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GENERALIZATIONS OF SOME ZYGMUND-TYPE INTEGRAL INEQUALITIES FOR POLAR DERIVATIVES OF A COMPLEX POLYNOMIAL

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Abstract. We prove some results for algebraic polynomials in the complex plane that relate the L^γ-norm of the polar derivative of a complex polynomial and the polynomial under some conditions. The obtained results include several interesting generalizations of some Zygmund-type integral inequalities for polynomials and derive polar derivative analogues of some classical Bernstein-type inequalities for the sup-norms on the unit disk as well.

1. Introduction

Let $\mathbb{P}_n$ be the class of complex polynomials $P(z) := \sum_{v=0}^{n} a_v z^v$ of degree n. The study of Bernstein-type inequalities that relate the norm of a polynomial to that of its derivative and their various versions are a classical topic in analysis. Over a period, these inequalities have been generalized in different domains, in different norms and for different classes of functions. Here, we study some of the new inequalities centered around Bernstein-type inequalities that relate the L^γ-norm of the polar derivatives and the polynomial under some conditions.

For $P \in \mathbb{P}_n$ and $\alpha \in \mathbb{C}$, define

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z).$$

Note that $D_\alpha P(z)$ is a polynomial of degree at most $n-1$. This is the so-called polar derivative of $P(z)$ with respect to α (see [**16**]). It generalizes the ordinary derivative in the following sense:

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} := P'(z),$$

uniformly with respect to z for $|z| \leqslant R,\ R > 0$.

We can construct a sequence of polar derivatives for $P \in \mathbb{P}_n$ as follows:

$$D_{\alpha_1} P(z) = nP(z) + (\alpha_1 - z)P'(z),$$

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$$\vdots$$

$$\begin{aligned}D_{\alpha_k}D_{\alpha_{k-1}}\dots D_{\alpha_1}P(z) &= (n-k+1)D_{\alpha_{k-1}}D_{\alpha_{k-2}}\dots D_{\alpha_1}P(z)\\ &\quad + (\alpha_k - z)(D_{\alpha_{k-1}}D_{\alpha_{k-2}}\dots D_{\alpha_1}P(z))', \ \ k=2,3,\dots,n.\end{aligned}$$

The points $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{C}$, $k = 1, 2, 3, \dots, n$ may or may not be distinct. Like the k^{th} ordinary derivative $P^{(k)}(z)$ of $P(z)$, the k^{th} polar derivative $D_{\alpha_k}D_{\alpha_{k-1}}\dots D_{\alpha_1}P(z)$ of $P(z)$ is a polynomial of degree at most $n-k$.

For $P \in \mathbb{P}_n$, we have

$$\max_{|z|=1}|P'(z)| \leqslant n\max_{|z|=1}|P(z)| \tag{1.1}$$

and for every $r \geqslant 1$,

$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})\right|^r d\theta\right\}^{1/r} \leqslant n\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{1/r}. \tag{1.2}$$

Inequality (1.1) is a classical result of Bernstein [**17**], whereas inequality (1.2) is due to Zygmund [**23**]. Arestov [**1**] proved that (1.2) remains true for $0 < r < 1$ as well. If we let $r \to \infty$ in (1.2), then we get (1.1). Equality holds in (1.1) and (1.2) only for $P(z) = \lambda z^n$, $\lambda \neq 0$. Noting that these extremal polynomials have all zeros at the origin, so it is natural to seek improvements under appropriate condition on the zeros of $P(z)$. If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then (1.1) and (1.2) can be improved. In fact, if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then (1.1) and (1.2) can be respectively replaced by

$$\max_{|z|=1}|P'(z)| \leqslant \frac{n}{2}\max_{|z|=1}|P(z)|, \tag{1.3}$$

$$\left\{\int_0^{2\pi}\left|P'(e^{i\theta})\right|^r d\theta\right\}^{1/r} \leqslant nC_r\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{1/r}, \tag{1.4}$$

where

$$C_r = \left\{\frac{1}{2\pi}\int_0^{2\pi}\left|1 + e^{i\gamma}\right|^r d\gamma\right\}^{-1/r}. \tag{1.5}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [**15**], whereas (1.4) was proved by de-Bruijn [**8**] for $r \geqslant 1$. Further, Rahman and Schmeisser [**21**] have shown that (1.4) holds for $0 < r < 1$ as well. If we let $r \to \infty$ in inequality (1.4), we get (1.3).

The literature on polynomial inequalities is vast and growing and over the last four decades many different authors produced a large number of different versions and generalizations of the above inequalities. Many of these generalizations involve the comparison of polar derivative $D_\alpha P(z)$ with various choices of $P(z)$, α and other parameters. More information on this topic can be found in the books of Milovanović et al. [**17**] and Marden [**16**]. Aziz was among the first to extend some of the above inequalities by replacing the derivative with the polar derivatives of polynomials. In fact in 1988, Aziz [**2**] extended (1.3) to the polar derivative of a polynomial and proved that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leqslant \frac{n}{2}(|\alpha| + 1) \max_{|z|=1} |P(z)|. \tag{1.6}$$

Inequality (1.6) was sharpened by Dewan et al. [**9**, Theorem 1 for $t = k = 1$], by proving that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant 1$,

$$\max_{|z|=1} |D_\alpha P(z)| \leqslant \frac{n}{2}\big\{(|\alpha| + 1) \max_{|z|=1} |P(z)| - (|\alpha| - 1) \min_{|z|=1} |P(z)|\big\}. \tag{1.7}$$

As an L^r analogue of (1.6) and a generalization of (1.4), Aziz and Rather [**5**] proved that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geqslant 1$ and $r \geqslant 1$,

$$\left\{ \int_0^{2\pi} \Big|D_\alpha P(e^{i\theta})\Big|^r d\theta \right\}^{1/r} \leqslant n(|\alpha| + 1)C_r \left\{ \int_0^{2\pi} \Big|P(e^{i\theta})\Big|^r d\theta \right\}^{1/r}, \tag{1.8}$$

where here and throughout C_r is defined by (1.5).

