PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 110(124) (2021), 47–55 DOI: https://doi.org/10.2298/PIM2124047A

ON GRADED I_e-PRIME SUBMODULES OF GRADED MODULES OVER GRADED COMMUTATIVE RINGS

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Abstract. Let G be a group with identity e. Let R be a G-graded commutative ring with identity and M a graded R-module. We introduce the concept of graded I_e-prime submodule as a generalization of a graded prime submodule for $I = \bigoplus_{g\in G} I_g$ a fixed graded ideal of R. We give a number of results concerning this class of graded submodules and their homogeneous components. A proper graded submodule N of M is said to be a graded I_e-prime submodule of M if whenever $r_g \in h(R)$ and $m_h \in h(M)$ with $r_g m_h \in N - I_e N$, then either $r_g \in (N :_R M)$ or $m_h \in N$.

1. Introduction and preliminaries

Throughout this paper all rings are commutative, with identity and all modules are unitary.

Graded prime submodules of graded modules over graded commutative rings, have been introduced and studied by many authors, (see for example [**1**, **3**, **4**, **6**, **7**, **10**, **16**]). Then, many generalizations of graded prime submodules were studied such as graded primary, graded classical prime, graded weakly prime and graded 2-absorbing submodules (see for example [**2**, **5**, **9**, **16**]). Akray and Hussein in [**8**] introduced the concept of I-prime submodule over a commutative ring as a new generalization of prime submodule.

The scope of this paper is devoted to the theory of graded modules over graded commutative rings. Here, we introduce the concept of graded I_e-prime submodule as a new generalization of a graded prime submodule. A number of results concerning this class of graded submodules and their homogeneous components are given. For example, we give a characterization of graded I_e-prime submodule (see Theorem 2.8). We also study the behaviour of graded I_e-prime submodule under localization (see Theorem 2.9).

2010 *Mathematics Subject Classification:* 13A02; 16W50L.

Key words and phrases: graded I_e-prime submodules, graded prime submodules, graded I_e-prime ideals.

Communicated by Nebojša Ikodinović.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [**12**–**15**] for these basic properties and more information on graded rings and modules.

Let G be a multiplicative group with identity element e. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_g of R indexed by the elements $g \in G$ such that $R = \bigoplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The elements of R_g are called homogeneous of degree g and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{g\in G} R_g$. If $r \in R$, then r can be written uniquely as $\sum_{g\in G} r_g$, where r_g is called a homogeneous component of r in R_g. Moreover, R_e is a subring of R and $1 \in R_e$. Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \sum_{g\in G}(I \cap R_g) := \sum_{g\in G} I_g$, (see [**15**]). Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. A left R-module M is said to be a graded R-module (or G-graded R-module) if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M such that $M = \bigoplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Similarly, if an element of M belongs to $\cup_{g\in G} M_g = h(M)$, then it is called a homogeneous. Note that M_g is an R_e-module for every $g \in G$. Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. A submodule N of M is said to be a graded submodule of M if $N = \bigoplus_{g\in G}(N \cap M_g) := \bigoplus_{g\in G} N_g$. In this case, N_g is called the g-component of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g := (M_g + N)/N$ for $g \in G$, (see [**15**]).

The graded radical of a graded ideal I, denoted by $Gr(I)$, is the set of all $r = \sum_{g\in G} r_g \in R$ such that for each $g \in G$ there exists $n_g \in \mathbb{N}$ with $r_g^{n_g} \in I$. Note that, if x is a homogeneous element, then $x \in Gr(I)$ if and only if $x^n \in I$ for some $n \in \mathbb{N}$, (see [**17**]).

Let R be a G-graded ring and M be a graded R-module. It is shown in [**10**, Lemma 2.1] that if N is a graded submodule of M, then $(N :_R M) = \{r \in R : rN \subseteq M\}$ is a graded ideal of R. Also, for any $r_g \in h(R)$, the graded submodule $\{m \in M : r_g m \in N\}$ will be denoted by $(N :_M r_g)$. The *graded radical* of a graded submodule N of M, denoted by $Gr_M(N)$, is defined to be the intersection of all graded prime submodules of M containing N. If N is not contained in any graded prime submodule of M, then $Gr_M(N) = M$, (see [**16**]).

