\int_0^1 g_1(t) dt} \int_0^1 t g_1(t) dt \text{ and } \beta = \frac{1}{\int_0^1 g_2(s) ds} \int_0^1 s g_2(s) ds.$$

Then, one has the following inequalities

$$\begin{aligned} (2.1) \quad & f(\lambda a + (1 - \lambda)b, \beta c + (1 - \beta)d) \\ & \leq \frac{1}{2} \left[ \frac{1}{G_1} \int_a^b f(x, \beta c + (1 - \beta)d) g_1\left(\frac{b-x}{b-a}\right) dx \right. \\ & \quad \left. + \frac{1}{G_2} \int_c^d f(\lambda a + (1 - \lambda)b, y) g_2\left(\frac{d-y}{d-c}\right) dy \right] \\ & \leq \frac{1}{G_1 G_2} \int_a^b \int_c^d f(x, y) g_1\left(\frac{b-x}{b-a}\right) g_2\left(\frac{d-y}{d-c}\right) dy dx \\ & \leq \frac{1}{2} \left[ \frac{\beta}{G_1} \int_a^b f(x, c) g_1\left(\frac{b-x}{b-a}\right) dx + \frac{(1-\beta)}{G_1} \int_a^b f(x, d) g_1\left(\frac{b-x}{b-a}\right) dx \right. \\ & \quad \left. + \frac{\lambda}{G_2} \int_c^d f(a, y) g_2\left(\frac{d-y}{d-c}\right) dy + \frac{(1-\lambda)}{G_2} \int_c^d f(b, y) g_2\left(\frac{d-y}{d-c}\right) dy \right] \\ & \leq \beta \lambda f(a, c) + \lambda(1 - \beta) f(a, d) + (1 - \lambda) \beta f(b, c) + (1 - \beta)(1 - \lambda) f(b, d), \end{aligned}$$

where

$$G_1 = \int_a^b g_1\left(\frac{b-x}{b-a}\right) dx, \quad G_2 = \int_c^d g_2\left(\frac{d-y}{d-c}\right) dy.$$

**PROOF.** Since  $f$  is co-ordinated convex on  $\Delta$ , if we define the mappings  $f_x : (c, d) \rightarrow \mathbb{R}$ ,  $f_x(y) = f(x, y)$ , then  $f_x(y)$  is convex on  $(c, d)$  for all  $x \in (a, b)$ . If we apply inequality (1.3) for the convex function  $f_x(y)$ , then we have

$$(2.2) \quad f_x(\beta c + (1 - \beta)d) \leq \frac{\int_c^d f_x(y) g_2\left(\frac{d-y}{d-c}\right) dy}{\int_c^d g_2\left(\frac{d-y}{d-c}\right) dy} \leq \beta f_x(c) + (1 - \beta) f_x(d).$$

That is,

$$(2.3) \quad f(x, \beta c + (1 - \beta)d) \leq \frac{\int_c^d f(x, y) g_2 \left( \frac{d-y}{d-c} \right) dy}{\int_c^d g_2 \left( \frac{d-y}{d-c} \right) dy} \leq \beta f(x, c) + (1 - \beta) f(x, d).$$

Multiplying by  $g_1 \left( \frac{b-x}{b-a} \right)$  inequality (2.3) and integrating with respect to  $x$  from  $a$  to  $b$ , and by dividing  $G_1 = \int_a^b g_1 \left( \frac{b-x}{b-a} \right) dx$ , we obtain

$$(2.4) \quad \begin{aligned} & \frac{1}{G_1} \int_a^b f(x, \beta c + (1 - \beta)d) g_1 \left( \frac{b-x}{b-a} \right) dx \\ & \leq \frac{1}{G_1 G_2} \int_a^b \int_c^d f(x, y) g_1 \left( \frac{b-x}{b-a} \right) g_2 \left( \frac{d-y}{d-c} \right) dy dx \\ & \leq \frac{\beta}{G_1} \int_a^b f(x, c) g_1 \left( \frac{b-x}{b-a} \right) dx + \frac{(1 - \beta)}{G_1} \int_a^b f(x, d) g_1 \left( \frac{b-x}{b-a} \right) dx. \end{aligned}$$