Further, the following more general result which besides provides an L^r analogue of (1.7) also extends inequality (1.8) for $r \in (0, 1)$ was proved by Mir and Baba [**20**]. More precisely, they proved that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\alpha, \delta \in \mathbb{C}$ with $|\alpha| \geqslant 1$, $|\delta| \leqslant 1$ and $r > 0$,

$$\begin{aligned} &\left\{ \int_0^{2\pi} \Big|e^{i\theta} D_\alpha P(e^{i\theta}) + \frac{mn}{2}\delta\big(|\alpha| - 1\big)\Big|^r d\theta \right\}^{1/r} \\ &\qquad\qquad \leqslant nC_r(|\alpha| + 1) \left\{ \int_0^{2\pi} \Big|P(e^{i\theta})\Big|^r d\theta \right\}^{1/r}, \end{aligned} \tag{1.9}$$

where here and throughout $m = \min_{|z|=1} |P(z)|$.

Recently, Mir and Hussain [**19**] proved the following generalization of (1.9) by using a parameter β and established that if $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\alpha, \beta, \delta \in \mathbb{C}$ with $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$, $|\delta| \leqslant 1$ and $r \geqslant 1$,

$$\begin{aligned} &\left\{ \int_0^{2\pi} \Big| e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta \frac{(|\alpha| - 1)}{2} P(e^{i\theta}) \right. \\ &\qquad \left. + \frac{mn}{2}\delta\Big(\Big|\alpha + \beta\frac{(|\alpha| - 1)}{2}\Big| - \Big|e^{i\theta} + \beta\frac{(|\alpha| - 1)}{2}\Big|\Big)\Big|^r d\theta \right\}^{1/r} \\ &\qquad \leqslant nC_r\Big((|\alpha| + 1) + |\beta|(|\alpha| - 1)\Big) \left\{ \int_0^{2\pi} \Big|P(e^{i\theta})\Big|^r d\theta \right\}^{1/r}. \end{aligned} \tag{1.10}$$

One can see in the literature (for example, refer [**14, 18, 19, 22**]), the latest research and development in this direction. Ideally, it is natural to seek integral inequalities analogous to the above inequalities for the k^{th} polar derivative of polynomials. This naturally leads us to establish some general Zygmund-type integral inequalities which in particular yield the above mentioned inequalities and related inequalities as special cases. Throughout this paper, we use the following notations:

$$P_k(z) := D_{\alpha_k} D_{\alpha_{k-1}} \dots D_{\alpha_1} P(z),$$

$$P_0(z) := P(z), \quad \Lambda_t := \alpha_1\alpha_2\dots\alpha_t,$$
$$A_{\alpha_t} := (|\alpha_1|-1)(|\alpha_2|-1)\dots(|\alpha_t|-1),$$
$$B_{\alpha_t} := (|\alpha_1|+1)(|\alpha_2|+1)\cdots(|\alpha_t|+1),$$
$$n_t := n(n-1)\cdots(n-t+1).$$

Definition. Given a polynomial $P(z)=\sum_{v=0}^{n} a_v z^v \in \mathbb{P}_n$, we associate with it the polynomials

$$\bar{P}(z) := \overline{P(\bar{z})} = \sum_{v=0}^{n} \overline{a_v} z^v \quad \text{and} \quad Q(z) := z^n\overline{P(1/\bar{z})} = \sum_{v=0}^{n} \overline{a_{n-v}} z^v.$$

If $P(z)\equiv \zeta Q(z)$, where $|\zeta|=1$, then $P(z)$ is said to be self-inversive.

2. Lemmas

In this section, we provide the following lemmas that will be used in the later sections for proving our main results.

Lemma 2.1. *Let P and Q be two polynomials with $Q\in\mathbb{P}_n$ and $\deg P \leqslant \deg Q(z)$. If $Q(z)$ has all its zeros in $|z|\leqslant 1$ and $|P(z)|\leqslant|Q(z)|$ for $|z|=1$, then for all $\beta,\alpha_j\in\mathbb{C}$ with $|\beta|\leqslant 1$, $|\alpha_j|\geqslant 1$, $j=1,2,\dots,t$ and $t\leqslant n-1$,*

$$\Big|z^tP_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}P(z)\Big| \leqslant \Big|z^tQ_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(z)\Big| \quad \textit{for } |z|\geqslant 1.$$

The above lemma is due to Bidkham and Soleiman Mezerji [**7**]. By applying it to the polynomials $P(z)$ and $z^n\min_{|z|=1}P(z)$, we get the following result.

Lemma 2.2. *If $P\in\mathbb{P}_n$ and $P(z)$ has all its zeros in $|z|\leqslant 1$, then for all $\alpha_j,\beta\in\mathbb{C}$ with $|\alpha_j|\geqslant 1$, $|\beta|\leqslant 1$, $1\leqslant j\leqslant t$, $t\leqslant n-1$ and $|z|=1$,*

$$\Big|z^tP_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}P(z)\Big| \geqslant n_t\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|m.$$

Lemma 2.3. *If $P\in\mathbb{P}_n$, then for every $\alpha\in\mathbb{C}$ and $r>0$,*

$$\bigg\{\int_0^{2\pi}\big|D_\alpha P(e^{i\theta})\big|^r d\theta\bigg\}^{1/r} \leqslant n(|\alpha|+1)\bigg\{\int_0^{2\pi}\big|P(e^{i\theta})\big|^r d\theta\bigg\}^{1/r}.$$

The above lemma is due to Aziz and Rather [**5**].