Let R be a G-graded ring and $S \subseteq h(R)$ be a multiplicatively closed subset of R. Then the ring of fractions $S^{-1}R$ is a graded ring which is called the graded ring of fractions. Indeed, $S^{-1}R = \bigoplus_{g\in G}(S^{-1}R)_g$ where $(S^{-1}R)_g = \{r/s : r \in R, s \in S$ and $g = (\deg s)^{-1}(\deg r)\}$. Let M be a graded module over a G-graded ring R and $S \subseteq h(R)$ be a multiplicatively closed subset of R. The module of fractions $S^{-1}M$ over a graded ring $S^{-1}R$ is a graded module which is called the module of fractions, if $S^{-1}M = \bigoplus_{g\in G}(S^{-1}M)_g$ where $(S^{-1}M)_g = \{m/s : m \in M, s \in S$ and $g = (\deg s)^{-1}(\deg m)\}$. We write $h(S^{-1}R) = \cup_{g\in G}(S^{-1}R)_g$ and $h(S^{-1}M) = \cup_{g\in G}(S^{-1}M)_g$. For any graded submodule N of M, the graded submodule $S^{-1}N$ of $S^{-1}M$ is defined by $S^{-1}N = \{\alpha \in S^{-1}M : \alpha = m/s$ for $m \in N$ and $s \in S\}$ and $S^{-1}N \neq S^{-1}M$ if and only if $S \cap (N :_R M) = \emptyset$, (see [**15**]).

2. Results

DEFINITION 2.1. Let R be a G-graded ring, M a graded R-module, $I = \bigoplus_{g\in G} I_g$ a graded ideal of R, $N = \bigoplus_{g\in G} N_g$ a graded submodule of M and $g \in G$.

(i) We say that N_g is a g-I_e-prime submodule of the R_e-module M_g if $N_g \neq M_g$; and whenever $r_e \in R_e$ and $m_g \in M_g$ with $r_e m_g \in N_g - I_e N_g$, then either $m_g \in N_g$ or $r_e \in (N_g :_{R_e} M_g)$.

(ii) We say that N is a graded I_e-prime submodule of M if $N \neq M$; and whenever $r_h \in h(R)$ and $m_\lambda \in h(M)$ with $r_h m_\lambda \in N - I_e N$, then either $m_\lambda \in N$ or $r_h \in (N :_R M)$.

DEFINITION 2.2. Let R be a G-graded ring and $I = \bigoplus_{g\in G} I_g$ and $J = \bigoplus_{g\in G} J_g$ graded ideals of R. Then J_e is said to be an e-I_e-prime ideal of R_e, if $J_e \neq R_e$; and whenever $r_e s_e \in J_e - I_e J_e$, where $r_e, s_e \in R_e$, then either $r_e \in J_e$ or $s_e \in J_e$.

Let R be a G-graded ring, M a graded R-module and $I = \bigoplus_{g\in G} I_g$ a graded ideal of R. Recall from [**10**] that a proper graded submodule $N = \bigoplus_{g\in G} N_g$ of a graded R-module M is said to be *a graded prime submodule of* M if whenever $r_g m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then either $m_h \in N$ or $r_g \in (N :_R M)$. It is easy to see that every graded prime submodule of M is a graded I_e-prime submodule. The following example shows that the converse is not true in general.

EXAMPLE 2.1. Let $G = \mathbb{Z}_2$ and $R = \mathbb{Z}$ be a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}_{12}$ be a graded R-module with $M_0 = \mathbb{Z}_{12}$ and $M_1 = \{0\}$. Now, consider a graded ideal $I = 4\mathbb{Z}$ of R and a graded submodule $N = \langle 4\rangle$ of M. We conclude that N is not a graded prime submodule of M since $2 \cdot 2 = 4 \in N$ but neither $2 \in N$ nor $2 \in (N :_R M) = 4\mathbb{Z}$. However, N is a graded I_e-prime submodule of M since $N - I_0 N = \langle 4\rangle - 4\mathbb{Z} \cdot \langle 4\rangle = \emptyset$.

Recall from [**10**] that if $N = \bigoplus_{g\in G} N_g$ is a graded submodule of a graded R-module M and $g \in G$, then N_g is called *a g-prime submodule of an R_e-module M_g* if $N_g \neq M_g$; and whenever $r_e m_g \in N_g$ where $r_e \in R_e$ and $m_g \in M_g$, then either $m_g \in N_g$ or $r_e \in (N_g :_{R_e} M_g)$.