Similarly, as  $f$  is co-ordinated convex on  $\Delta$ , if we define the mappings  $f_y : (a, b) \rightarrow \mathbb{R}$ ,  $f_y(x) = f(x, y)$ , then  $f_y(x)$  is convex on  $(a, b)$  for all  $y \in (c, d)$ . Utilizing inequality (1.3) for the convex function  $f_y(x)$ , then we obtain the inequality

$$(2.5) \quad f_y(\lambda a + (1 - \lambda)b) \leq \frac{\int_a^b f_y(x) g_1 \left( \frac{b-x}{b-a} \right) dx}{\int_a^b g_1 \left( \frac{b-x}{b-a} \right) dx} \leq \lambda f_y(a) + (1 - \lambda) f_y(b),$$

i.e.,

$$(2.6) \quad f(\lambda a + (1 - \lambda)b, y) \leq \frac{\int_a^b f(x, y) g_1 \left( \frac{b-x}{b-a} \right) dx}{\int_a^b g_1 \left( \frac{b-x}{b-a} \right) dx} \leq \lambda f(a, y) + (1 - \lambda) f(b, y).$$

Multiplying by  $g_2 \left( \frac{d-y}{d-c} \right)$  inequality (2.6) and integrating with respect to  $y$  on  $(c, d)$ , and by dividing  $G_2 = \int_c^d g_2 \left( \frac{d-y}{d-c} \right) dy$ , we get

$$(2.7) \quad \begin{aligned} & \frac{1}{G_2} \int_c^d f(\lambda a + (1 - \lambda)b, y) g_2 \left( \frac{d-y}{d-c} \right) dy \\ & \leq \frac{1}{G_1 G_2} \int_a^b \int_c^d f(x, y) g_1 \left( \frac{b-x}{b-a} \right) g_2 \left( \frac{d-y}{d-c} \right) dy dx \\ & \leq \frac{\lambda}{G_2} \int_c^d f(a, y) g_2 \left( \frac{d-y}{d-c} \right) dy + \frac{(1 - \lambda)}{G_2} \int_c^d f(b, y) g_2 \left( \frac{d-y}{d-c} \right) dy. \end{aligned}$$

Summing inequalities (2.4) and (2.7), we obtain the second and third inequalities in (2.1).

Since  $f(x, \beta c + (1 - \beta)d)$  is convex on  $(a, b)$  and  $g_1(x)$  is positive and integrable, using the Jensen integral inequality, we have

$$(2.8) \quad \frac{1}{G_1} \int_a^b f(x, \beta c + (1 - \beta)d) g_1 \left( \frac{b-x}{b-a} \right) dx$$

$$\begin{aligned}
&= \frac{1}{\int_0^1 g_1(t) dt} \int_0^1 f(at + (1-t)b, \beta c + (1-\beta)d) g_1(t) dt \\
&\geq f\left(\frac{1}{\int_0^1 g_1(t) dt} \int_0^1 (at + (1-t)b) g_1(t) dt, \beta c + (1-\beta)d\right) \\
&= f(\lambda a + (1-\lambda)b, \beta c + (1-\beta)d).
\end{aligned}$$

And similarly, since  $f(\lambda a + (1-\lambda)b, y)$  is convex on  $(c, d)$  and  $g_2(y)$  is positive and integrable, using the Jensen integral inequality, we get

$$\begin{aligned}
(2.9) \quad &\frac{1}{G_2} \int_c^d f(\lambda a + (1-\lambda)b, y) g_2\left(\frac{d-y}{d-c}\right) dy \\
&\geq f\left(\lambda a + (1-\lambda)b, \frac{1}{\int_0^1 g_2(s) ds} \int_0^1 (sc + (1-s)d) g_2(s) ds\right) \\
&= f(\lambda a + (1-\lambda)b, \beta c + (1-\beta)d).
\end{aligned}$$

Summing inequalities (2.8) and (2.9), we obtain the first inequality in (2.1).

Since  $f(x, c)$  and  $f(x, d)$  are convex on  $(a, b)$  and  $g_1(x)$  is positive, integrable, we have the following inequality

$$\begin{aligned}
(2.10) \quad &\frac{\beta}{G_1} \int_a^b f(x, c) g_1\left(\frac{b-x}{b-a}\right) dx + \frac{(1-\beta)}{G_1} \int_a^b f(x, d) g_1\left(\frac{b-x}{b-a}\right) dx \\
&= \frac{\beta}{\int_0^1 g_1(t) dt} \int_0^1 f(at + (1-t)b, c) g_1(t) dt \\
&\quad + \frac{(1-\beta)}{\int_0^1 g_1(t) dt} \int_0^1 f(at + (1-t)b, d) g_1(t) dt \\
&\leq \beta \left[ f(a, c) \frac{1}{\int_0^1 g_1(t) dt} \int_0^1 t g_1(t) dt + f(b, c) \frac{1}{\int_0^1 g_1(t) dt} \int_0^1 (1-t) g_1(t) dt \right] \\
&\quad + (1-\beta) \left[ f(a, d) \frac{1}{\int_0^1 g_1(t) dt} \int_0^1 t g_1(t) dt + f(b, d) \frac{1}{\int_0^1 g_1(t) dt} \int_0^1 (1-t) g_1(t) dt \right] \\
&= \beta \lambda f(a, c) + \beta(1-\lambda) f(b, c) + (1-\beta) \lambda f(a, d) + (1-\beta)(1-\lambda) f(b, d).
\end{aligned}$$