Lemma 2.4. *If $P\in\mathbb{P}_n$ and $Q(z)=z^n\overline{P(1/\bar{z})}$, then for every $r>0$ and γ real,*

$$\int_0^{2\pi}\int_0^{2\pi}\big|P'(e^{i\theta})+e^{i\gamma}Q'(e^{i\theta})\big|^r d\theta d\gamma \leqslant 2\pi n^r\int_0^{2\pi}\big|P(e^{i\theta})\big|^r d\theta.$$

The above lemma is due to Aziz and Rather [**4**].

Lemma 2.5. *If $P\in\mathbb{P}_n$ and $P(z)\neq 0$ in $|z|<1$, then for all $\beta,\alpha_j\in\mathbb{C}$, $1\leqslant j\leqslant t$, $t\leqslant n-1$ with $|\beta|\leqslant 1$, $|\alpha_j|\geqslant 1$ and $|z|=1$,*

$$\Big|z^tP_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}P(z)\Big| \leqslant \Big|z^tQ_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(z)\Big| \tag{2.1}$$

$$- mn_t\Big\{\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|-\Big|z^t+\beta\frac{A_{\alpha_t}}{2^t}\Big|\Big\},$$

where $Q(z)=z^n\overline{P(1/\bar{z})}$.

Proof. Recall that $P(z)$ has all zeros in $|z|\geqslant 1$. If $P(z)$ has a zero on $|z|=1$, then $m=0$ and the result follows by Lemma 2.1 in this case. Henceforth, we suppose that all the zeros of $P(z)$ lie in $|z|>1$ and so $m>0$. We have $|\lambda m|<|P(z)|$ on $|z|=1$ for any λ with $|\lambda|<1$. It follows by Rouche's theorem that the polynomial $G(z)=P(z)-\lambda m$ has no zeros in $|z|<1$. Therefore, the polynomial $H(z)=z^n\overline{G(1/\bar{z})}=Q(z)-\bar{\lambda}mz^n$ will have all its zeros in $|z|\leqslant 1$. Also $|G(z)|=|H(z)|$ for $|z|=1$. On applying Lemma 2.1, we get for every β,α_j with $|\beta|\leqslant 1$, $|\alpha_j|\geqslant 1$, $j=1,2,\dots,t$ and $t\leqslant n-1$,

$$\Big|z^tG_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}G(z)\Big|\leqslant\Big|z^tH_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}H(z)\Big|\quad\text{for } |z|\geqslant 1.$$

Equivalently,

$$\Big|z^tP_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}P(z)-\lambda mn_t\Big(z^t+\beta\frac{A_{\alpha_t}}{2^t}\Big)\Big| \leqslant\Big|z^tQ_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(z)-\bar{\lambda}mn_tz^n\Big(\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big)\Big|. \tag{2.2}$$

Since $Q(z)$ has all zeros in $|z|\leqslant 1$ and $\min_{|z|=1}|Q(z)|=\min_{|z|=1}|P(z)|=m$, by Lemma 2.2, we have

$$\Big|z^tQ_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(z)\Big|\geqslant n_t\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|m|z|^n\quad\text{for } |z|\geqslant 1. \tag{2.3}$$

Now, by choosing a suitable argument of λ on the right-hand side of (2.2), in view of (2.3), we get for $|z|=1$,

$$\Big|z^tP_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}P(z)\Big|-|\lambda|mn_t\Big|z^t+\beta\frac{A_{\alpha_t}}{2^t}\Big| \leqslant\Big|z^tQ_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(z)\Big|-|\lambda|mn_t\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|.$$

Letting $|\lambda|\to 1$, we get for $|z|=1$,

$$\Big|z^tP_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}P(z)\Big|-mn_t\Big|z^t+\beta\frac{A_{\alpha_t}}{2^t}\Big| \leqslant\Big|z^tQ_t(z)+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(z)\Big|-mn_t\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|,$$

which is equivalent to (2.1). □

Lemma 2.6. *If A,B,C are non-negative real numbers such that $B+C\leqslant A$, then for every real number α,* $\big|(A-C)e^{i\alpha}+(B+C)\big|\leqslant\big|Ae^{i\alpha}+B\big|$.

The above lemma is due to Aziz and Shah [**6**].

Lemma 2.7. *Let z_1,z_2 be two complex numbers independent of β, where β is real. Then for $r>0$,*

$$\int_0^{2\pi}\big|z_1+z_2e^{i\beta}\big|^r d\beta=\int_0^{2\pi}\big||z_1|+|z_2|e^{i\beta}\big|^r d\beta.$$

The above lemma is due to Govil and Kumar [**12**].