THEOREM 2.1. *Let R be a G-graded ring, M a graded R-module, $I = \bigoplus_{g\in G} I_g$ a graded ideal of R, $N = \bigoplus_{g\in G} N_g$ a graded submodule of M and $g \in G$. If N_g is a g-I_e-prime submodule of M_g, then either N_g is a g-prime submodule of M_g or $(N_g :_{R_e} M_g)N_g \subseteq I_e N_g$.*

PROOF. Suppose that N_g is a g-I_e-prime submodule of M_g such that $(N_g :_{R_e} M_g)N_g \subseteq I_e N_g$. Now, let $r_e \in R_e$ and $m_g \in M_g$ with $r_e m_g \in N_g$. If $r_e m_g \notin I_e N_g$, then either $m_g \in N_g$ or $r_e \in (N_g :_{R_e} M_g)$ as N_g is a g-I_e-prime submodule of M_g. Assume that $r_e m_g \in I_e N_g$. If $r_e N_g \subseteq I_e N_g$, then there exists $x_g \in N_g$ such that $r_e x_g \notin I_e N_g$, so we get $r_e(m_g + x_g) \in N_g - I_e N_g$ and then either $m_g + x_g \in N_g$ or $r_e \in (N_g :_{R_e} M_g)$ as N_g is a g-I_e-prime submodule of M_g. Hence, either $m_g \in N_g$ or $r_e \in (N_g :_{R_e} M_g)$. If $(N_g :_{R_e} M_g)m_g \subseteq I_e N_g$, there exists $t_e \in (N_g :_{R_e} M_g)$ such that $t_e m_g \notin I_e N_g$, so we get $(r_e + t_e)m_g \in N_g - I_e N_g$ and then either $m_g \in N_g$ or $r_e + t_e \in (N_g :_{R_e} M_g)$ as N_g is a g-I_e-prime submodule of M_g. Hence, either

$m_g \in N_g$ or $r_e \in (N_g :_{R_e} M_g)$. Now, we can assume that $r_e N_g \subseteq I_e N_g$ and $(N_g :_{R_e} M_g) m_g \subseteq I_e N_g$. But $(N_g :_{R_e} M_g) N_g \subseteq I_e N_g$, so there exist $s_e \in (N_g :_{R_e} M_g)$ and $l_g \in N_g$ such that $s_e l_g \notin I_e N_g$. Thus $(r_e + s_e)(m_g + l_g) \in N_g - I_e N_g$ gives either $m_g + l_g \in N_g$ or $r_e + s_e \in (N_g :_{R_e} M_g)$ as N_g is a g-I_e-prime submodule of M_g. Hence, either $m_g \in N_g$ or $r_e \in (N_g :_{R_e} M_g)$. Therefore, N_g is a g-prime submodule of M_g. □

Corollary 2.1. *Let R be a G-graded ring, M a graded R-module, $N = \bigoplus_{g \in G} N_g$ a graded submodule of M and $g \in G$. If N_g is a g-0-prime submodule of M_g such that $(N_g :_{R_e} M_g) N_g \neq 0$, then N_g is a g-prime submodule of M_g.*

Proof. Take $I = 0$ in the Theorem 2.1. □

Recall from [**11**] that a graded R-module M is called a graded multiplication if for each graded submodule N of M, we have $N = IM$ for some graded ideal I of R. If N is graded submodule of a graded multiplication module M, then $N = (N :_R M)M$. Let N and K be two graded submodules of a graded multiplication R-module M with $N = I_1 M$ and $K = I_2 M$ for some graded ideals I_1 and I_2 of R. The product of N and K denoted by NK is defined by $NK = I_1 I_2 M$.

Corollary 2.2. *Let R be a G-graded ring, M a graded R-module, $I = \bigoplus_{g \in G} I_g$ a graded ideal of R, $N = \bigoplus_{g \in G} N_g$ a graded submodule of M and $g \in G$. If M_g is a multiplication R_e-module and N_g is a g-I_e-prime submodule which is not a g-prime submodule of M_g, then $N_g^2 \subseteq I_e N_g$.*

Proof. Suppose that N_g is a g-I_e-prime submodule of M_g which is not a g-prime, so by Theorem 2.1, we get $(N_g :_{R_e} M_g) N_g \subseteq I_e N_g$. Now, as M_g is a multiplication R_e-module we get $N_g^2 = (N_g :_{R_e} M_g)^2 M_g \subseteq (N_g :_{R_e} M_g) N_g \subseteq I_e N_g$. Therefore, $N_g^2 \subseteq I_e N_g$. □

Theorem 2.2. *Let R be a G-graded ring, M a graded R-module, $I = \bigoplus_{g \in G} I_g$ a graded ideal of R, $N = \bigoplus_{g \in G} N_g$ a graded submodule of M and $g \in G$. Then the following statements are equivalent:*