And similarly, since  $f(a, y)$  and  $f(b, y)$  are convex on  $(c, d)$  and  $g_2(y)$  is positive, integrable, we have the following inequality

$$\begin{aligned}
(2.11) \quad &\frac{\lambda}{G_2} \int_c^d f(a, y) g_2\left(\frac{d-y}{d-c}\right) dy + \frac{(1-\lambda)}{G_2} \int_c^d f(b, y) g_2\left(\frac{d-y}{d-c}\right) dy \\
&= \frac{\lambda}{\int_0^1 g_2(s) ds} \int_0^1 f(a, cs + (1-s)d) g_2(s) ds \\
&\quad + \frac{(1-\lambda)}{\int_0^1 g_2(s) ds} \int_0^1 f(b, cs + (1-s)d) g_2(s) ds
\end{aligned}$$

$$\leq \beta\lambda f(a, c) + \lambda(1 - \beta)f(a, d) + (1 - \lambda)\beta f(b, c) + (1 - \beta)(1 - \lambda)f(b, d).$$

By summing resulting inequalities (2.10) and (2.11), then we obtain the last inequality in (2.1). This completes the proof.  $\square$

REMARK 2.1. Under assumptions of Theorem 2.1 with  $g_1(t) = 1$  and  $g_2(s) = 1$  for all  $t, s \in (0, 1)$ , inequalities (2.1) reduce to inequalities (1.5) which were proved by Dragomir in [8].

THEOREM 2.2. Under assumptions of Theorem 2.1, let

$$\lambda = \frac{1}{\int_a^b (b-x)g_1(x)dx} \int_a^b (b-x)g_1(a+b-x)dx,$$

$$\beta = \frac{1}{\int_c^d (d-y)g_2(y)dy} \int_c^d (d-y)g_2(c+d-y)dy.$$

Then, one has the following inequalities

$$\begin{aligned} & f\left(\frac{a+\lambda b}{1+\lambda}, \frac{c+\beta d}{1+\beta}\right) \\ & \leq \frac{1}{2} \left[ \frac{1}{G_3} \int_a^b f\left(x, \frac{c+\beta d}{1+\beta}\right) g_1(x) dx + \frac{1}{G_4} \int_c^d f\left(\frac{a+\lambda b}{1+\lambda}, y\right) g_2(y) dy \right] \\ & \leq \frac{1}{G_3 G_4} \int_a^b \int_c^d f(x, y) g_1(x) g_2(y) dy dx \\ & \leq \frac{1}{2} \left[ \frac{1}{(1+\beta)G_3} \int_a^b f(x, c) g_1(x) dx + \frac{\beta}{(1+\beta)G_3} \int_a^b f(x, d) g_1(x) dx \right. \\ & \quad \left. + \frac{1}{(1+\lambda)G_4} \int_c^d f(a, y) g_2(y) dy + \frac{\lambda}{(1+\lambda)G_4} \int_c^d f(b, y) g_2(y) dy \right] \\ & \leq \frac{f(a, c) + \beta f(a, d) + \lambda f(b, c) + \lambda \beta f(b, d)}{(1+\lambda)(1+\beta)}, \end{aligned}$$

where  $G_3 = \int_a^b g_1(x) dx$  and  $G_4 = \int_c^d g_2(y) dy$ .

PROOF. Based on the assumption that  $g_1$  and  $g_2$  are nonnegative, integrable functions on  $(a, b)$  and  $(c, d)$ , respectively, one can show that  $\varphi_1(t) = g_1(b - (b-a)t)$  and  $\varphi_2(s) = g_2(d - (d-c)s)$  are nonnegative, integrable functions on  $(0, 1)$ . Thus, by using Theorem 2.1, we can write the following inequalities