LEMMA 2.8. *If $a, b \in \mathbb{C}$ with $|b| \geqslant |a|$, then for $r > 0$ and γ real, we have*

$$\int_0^{2\pi} \left|a + e^{i\gamma} b\right|^r d\gamma \geqslant |a|^r \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^r d\gamma. \tag{2.4}$$

PROOF. If $a = 0$, then (2.4) is obvious. Henceforth, we assume that $a \neq 0$. Now for every real γ and $t \geqslant 1$, it can be easily verified that $|t + e^{i\gamma}| \geqslant |1 + e^{i\gamma}|$ which by using Lemma 2.7 gives

$$\int_0^{2\pi} \left|1 + e^{i\gamma} \frac{b}{a}\right|^r d\gamma = \int_0^{2\pi} \left|1 + e^{i\gamma} \left|\frac{b}{a}\right|\right|^r d\gamma = \int_0^{2\pi} \left|\left|\frac{b}{a}\right| + e^{i\gamma}\right|^r d\gamma \geqslant \int_0^{2\pi} \left|1 + e^{i\gamma}\right|^r d\gamma,$$

which is equivalent to (2.4). □

LEMMA 2.9. *If $P \in \mathbb{P}_n$ is self-inversive, then for $0 \leqslant \theta < 2\pi$, we have*

$$|P'(e^{i\theta})| \geqslant \frac{n}{2} \left|P(e^{i\theta})\right|.$$

The above lemma is due to Govil and Nyuydinkong [**10**].

3. Main Results and Proofs

Here, we first prove the following more general result which as special cases yield some interesting generalizations of (1.6)–(1.10).

THEOREM 3.1. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for all $\beta, \delta, \alpha_j \in \mathbb{C}$ with $|\beta| \leqslant 1$, $|\delta| \leqslant 1$, $|\alpha_j| \geqslant 1$, $1 \leqslant j \leqslant t$, $t \leqslant n - 1$ and $r \geqslant 1$,*

$$\left\{\int_0^{2\pi} \left|e^{it\theta} P_t(e^{i\theta}) + \beta \frac{n_t A_{\alpha_t}}{2^t} P(e^{i\theta}) + \frac{mn_t}{2} \delta\left(\left|\Lambda_t + \beta \frac{A_{\alpha_t}}{2^t}\right| - \left|e^{it\theta} + \beta \frac{A_{\alpha_t}}{2^t}\right|\right)\right|^r d\theta\right\}^{1/r}$$
$$\leqslant n_t C_r \Big(B_{\alpha_t} + \frac{|\beta| A_{\alpha_t}}{2^{t-1}}\Big) \left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^r d\theta\right\}^{1/r}. \tag{3.1}$$

PROOF. Recall that $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$. If $Q(z) = z^n \overline{P(1/\bar{z})}$, then $P(z) = z^n \overline{Q(1/\bar{z})}$ and it can be easily verified that for $0 \leqslant \theta < 2\pi$,

$$nP(e^{i\theta}) - e^{i\theta} P'(e^{i\theta}) = e^{i(n-1)\theta} \overline{Q'(e^{i\theta})},$$
$$nQ(e^{i\theta}) - e^{i\theta} Q'(e^{i\theta}) = e^{i(n-1)\theta} \overline{P'(e^{i\theta})}.$$

Hence

$$\begin{aligned} &nP(e^{i\theta}) + e^{i\gamma} nQ(e^{i\theta}) \\ &\quad = e^{i\theta} P'(e^{i\theta}) + e^{i(n-1)\theta} \overline{Q'(e^{i\theta})} + e^{i\gamma} \big(e^{i\theta} Q'(e^{i\theta}) + e^{i(n-1)\theta} \overline{P'(e^{i\theta})}\big) \\ &\quad = e^{i\theta} \big(P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta})\big) + e^{i(n-1)\theta} \big(\overline{Q'(e^{i\theta})} + e^{i\gamma} \overline{P'(e^{i\theta})}\big), \end{aligned}$$

which gives

$$\begin{aligned} n\left|P(e^{i\theta}) + e^{i\gamma} Q(e^{i\theta})\right| &\leqslant \left|P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta})\right| + \left|\overline{Q'(e^{i\theta})} + e^{i\gamma} \overline{P'(e^{i\theta})}\right| \\ &= 2\left|P'(e^{i\theta}) + e^{i\gamma} Q'(e^{i\theta})\right|. \end{aligned} \tag{3.2}$$

Also, we have

$$\begin{aligned}
&\Big|D_\alpha P(e^{i\theta}) + e^{i\gamma} D_\alpha Q(e^{i\theta})\Big| \\
&= \Big|nP(e^{i\theta}) + (\alpha - e^{i\theta})P'(e^{i\theta}) + e^{i\gamma}\big(nQ(e^{i\theta}) + (\alpha - e^{i\theta})Q'(e^{i\theta})\big)\Big| \\
&= \Big|\big(nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})\big) + e^{i\gamma}\big(nQ(e^{i\theta}) - e^{i\theta}Q'(e^{i\theta})\big) \\
&\qquad\qquad + \alpha\big(P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\big)\Big| \\
&= \Big|\big(\overline{Q'(e^{i\theta})} + e^{i\gamma}\overline{P'(e^{i\theta})}\big)e^{i(n-1)\theta} + \alpha\big(P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\big)\Big| \\
&\leqslant \Big|\overline{P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})}\Big| + |\alpha|\Big|P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big| \\
&= \big(|\alpha| + 1\big)\Big|P'(e^{i\theta}) + e^{i\gamma}Q'(e^{i\theta})\Big|. \qquad (3.3)
\end{aligned}$$