(i) *N_g is a g-I_e-prime submodule of M_g.*
(ii) *For $r_e \in R_e \backslash (N_g :_{R_e} M_g)$, $(N_g :_{M_g} r_e) = N_g \cup (I_e N_g :_{M_g} r_e)$.*
(iii) *For $r_e \in R_e \backslash (N_g :_{R_e} M_g)$, either $(N_g :_{M_g} r_e) = N_g$ or $(N_g :_{M_g} r_e) = (I_e N_g :_{M_g} r_e)$.*

Proof. (i) ⇒ (ii) Suppose that N_g is a g-I_e-prime submodule of M_g and $r_e \in R_e \backslash (N_g :_{R_e} M_g)$. Let $m_g \in (N_g :_{M_g} r_e)$, so $r_e m_g \in N_g$. If $r_e m_g \notin I_e N_g$, then $m_g \in N_g$. Now, if $r_e m_g \in I_e N_g$, then $m_g \in (I_e N_g :_{M_g} r_e)$. Hence, $(N_g :_{M_g} r_e) = N_g \cup (I_e N_g :_{M_g} r_e)$.

(ii) ⇒ (iii) If a submodule is a union of two submodules, then it is equal to one of them.

(iii) ⇒ (i) Let $r_e \in R_e$ and $m_g \in M_g$ such that $r_e m_g \in N_g - I_e N_g$ and $r_e \notin (N_g :_{R_e} M_g)$, so $m_g \in (N_g :_{M_g} r_e)$. Now, by (iii) we get either $(N_g :_{M_g} r_e) = N_g$ or $(N_g :_{M_g} r_e) = (I_e N_g :_{M_g} r_e)$. But $r_e m_g \notin I_e N_g$, so $m_g \notin (I_e N_g :_{M_g} r_e)$. Hence,

$(N_g :_{M_g} r_e) = N_g$ and then $m_g \in N_g$. Therefore, N_g is a g-I_e-prime submodule of M_g. $\square$

Recall from [**9**] that a proper graded submodule $N = \bigoplus_{g\in G} N_g$ of a graded R-module M is said to be *a graded weakly prime submodule of* M if whenever $0 \neq r_g m_h \in N$ where $r_g \in h(R)$ and $m_h \in h(M)$, then either $m_h \in N$ or $r_g \in (N :_R M)$.

THEOREM 2.3. *Let R be a G-graded ring, M a graded R-module, $I = \bigoplus_{g\in G} I_g$ a graded ideal of R and N a proper graded submodule of M. Then the following statements are equivalent:*

(i) *N is a graded I_e-prime submodule of M.*
(ii) *N/I_eN is a graded weakly prime submodule of M/I_eN.*

PROOF. (i) $\Rightarrow$ (ii) Suppose that N is a graded I_e-prime submodule of M. Let $r_g \in h(R)$ and $(m_h + I_eN) \in h(M/I_eN)$ with $0_{M/I_eN} \neq (r_g m_h + I_eN) \in N/I_eN$. This yields that $r_g m_h \in N - I_eN$. Hence, either $m_h \in N$ or $r_g M \subseteq N$ as N is a graded I_e-prime submodule of M. Then either $(m_h + I_eN) \in N/I_eN$ or $r_g(M/I_eN) \subseteq N/I_eN$. Therefore, N/I_eN is a graded weakly prime submodule of M/I_eN.

(ii) $\Rightarrow$ (i) Suppose that N/I_eN is a graded weakly prime submodule of M/I_eN. Let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g m_h \in N - I_eN$. Then it follows that $0_{M/I_eN} \neq (r_g m_h + I_eN) = r_g(m_h + I_eN) \in N/I_eN$. Thus, either $r_g \in (N/I_eN :_R M/I_eN)$ or $(m_h + I_eN) \in N/I_eN$ and then either $r_g \in (N :_R M)$ or $m_h \in N$. Therefore, N is a graded I_e-prime submodule of M. $\square$

LEMMA 2.1. *Let R be a G-graded ring, M a graded R-module and $I = \bigoplus_{g\in G} I_g$ a graded ideal of R and let $g \in G$. If M_g is a multiplication R_e-module and N_g is a g-I_e-prime submodule of M_g with $(N_g :_{R_e} M_g) \subseteq I_e$, then $Gr((I_eN_g :_{R_e} M_g))N_g = I_eN_g$.*