$$\begin{aligned} & f(\gamma a + (1-\gamma)b, \delta c + (1-\delta)d) \\ & \leq \frac{1}{2} \left[ \frac{1}{\int_a^b \varphi_1\left(\frac{b-x}{b-a}\right) dx} \int_a^b f\left(x, \delta c + (1-\delta)d\right) \varphi_1\left(\frac{b-x}{b-a}\right) dx \right. \\ & \quad \left. + \frac{1}{\int_c^d \varphi_2\left(\frac{d-y}{d-c}\right) dy} \int_c^d f\left(\gamma a + (1-\gamma)b, y\right) \varphi_2\left(\frac{d-y}{d-c}\right) dy \right] \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\int_a^b \int_c^d f(x, y) \varphi_1\left(\frac{b-x}{b-a}\right) \varphi_2\left(\frac{d-y}{d-c}\right) dy dx}{\int_a^b \int_c^d \varphi_1\left(\frac{b-x}{b-a}\right) \varphi_2\left(\frac{d-y}{d-c}\right) dy dx} \\
&\leq \frac{1}{2} \left[ \frac{\delta}{\int_a^b \varphi_1\left(\frac{b-x}{b-a}\right) dx} \int_a^b f(x, c) \varphi_1\left(\frac{b-x}{b-a}\right) dx \right. \\
&\quad + \frac{(1-\delta)}{\int_a^b \varphi_1\left(\frac{b-x}{b-a}\right) dx} \int_a^b f(x, d) \varphi_1\left(\frac{b-x}{b-a}\right) dx \\
&\quad + \frac{\gamma}{\int_c^d \varphi_2\left(\frac{d-y}{d-c}\right) dy} \int_c^d f(a, y) \varphi_2\left(\frac{d-y}{d-c}\right) dy \\
&\quad \left. + \frac{(1-\gamma)}{\int_c^d \varphi_2\left(\frac{d-y}{d-c}\right) dy} \int_c^d f(b, y) \varphi_2\left(\frac{d-y}{d-c}\right) dy \right] \\
&\leq \delta \gamma f(a, c) + \gamma(1-\delta)f(a, d) + (1-\gamma)\delta f(b, d) + (1-\delta)(1-\gamma)f(b, d)
\end{aligned}$$

i.e.,

$$\begin{aligned}
&f(\gamma a + (1-\gamma)b, \delta c + (1-\delta)d) \\
&\leq \frac{1}{2} \left[ \frac{1}{G_3} \int_a^b f(x, \delta c + (1-\delta)d) g_1(x) dx + \frac{1}{G_4} \int_c^d f(\gamma a + (1-\gamma)b, y) g_2(y) dy \right] \\
&\leq \frac{1}{G_3 G_4} \int_a^b \int_c^d f(x, y) g_1(x) g_2(y) dy dx \\
&\leq \frac{1}{2} \left[ \frac{\delta}{G_3} \int_a^b f(x, c) g_1(x) dx + \frac{(1-\delta)}{G_3} \int_a^b f(x, d) g_1(x) dx \right. \\
&\quad \left. + \frac{\gamma}{G_4} \int_c^d f(a, y) g_2(y) dy + \frac{(1-\gamma)}{G_4} \int_c^d f(b, y) g_2(y) dy \right] \\
&\leq \delta \gamma f(a, c) + \gamma(1-\delta)f(a, d) + (1-\gamma)\delta f(b, d) + (1-\delta)(1-\gamma)f(b, d),
\end{aligned}$$

where

$$\begin{aligned}
\gamma &= \frac{1}{\int_0^1 g_1(b - (b-a)t) dt} \int_0^1 t g_1(b - (b-a)t) dt \\
&= \frac{1}{\int_a^b (b-a) g_1(x) dx} \int_a^b (b-x) g_1(x) dx \\
&= \frac{\int_a^b (b-x) g_1(x) dx}{\int_a^b (b-x) g_1(x) dx + \int_a^b (b-x) g_1(a+b-x) dx} = \frac{1}{1+\lambda}
\end{aligned}$$

and similarly

$$\delta = \frac{1}{\int_0^1 g_2(d - (d-c)s) ds} \int_0^1 s g_2(d - (d-c)s) ds = \frac{1}{1+\beta}. \quad \square$$

COROLLARY 2.1. Under assumptions of Theorem 2.2, let  $g_1(x) = g_1(a + b - x)$  for any  $x \in (a, b)$  and  $g_2(y) = g_2(c + d - y)$  for any  $y \in (c, d)$ , then we have the following inequality

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ & \leq \frac{1}{2} \left[ \frac{1}{G_3} \int_a^b f\left(x, \frac{c+d}{2}\right) g_1(x) dx + \frac{1}{G_4} \int_c^d f\left(\frac{a+b}{2}, y\right) g_2(y) dy \right] \\ & \leq \frac{1}{G_3 G_4} \int_a^b \int_c^d f(x, y) g_1(x) g_2(y) dy dx \\ & \leq \frac{1}{4} \left[ \frac{1}{G_3} \int_a^b [f(x, c) + f(x, d)] g_1(x) dx + \frac{1}{G_4} \int_c^d [f(a, y) + f(b, y)] g_2(y) dy \right] \\ & \leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \end{aligned}$$

which is the same result as proved by Farid et al. in [12].

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