Further, since $F(z) = P(z) + e^{i\gamma}Q(z)$ is a polynomial of degree n, so that $F_t(z) = P_t(z) + e^{i\gamma}Q_t(z)$ is a polynomial of degree at most $n-t$, $t \leqslant n-1$, we have, by repeated application of Lemma 2.3, for $r \geqslant 1$

$$\begin{aligned}
\int_0^{2\pi} \big|D_{\alpha_t}D_{\alpha_{t-1}}\dots D_{\alpha_1}F(e^{i\theta})\big|^r d\theta &\leqslant (n-t+1)^r(|\alpha_t|+1)^r \\
&\quad \times \int_0^{2\pi} \big|D_{\alpha_{t-1}}D_{\alpha_{t-2}}\dots D_{\alpha_1}F(e^{i\theta})\big|^r d\theta.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
&\int_0^{2\pi} \big|D_{\alpha_t}D_{\alpha_{t-1}}\cdots D_{\alpha_1}P(e^{i\theta}) + e^{i\gamma}D_{\alpha_t}D_{\alpha_{t-1}}\cdots D_{\alpha_1}Q(e^{i\theta})\big|^r d\theta \qquad (3.4)\\
&\leqslant (n-t+1)^r(|\alpha_t|+1)^r \\
&\quad \times \int_0^{2\pi} \big|D_{\alpha_{t-1}}D_{\alpha_{t-2}}\dots D_{\alpha_1}P(e^{i\theta}) + e^{i\gamma}D_{\alpha_{t-1}}D_{\alpha_{t-2}}\dots D_{\alpha_1}Q(e^{i\theta})\big|^r d\theta \\
&\ \vdots \\
&\leqslant (n-t+1)^r(n-t+2)^r\dots(n-1)^r(|\alpha_t|+1)^r(|\alpha_{t-1}|+1)^r\dots(|\alpha_2|+1)^r \\
&\qquad \times \int_0^{2\pi} \big|D_{\alpha_1}P(e^{i\theta}) + e^{i\gamma}D_{\alpha_1}Q(e^{i\theta})\big|^r d\theta.
\end{aligned}$$

Integrating both sides of (3.4) with respect to γ from 0 to 2π, we get, with the help of Lemma 2.4 and inequality (3.3), that for each $r \geqslant 1$,

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \big|P_t(e^{i\theta}) + e^{i\gamma}Q_t(e^{i\theta})\big|^r d\theta\, d\gamma \qquad (3.5)\\
&\leqslant 2\pi(n-t+1)^r(n-t+2)^r\dots(n-1)^r n^r \\
&\quad \times (|\alpha_t|+1)^r(|\alpha_{t-1}|+1)^r\dots(|\alpha_2|+1)^r(|\alpha_1|+1)^r \int_0^{2\pi} \big|P(e^{i\theta})\big|^r d\theta.
\end{aligned}$$

By Lemma 2.5, we have for each θ, $0\leqslant\theta<2\pi$ and for all β,α_j, $1\leqslant j\leqslant t$, $t\leqslant n-1$ with $|\beta|\leqslant 1$, $|\alpha_j|\geqslant 1$,

$$\Big|e^{it\theta}P_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}P(e^{i\theta})\Big|\leqslant\Big|e^{it\theta}Q_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(e^{i\theta})\Big| - mn_t\Big(\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|-\Big|e^{it\theta}+\beta\frac{A_{\alpha_t}}{2^t}\Big|\Big).$$

This implies

$$\begin{aligned}&\Big|e^{it\theta}P_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}P(e^{i\theta})\Big|+\frac{mn_t}{2}\Big(\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|-\Big|e^{it\theta}+\beta\frac{A_{\alpha_t}}{2^t}\Big|\Big)\\ &\leqslant\Big|e^{it\theta}Q_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(e^{i\theta})\Big|-\frac{mn_t}{2}\Big(\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|-\Big|e^{it\theta}+\beta\frac{A_{\alpha_t}}{2^t}\Big|\Big).\end{aligned}\tag{3.6}$$

Taking in Lemma 2.6

$$A=\Big|e^{it\theta}Q_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(e^{i\theta})\Big|,\quad B=\Big|e^{it\theta}P_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}P(e^{i\theta})\Big|,$$
$$C=\frac{mn_t}{2}\Big(\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|-\Big|e^{it\theta}+\beta\frac{A_{\alpha_t}}{2^t}\Big|\Big)$$

we get $B+C\leqslant A-C\leqslant A$. Hence for every real γ, by using Lemma 2.6, we get

$$\begin{aligned}&\Bigg|\Big\{\Big|e^{it\theta}Q_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(e^{i\theta})\Big|-\frac{mn_t}{2}\Big(\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|-\Big|e^{it\theta}+\beta\frac{A_{\alpha_t}}{2^t}\Big|\Big)\Big\}e^{i\gamma}\\ &+\Big\{\Big|e^{it\theta}P_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}P(e^{i\theta})\Big|+\frac{mn_t}{2}\Big(\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|-\Big|e^{it\theta}+\beta\frac{A_{\alpha_t}}{2^t}\Big|\Big)\Big\}\Bigg|\\ &\leqslant\Bigg|\Big|e^{it\theta}Q_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(e^{i\theta})\Big|e^{i\gamma}+\Big|e^{it\theta}P_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}P(e^{i\theta})\Big|\Bigg|.\end{aligned}$$

This implies for each $r\geqslant 1$,

$$\begin{aligned}&\int_0^{2\pi}\big|F(\theta)+e^{i\gamma}G(\theta)\big|^r d\theta\\ &\leqslant\int_0^{2\pi}\Bigg|\Big|e^{it\theta}P_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}P(e^{i\theta})\Big|+e^{i\gamma}\Big|e^{it\theta}Q_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(e^{i\theta})\Big|\Bigg|^r d\theta,\end{aligned}\tag{3.7}$$

where

$$F(\theta)=\Big|e^{it\theta}P_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}P(e^{i\theta})\Big|+\frac{mn_t}{2}\Big(\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|-\Big|e^{it\theta}+\beta\frac{A_{\alpha_t}}{2^t}\Big|\Big),$$
$$G(\theta)=\Big|e^{it\theta}Q_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(e^{i\theta})\Big|-\frac{mn_t}{2}\Big(\Big|\Lambda_t+\beta\frac{A_{\alpha_t}}{2^t}\Big|-\Big|e^{it\theta}+\beta\frac{A_{\alpha_t}}{2^t}\Big|\Big).$$