PROOF. Suppose that N_g is a g-I_e-prime submodule of M_g with $(N_g :_{R_e} M_g) \subseteq I_e$. Clearly, $I_eN_g \subseteq (I_eN_g :_{R_e} M_g)N_g \subseteq Gr((I_eN_g :_{R_e} M_g))N_g$. Now, let $r_e \in Gr((I_eN_g :_{R_e} M_g))$. If $r_e \in I_e$, then $r_eN_g \subseteq I_eN_g$. So we can assume that $r_e \notin I_e$, which follows that $r_e \notin (N_g :_{R_e} M_g)$, then by Theorem 2.2, either $(N_g :_{M_g} r_e) = N_g$ or $(N_g :_{M_g} r_e) = (I_eN_g :_{M_g} r_e)$. If $(N_g :_{M_g} r_e) = (I_eN_g :_{M_g} r_e)$, then $r_eN_g \subseteq r_e(N_g :_{M_g} r_e) = r_e(I_eN_g :_{M_g} r_e) \subseteq I_eN_g$. On the other hand, if $(N_g :_{M_g} r_e) = N_g$, let $n \geqslant 2$ be the smallest integer such that $r_e^n \in (I_eN_g :_{R_e} M_g)$. Since $I_eN_g \subseteq N_g$, $r_e^n M_g = r_e(r_e^{n-1}M_g) \subseteq N_g$ which yields that $(r_e^{n-1}M_g) \subseteq N_g$. Thus, after a finite number of steps we get $r_eM_g \subseteq N_g$ which is a contradiction. Hence, $Gr((I_eN_g :_{R_e} M_g))N_g \subseteq I_eN_g$. Therefore, $Gr((I_eN_g :_{R_e} M_g))N_g = I_eN_g$. $\square$

THEOREM 2.4. *Let R be a G-graded ring, M a graded R-module, $I = \bigoplus_{g\in G} I_g$ a graded ideal of R, $N = \bigoplus_{g\in G} N_g$ a graded submodule of M and $g \in G$ with $(I_eN_g :_{R_e} M_g) = I_e(N_g :_{R_e} M_g)$. If N_g is a g-I_e-prime submodule of M_g, then $(N_g :_{R_e} M_g)$ is an e-I_e-prime ideal of R_e.*

PROOF. Suppose that N_g is a g-I_e-prime submodule of M_g. Now, let $r_e, s_e \in R_e$ such that $r_e s_e \in (N_g :_{R_e} M_g) - I_e(N_g :_{R_e} M_g)$ and $r_e \notin (N_g :_{R_e} M_g)$. By Theorem 2.2 since $r_e \notin (N_g :_{R_e} M_g)$, we get either $(N_g :_{M_g} r_e) = N_g$ or $(N_g :_{M_g} r_e) = (I_e N_g :_{M_g} r_e)$. If $r_e s_e M_g \subseteq I_e N_g$, then $r_e s_e \in (I_e N_g :_{R_e} M_g) = I_e(N_g :_{R_e} M_g)$, a contradiction. Hence, $r_e s_e M_g \subseteq I_e N_g$. Since $s_e M_g \subseteq (N_g :_{M_g} r_e)$ and $s_e M_g \subseteq (I_e N_g :_{M_g} r_e)$, we get $s_e M_g \subseteq N_g$. So $s_e \in (N_g :_{R_e} M_g)$. □

THEOREM 2.5. *Let R be a G-graded ring, $I = \bigoplus_{g\in G} I_g$ a graded ideal of R and $J = \bigoplus_{g\in G} J_g$ a proper graded ideal of R. Then the following statements are equivalent:*

(i) *J_e is an e-I_e-prime ideal of R_e.*
(ii) *For $r_e \in R_e - J_e$, $(J_e :_{R_e} r_e) = J_e \cup (I_e J_e :_{R_e} r_e)$.*
(iii) *For $r_e \in R_e - J_e$, $(J_e :_{R_e} r_e) = J_e$ or $(J_e :_{R_e} r_e) = (I_e J_e :_{R_e} r_e)$.*
(iv) *For any two graded ideals $K = \bigoplus_{g\in G} K_g$ and $L = \bigoplus_{h\in G} L_h$ of R with $K_e L_e \subseteq J_e$ and $K_e L_e \subseteq I_e J_e$, it follows that either $K_e \subseteq J_e$ or $L_e \subseteq J_e$.*

PROOF. (i) ⇒ (ii) Suppose that J_e is an e-I_e-prime ideal of R_e and $r_e \in R_e - J_e$. It is easy to see that $J_e \cup (I_e J_e :_{R_e} r_e) \subseteq (J_e :_{R_e} r_e)$. Now, let $s_e \in (J_e :_{R_e} r_e)$, so $r_e s_e \in J_e$. If $r_e s_e \in J_e - I_e J_e$, then $s_e \in J_e$. If $r_e s_e \in I_e J_e$, then $s_e \in (I_e J_e :_{R_e} r_e)$, which implies that $(J_e :_{R_e} r_e) \subseteq J_e \cup (I_e J_e :_{R_e} r_e)$ and hence $(J_e :_{R_e} r_e) = J_e \cup (I_e J_e :_{R_e} r_e)$.