Integrating both sides of (3.7) with respect to γ from 0 to 2π, we get

$$\begin{aligned}&\int_0^{2\pi}\int_0^{2\pi}\big|F(\theta)+e^{i\gamma}G(\theta)\big|^r d\theta\,d\gamma\\ &\leqslant\int_0^{2\pi}\Big\{\int_0^{2\pi}\Bigg|\Big|e^{it\theta}P_t(e^{it\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}P(e^{i\theta})\Big|+e^{i\gamma}\Big|e^{it\theta}Q_t(e^{i\theta})+\beta\frac{n_tA_{\alpha_t}}{2^t}Q(e^{i\theta})\Big|\Bigg|^r d\gamma\Big\}d\theta\end{aligned}$$

$$= \int_0^{2\pi}\Big\{\int_0^{2\pi}\Big|e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_tA_{\alpha_t}}{2^t}P(e^{i\theta}) + e^{i\gamma}\Big(e^{it\theta}Q_t(e^{i\theta}) + \beta\frac{n_tA_{\alpha_t}}{2^t}Q(e^{i\theta})\Big)\Big|^r d\gamma\Big\}d\theta$$

$$= \int_0^{2\pi}\Big\{\int_0^{2\pi}\Big|e^{it\theta}\Big(P_t(e^{i\theta}) + e^{i\gamma}Q_t(e^{i\theta})\Big) + \beta\frac{n_tA_{\alpha_t}}{2^t}\Big(P(e^{i\theta}) + e^{i\gamma}Q(e^{i\theta})\Big)\Big|^r d\theta\Big\}d\gamma.$$

Therefore, it follows by Minkowski's inequality that

$$\Big\{\int_0^{2\pi}\int_0^{2\pi}\big|F(\theta) + e^{i\gamma}G(\theta)\big|^r d\theta\, d\gamma\Big\}^{1/r}$$

$$\leqslant \Big\{\int_0^{2\pi}\int_0^{2\pi}\Big|e^{it\theta}\Big(P_t(e^{i\theta}) + e^{i\gamma}Q_t(e^{i\theta})\Big) + \beta\frac{n_tA_{\alpha_t}}{2^t}\Big(P(e^{i\theta}) + e^{i\gamma}Q(e^{i\theta})\Big)\Big|^r d\theta\, d\gamma\Big\}^{1/r}$$

$$\leqslant \Big\{\int_0^{2\pi}\int_0^{2\pi}\Big|P_t(e^{i\theta}) + e^{i\gamma}Q_t(e^{i\theta})\Big|^r d\theta\, d\gamma\Big\}^{1/r} + |\beta|\frac{n_tA_{\alpha_t}}{2^t}\Big\{\int_0^{2\pi}\int_0^{2\pi}\Big|P(e^{i\theta}) + e^{i\gamma}Q(e^{i\theta})\Big|^r d\theta\, d\gamma\Big\}^{1/r},$$

which gives on using (3.2), (3.5) and Lemma 2.4 that for every β with $|\beta| \leqslant 1$, $r \geqslant 1$ and γ real,

$$\Big\{\int_0^{2\pi}\int_0^{2\pi}\big|F(\theta) + e^{i\gamma}G(\theta)\big|^r d\theta\, d\gamma\Big\}^{1/r} \leqslant (2\pi)^{1/r}n_t\Big\{B_{\alpha_t} + |\beta|\frac{A_{\alpha_t}}{2^{t-1}}\Big\}\Big\{\int_0^{2\pi}\big|P(e^{i\theta})\big|^r d\theta\Big\}^{1/r}. \tag{3.8}$$

If we take $a = F(\theta)$ and $b = G(\theta)$, since $|b| \geqslant |a|$ from (3.6), we get from Lemma 2.8 that

$$\int_0^{2\pi}\big|F(\theta) + e^{i\gamma}G(\theta)\big|^r d\gamma \geqslant |F(\theta)|^r\int_0^{2\pi}\big|1 + e^{i\gamma}\big|^r d\gamma. \tag{3.9}$$

Integrating both sides of (3.9) with respect to θ from 0 to 2π, we get from (3.8) that

$$\Big\{\int_0^{2\pi}\big|1 + e^{i\gamma}\big|^r d\gamma\int_0^{2\pi}\big|F(\theta)\big|^r d\theta\Big\}^{1/r} \leqslant (2\pi)^{1/r}n_t\Big\{B_{\alpha_t} + |\beta|\frac{A_{\alpha_t}}{2^{t-1}}\Big\}\Big\{\int_0^{2\pi}\Big|P(e^{i\theta})\Big|^r d\theta\Big\}^{1/r}. \tag{3.10}$$

Now using the fact that for every $\delta \in \mathbb{C}$ with $|\delta| \leqslant 1$,

$$\Big|e^{it\theta}P_t(e^{i\theta}) + \beta\frac{n_tA_{\alpha_t}}{2^t}P(e^{i\theta}) + \frac{\delta mn_t}{2}\Big(\Big|\Lambda_t + \beta\frac{A_{\alpha_t}}{2^t}\Big| - \Big|e^{it\theta} + \beta\frac{A_{\alpha_t}}{2^t}\Big|\Big)\Big| \leqslant |F(\theta)|,$$

the inequality (3.1) follows from (3.10). □

If we take $\alpha_1 = \alpha_2 = \ldots = \alpha_t = \alpha$, then dividing both sides of (3.1) by $|\alpha|^t$ and leting $|\alpha| \to \infty$, we get the following result.

Corollary 3.1. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$ then for all $\beta, \delta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $|\delta| \leqslant 1$, $1 \leqslant t \leqslant n-1$ and $r \geqslant 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{it\theta} P^t(e^{i\theta}) + \beta \frac{n_t}{2^t} P(e^{i\theta}) + \frac{mn_t}{2} \delta \Big(\Big|1 + \frac{\beta}{2^t}\Big| - \Big|\frac{\beta}{2^t}\Big| \Big) \right|^r d\theta \right\}^{1/r} \leqslant n_t C_r \Big(1 + \frac{|\beta|}{2^{t-1}}\Big) \left\{ \int_0^{2\pi} \big|P(e^{i\theta})\big|^r d\theta \right\}^{1/r}. \tag{3.11}$$

Now, we present and discuss some consequences of Theorem 3.1 and Corollary 3.1. First, we point out that inequalities involving polynomials in the sup-norm on the unit circle in the complex plane are a special case of the polynomial inequalities involving the integral norm. For example, if we let $r \to \infty$ in (3.1) and (3.11) and choose the argument of δ suitably with $|\delta| = 1$, noting that $C_r \to \frac{1}{2}$, we get the following results.

Corollary 3.2. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for all $\beta, \alpha_j \in \mathbb{C}$ with $|\beta| \leqslant 1$, $|\alpha_j| \geqslant 1$, $1 \leqslant j \leqslant t$, $t \leqslant n-1$,*

$$\max_{|z|=1} \left| z^t P_t(z) + \frac{\beta n_t A_{\alpha t}}{2^t} P(z) \right| \leqslant \frac{n_t}{2} \Big\{ \Big(B\alpha_t + \frac{|\beta| A_{\alpha_t}}{2^{t-1}} \Big) \max_{|z|=1} |P(z)| - \Big(\Big| \Lambda_t + \frac{\beta A_{\alpha_t}}{2^t} \Big| - \Big| z^t + \frac{\beta A_{\alpha_t}}{2^t} \Big| \Big) m \Big\}. \tag{3.12}$$

Corollary 3.3. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$ then for all $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$, $1 \leqslant t \leqslant n-1$,*

$$\max_{|z|=1} \left| z^t P^t(z) + \frac{\beta n_t}{2^t} P(z) \right| \leqslant \frac{n_t}{2} \Big\{ \Big(1 + \frac{|\beta|}{2^{t-1}} \Big) \max_{|z|=1} |P(z)| - \Big(\Big| 1 + \frac{\beta}{2^t} \Big| - \Big| \frac{\beta}{2^t} \Big| \Big) m \Big\}. \tag{3.13}$$

If we take $t = 1$ in (3.12), we get the following generalization of (1.7).

Corollary 3.4. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\alpha, \beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $|\alpha| \geqslant 1$,*

$$\max_{|z|=1} \left| zD_\alpha P(z) + \frac{n\beta}{2}(|\alpha| - 1) P(z) \right| \leqslant \frac{n}{2} \Big\{ \Big((|\alpha| + 1) + |\beta|(|\alpha| - 1) \Big) \max_{|z|=1} |P(z)| - \Big(\Big| \alpha + \frac{\beta(|\alpha| - 1)}{2} \Big| - \Big| z + \frac{\beta(|\alpha| - 1)}{2} \Big| \Big) m \Big\}. \tag{3.14}$$

Remark 3.1. Clearly (3.14) is a generalization of (1.7). The special case of Theorem 3.1 for $t = 1$ gives (1.10). Inequality (3.13) generalizes a result of Aziz and Dawood [**3**]. If we take $\beta = 0$ and $t = 1$ in (3.1), we obtain (1.9) for $r \geqslant 1$.

We now mention the following result that follows by taking $\delta = 0$ and $t = 1$ in (3.1) and provides a generalization of (1.8).

COROLLARY 3.5. *If $P \in \mathbb{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$ and $r \geqslant 1$,*

$$\left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha P(e^{i\theta}) + n\beta \frac{(|\alpha|-1)}{2} P(e^{i\theta}) \right|^r d\theta \right\}^{1/r} \leqslant nC_r \Big((|\alpha|+1) + |\beta|(|\alpha|-1) \Big) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{1/r}. \tag{3.15}$$

REMARK 3.2. If we take $\beta = 0$ in (3.15), then we get (1.8).

For the class of self-inversive polynomials, we prove the following result.

THEOREM 3.2. *If $P \in \mathbb{P}_n$ is self-inversive and $\beta \in \mathbb{C}$ with $|\beta| \leqslant 1$ and $r \geqslant 1$, then for any α with $|\alpha| \geqslant 1$,*

$$\begin{aligned} \frac{n}{2}(|\alpha|-1)(1-|\beta|) &\left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{1/r} \\ &\leqslant \left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha(e^{i\theta}) + \frac{n\beta(|\alpha|-1)}{2} P(e^{i\theta}) \right|^r d\theta \right\}^{1/r} \\ &\leqslant nC_r \Big((|\alpha|+1) + |\beta|(|\alpha|-1) \Big) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{1/r} \end{aligned} \tag{3.16}$$

and for any α with $|\alpha| \leqslant 1$

$$\begin{aligned} \frac{n}{2}(1-|\alpha|)(1-|\beta|) &\left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{1/r} \\ &\leqslant \left\{ \int_0^{2\pi} \left| e^{i\theta} D_\alpha(e^{i\theta}) + \frac{n\beta(1-|\alpha|)}{2} P(e^{i\theta}) \right|^r d\theta \right\}^{1/r} \\ &\leqslant nC_r \Big((|\alpha|+1) + |\beta|(|1-\alpha|) \Big) \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^r d\theta \right\}^{1/r} \end{aligned} \tag{3.17}$$