(ii) ⇒ (iii) Note that if an ideal is a union of two ideals, then it is equal to one of them.

(iii) ⇒ (iv) Let $K = \bigoplus_{g\in G} K_g$ and $L = \bigoplus_{h\in G} L_h$ be two graded ideals of R such that $K_e L_e \subseteq J_e$, $K_e L_e \subseteq I_e J_e$ and neither $K_e \subseteq J_e$ nor $L_e \subseteq J_e$. Let $k_e \in K_e$. If $k_e \notin J_e$, then $k_e L_e \subseteq J_e$ gives $L_e \subseteq (J_e :_{R_e} k_e)$. Now, by (iii) since $k_e \in R_e - J_e$, $(J_e :_{R_e} k_e) = J_e$ or $(J_e :_{R_e} k_e) = (I_e J_e :_{R_e} k_e)$. But $L_e \subseteq J_e$, so $L_e \subseteq (I_e J_e :_{R_e} k_e)$. Hence, $k_e L_e \subseteq I_e J_e$. Now, if $k_e \in J_e$, then since $K_e \subseteq J_e$, there exists $k'_e \in K_e - J_e$, so we have $(k_e + k'_e) L_e \subseteq J_e$ and by using the first case we get $k'_e L_e \subseteq I_e J_e$. Since $k_e + k'_e \in R_e - J_e$ and $L_e \subseteq J_e$, $L_e \subseteq (I_e J_e :_{R_e} k_e + k'_e)$ and then $(k_e + k'_e L_e \subseteq I_e J_e$. Hence $k_e L_e \subseteq I_e J_e$ since $k'_e L_e \subseteq I_e J_e$. Thus, $K_e L_e \subseteq I_e J_e$, a contradiction.

(iv) ⇒ (i) Let $r_e, s_e \in R_e$ such that $r_e s_e \in J_e - I_e J_e$. Then $K = (r_e)$ and $L = (s_e)$ are graded ideals of R generated by r_e and s_e, respectively. Now, $K_e L_e \subseteq J_e$ and $K_e L_e \subseteq I_e J_e$. So either $K_e \subseteq J_e$ or $L_e \subseteq J_e$ and hence $r_e \in J_e$ or $s_e \in J_e$. □

THEOREM 2.6. *Let R be a G-graded ring, M a graded R-module, $I = \bigoplus_{g\in G} I_g$ a graded ideal of R, $N = \bigoplus_{g\in G} N_g$ a proper graded submodule of M and $g \in G$ such that $(I_e N_g :_{R_e} M_g) = I_e(N_g :_{R_e} M_g)$. If N_g is a g-I_e-prime submodule of M_g and M_g is a multiplication R_e-module, then for any graded submodules $K = \bigoplus_{h\in G} K_h$ and $L = \bigoplus_{h\in G} L_h$ with $K_g L_g \subseteq N_g$ and $K_g L_g \subseteq I_e N_g$, it holds either $K_g \subseteq N_g$ or $L_g \subseteq N_g$.*

PROOF. Suppose that N_g is a g-I_e-prime submodule of M_g so by Theorem 2.4 we get that $(N_g :_{R_e} M_g)$ is an e-I_e-prime ideal of R_e. Now, let $K = \bigoplus_{h\in G} K_h$ and $L = \bigoplus_{h\in G} L_h$ be two graded submodules of M such that $K_g L_g \subseteq N_g$ and $K_g L_g \subseteq I_e N_g$ and neither $K_g \subseteq N_g$ nor $L_g \subseteq N_g$. Hence, $K_g L_g = (K_g :_{R_e}$