PROOF. Since $P \in \mathbb{P}_n$ is a self-inversive polynomial, therefore, $P(z) \equiv \zeta Q(z)$, where $|\zeta| = 1$ and $Q(z) = z^n \overline{P(1\bar{z})} \in \mathbb{P}_n$. Then for any $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$, we have

$$\left| zD_\alpha P(z) + \frac{n\beta}{2}(|\alpha|-1)P(z) \right| = \left| zD_\alpha Q(z) + \frac{n\beta}{2}(|\alpha|-1)Q(z) \right| \tag{3.18}$$

and for any $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, we have

$$\left| zD_\alpha P(z) + \frac{n\beta}{2}(1-|\alpha|)P(z) \right| = \left| zD_\alpha Q(z) + \frac{n\beta}{2}(1-|\alpha|)Q(z) \right| \tag{3.19}$$

for all $z \in \mathbb{C}$. The inequalities on the right hand sides of (3.16) and (3.17) can be obtained by proceeding similarly as in the proof of Theorem 3.1 and using (3.18) and (3.19). To prove the inequalities on the left hand sides of (3.16) and (3.18), note that for any α, β with $|\alpha| \geqslant 1$, $|\beta| \leqslant 1$, we have for $0 \leqslant \theta < 2\pi$,

$$\left| e^{i\theta} D_\alpha P(e^{i\theta}) + \frac{n\beta}{2}(|\alpha|-1)P(e^{i\theta}) \right| \tag{3.20}$$

$$\geqslant \left|D_\alpha P(e^{i\theta})\right| - \frac{n|\beta|}{2}(|\alpha|-1)|P(e^{i\theta})|$$
$$= |\alpha P'(e^{i\theta}) + nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| - \frac{n|\beta|}{2}(|\alpha|-1)|P(e^{i\theta})|$$
$$\geqslant |\alpha||P'(e^{i\theta})| - |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})| - \frac{n|\beta|}{2}(|\alpha|-1)|P(e^{i\theta})|.$$

It is easy to verify that for $0 \leqslant \theta < 2\pi$, $|P'(e^{i\theta})| = |nP(e^{i\theta}) - e^{i\theta}P'(e^{i\theta})|$, which on using in (3.20) gives

$$\left|e^{i\theta}D_\alpha P(e^{i\theta}) + \frac{n\beta}{2}(|\alpha|-1)P(e^{i\theta})\right| \geqslant (|\alpha|-1)|P'(e^{i\theta})| - \frac{n|\beta|}{2}(|\alpha|-1)|P(e^{i\theta})|. \tag{3.21}$$

Similarly, for $|\alpha| \leqslant 1$, we have

$$\left|e^{i\theta}D_\alpha P(e^{i\theta}) + \frac{n\beta}{2}(1-|\alpha|)P(e^{i\theta})\right| \geqslant (1-|\alpha|)|P'(e^{i\theta})| - \frac{n|\beta|}{2}(1-|\alpha|)|P(e^{i\theta})|. \tag{3.22}$$

Inequalities (3.21) and (3.22) when combined with Lemma 2.9, give

$$\left|e^{i\theta}D_\alpha P(e^{i\theta}) + \frac{n\beta}{2}(|\alpha|-1)P(e^{i\theta})\right| \geqslant \frac{n}{2}(|\alpha|-1)(1-|\beta|)|P(e^{i\theta})| \text{ for } |\alpha| \geqslant 1,$$
$$\left|e^{i\theta}D_\alpha P(e^{i\theta}) + \frac{n\beta}{2}(1-|\alpha|)P(e^{i\theta})\right| \geqslant \frac{n}{2}(1-|\alpha|)(1-|\beta|)|P(e^{i\theta})| \text{ for } |\alpha| \leqslant 1,$$

wherefrom we can obtain the inequalities on the left-hand sides of (3.16) and (3.17). □

If in (3.16), we divide throughout by $|\alpha|$ and make $|\alpha| \to \infty$, then we get the following result.

Corollary 3.6. *If $P \in \mathbb{P}_n$ is self-inversive, then for every $\beta \in \mathbb{C}$, with $|\beta| \leqslant 1$ and $r \geqslant 1$,*

$$\frac{n}{2}(1-|\beta|)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{1/r} \leqslant \left\{\int_0^{2\pi}\left|e^{i\theta}P'(e^{i\theta}) + \frac{n\beta}{2}P(e^{i\theta})\right|^r d\theta\right\}^{1/r} \leqslant nC_r(1+|\beta|)\left\{\int_0^{2\pi}\left|P(e^{i\theta})\right|^r d\theta\right\}^{1/r}. \tag{3.23}$$

Remark 3.3. For $\beta = 0$, Corollary 3.6 reduces to a result of Govil and Jain [**11**, Corollary 1]. Inequalities (3.16) and (3.17) also represent a generalization of a result due to Aziz and Rather [**5**, Theorem 3].

Making $r \to \infty$ in (3.23) and noting that $C_r \to \frac{1}{2}$ as $r \to \infty$, we get the following generalization of a result due to O'Hara and Rodriguez [**13**].

Corollary 3.7. *If $P \in \mathbb{P}_n$ is self-inversive, then for any $|\beta| \leqslant 1$, we have*

$$\frac{n}{2}(1-|\beta|)\max_{|z|=1}|P(z)| \leqslant \max_{|z|=1}\left|zP'(z) + n\frac{\beta}{2}P(z)\right| \leqslant \frac{n}{2}(1+|\beta|)\max_{|z|=1}|P(z)|.$$

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