$M_g)(L_g :_{R_e} M_g)M_g \subseteq N_g$ and then $(K_g :_{R_e} M_g)(L_g :_{R_e} M_g) \subseteq (N_g :_{R_e} M_g)$. Now, $K_g = (K_g :_{R_e} M_g)M_g \subseteq N_g$ gives $(K_g :_{R_e} M_g) \subseteq (N_g :_{R_e} M_g)$, and $L_g = (L_g :_{R_e} M_g)M_g \subseteq N_g$ gives $(L_g :_{R_e} M_g) \subseteq (N_g :_{R_e} M_g)$. So by Theorem 2.5, we get $(K_g :_{R_e} M_g)(L_g :_{R_e} M_g) \subseteq I_e(N_g :_{R_e} M_g) = (I_eN_g :_{R_e} M_g)$ which yields that $K_gL_g = (K_g :_{R_e} M_g)(L_g :_{R_e} M_g)M_g \subseteq I_eN_g$, a contradiction. Therefore, either $K_g \subseteq N_g$ or $L_g \subseteq N_g$. □

Suppose M_g is a multiplication R_e-module and m_{1_g}, $m_{2_g} \in M_g$. Then we can define the product of m_{1_g} and m_{2_g} as $m_{1_g}m_{2_g} = R_em_{1_g}\ R_em_{2_g} = (R_em_{1_g} :_{R_e} M_g)(R_em_{2_g} : M_g)M_g$. Thus we have the following corollary.

Corollary 2.3. *Let R be a G-graded ring, M a graded R-module, $I = \bigoplus_{g\in G} I_g$ a graded ideal of R, $N = \bigoplus_{g\in G} N_g$ a proper graded submodule of M and $g \in G$ such that $(I_eN_g :_{R_e} M_g) = I_e(N_g :_{R_e} M_g)$. If M_g is a multiplication R_e-module and N_g is a g-I_e-prime submodule of M_g, then for any $m_{1_g}, m_{2_g} \in M_g$ with $m_{1_g}m_{2_g} \in N_g - I_eN_g$, we have either $m_{1_g} \in N_g$ or $m_{2_g} \in N_g$.*

Proof. Let $m_{1_g}, m_{2_g} \in M_g$ such that $m_{1_g}m_{2_g} \in N_g - I_eN_g$. Then $K = (m_{1_g})$ and $L = (m_{2_g})$ are graded submodules of M generated by m_{1_g} and m_{2_g}, respectively. Since $K_gL_g \subseteq N_g$ and $K_gL_g \subseteq I_eN_g$, by Theorem 2.6, we get either $m_{1_g} \in N_g$ or $m_{2_g} \in N_g$. □

Theorem 2.7. *Let R be a G-graded ring, M_1 and M_2 be two graded R-modules, $I = \bigoplus_{g\in G} I_g$ a graded ideal of R and N_1 and N_2 be two graded submodules of M_1 and M_2, respectively. Then:*

(i) *If N_1 is a graded I_e-prime submodule of M_1, then $N_1 \times M_2$ is a graded I_e-prime submodule of $M_1 \times M_2$.*
(ii) *If N_2 is a graded I_e-prime submodule of M_2, then $M_1 \times N_2$ is a graded I_e-prime submodule of $M_1 \times M_2$.*

Proof. (i) Suppose that N_1 is a graded I_e-prime submodule of M_1. Now, let $r_g \in h(R)$ and $(m_{1_h}, m_{2_h}) \in h(M_1 \times M_2)$ such that $r_g(m_{1_h}, m_{2_h}) = (r_gm_{1_h}, r_gm_{2_h}) \in (N_1 \times M_2) - I_e(N_1 \times M_2) = (N_1 - I_eN_1) \times (M_2 - I_eM_2)$, which implies that $r_gm_{1_h} \in N_1 - I_eN_1$. Hence, either $m_{1_h} \in N_1$ or $r_gM_1 \subseteq N_1$ and then either $(m_{1_h}, m_{2_h}) \in N_1 \times M_2$ or $r_g(M_1 \times M_2) \subseteq N_1 \times M_2$. Therefore, $N_1 \times M_2$ is a graded I_e-prime submodule of $M_1 \times M_2$.

(ii) The proof is similar to (i). □

Theorem 2.8. *Let R be a G-graded ring, M a graded R-module, $I = \bigoplus_{g\in G} I_g$ a graded ideal of R and N a proper graded submodule of M. Let $K = \bigoplus_{h\in G} K_h$ be a graded submodule of M. Then the following statements are equivalent:*

(i) *N is a graded I_e-prime submodule of M.*
(ii) *For any $r_g \in h(R)$ and $h \in G$ with $r_gK_h \subseteq N$ and $r_gK_h \subseteq I_eN$, it follows either $K_h \subseteq N$ or $r_g \in (N :_R M)$.*

Proof. (i) ⇒ (ii) Suppose that N is a graded I_e-prime submodule of M. Now, let $r_g \in h(R)$ and $h \in G$ such that $r_gK_h \subseteq N$, $r_gK_h \subseteq I_eN$ and $r_g \notin (N :_R M)$. Let $k_h \in K_h$. If $r_gk_h \notin I_eN$, then $r_gk_h \in N - I_eN$. So $k_h \in N$ as N is a graded

I_e-prime submodule of M. Now, if $r_g k_h \in I_e N$, since $r_g K_h \subseteq I_e N$, there exists $k'_h \in K_h$ such that $r_g k'_h \notin I_e N$, but $r_g k'_h \in N - I_e N$ and $r_g \notin (N :_R M)$, so $k'_h \in N$. Hence, we get $r_g(k_h + k'_h) \in N - I_e N$, which yields that $k_h + k'_h \in N$ and then $k_h \in N$. Therefore, $K_h \subseteq N$.

(ii) $\Rightarrow$ (i) Let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g m_h \in N - I_e N$. Then $K = (m_h)$ is a graded submodule generated by m_h. Since $r_g K_h \subseteq N$ and $r_g K_h \subseteq I_e N$, by (ii), we get either $K_h \subseteq N$ or $r_g \in (N :_R M)$ and then either $m_h \in N$ or $r_g \in (N :_R M)$. Therefore, N is a graded I_e-prime submodule of M. □

Recall that a graded zero-divisor on a graded R-module M is an element $r_g \in h(R)$ for which there exists $m_h \in h(M)$ such that $m_h \neq 0$ but $r_g m_h = 0$. The set of all graded zero-divisors on M is denoted by $G\text{-}Zdv_R(M)$, see [**4**].

The following result studies the behavior of graded I_e-prime submodules under localization.

THEOREM 2.9. *Let R be a G-graded ring, M a graded R-module, $S \subseteq h(R)$ be a multiplicatively closed subset of R and $I = \bigoplus_{h \in G} I_h$ a graded ideal of R.*

(i) *If N is a graded I_e-prime submodule of M with $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a graded I_e-prime submodule of $S^{-1}M$.*

(ii) *If $S^{-1}N$ is a graded I_e-prime submodule of $S^{-1}M$ with $S \cap G\text{-}Zdv_R(M/N) = \emptyset$, then N is a graded I_e-prime submodule of M.*

PROOF. (i) Since $(N :_R M) \cap S = \emptyset$, $S^{-1}N$ is a proper graded submodule of $S^{-1}M$. Let $\frac{r_g}{s_1} \in h(S^{-1}R)$ and $\frac{m_h}{s_2} \in h(S^{-1}M)$ such that $\frac{r_g}{s_1}\frac{m_h}{s_2} \in S^{-1}N - I_e S^{-1}N$. Then there exists $t \in S$ such that $t r_g m_h \in N - I_e N$ which yields that either $t m_h \in N$ or $r_g \in (N :_R M)$ as N is a graded I_e-prime submodule of M. Hence either $\frac{m_h}{s_2} = \frac{t m_h}{t s_2} \in S^{-1}N$ or $\frac{r_g}{s_1} \in S^{-1}(N :_R M) = (S^{-1}N :_{S^{-1}R} S^{-1}M)$. Therefore, $S^{-1}N$ is a graded I_e-prime submodule of $S^{-1}M$.

(ii) Let $r_g \in h(R)$ and $m_h \in h(M)$ such that $r_g m_h \in N - I_e N$. Then $\frac{r_g}{1}\frac{m_h}{1} \in S^{-1}N - I_e S^{-1}N$. Since $S^{-1}N$ is a graded I_e-prime submodule of $S^{-1}M$, either $\frac{m_h}{1} \in S^{-1}N$ or $\frac{r_g}{1} \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$. If $\frac{m_h}{1} \in S^{-1}N$, then there exists $t \in S$ such that $s m_h \in N$. Since $S \cap G\text{-}Zdv_R(M/N) = \emptyset$, it follows $m_h \in N$. Now, if $\frac{r_g}{1} \in (S^{-1}N :_{S^{-1}R} S^{-1}M) = S^{-1}(N :_R M)$, then there exists $s \in S$ such that $s r_g M \subseteq N$ and hence $r_g \in (N :_R M)$ since $S \cap G\text{-}Zdv_R(M/N) = \emptyset$. Therefore, N is a graded I_e-prime submodule of M. □

Acknowledgements. The authors wish to thank sincerely the referees for their valuable comments and suggestions.

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(Received 08 06 2020)
(Revised 06 02 2021)