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A RELATION BETWEEN POROSITY CONVERGENCE AND PRETANGENT SPACES

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Abstract. The convergence of porosity is one of the relatively new concept in Mathematical analysis. It is completely structurally different from the other convergence concepts. Here we give a relation between porosity convergence and pretangent spaces.

1. Introduction

The notion of convergence, as one of the fundamental concepts in Mathematical analysis, has many generalizations such as statistical convergence [**14**, **23**], ideal convergence [**21**], convergence in measure [**26**], A-convergence for a matrix A [**15**, **19**, **20**], etc. Unlike all types of convergences given in the literature with different forms, porosity convergence as relatively new is defined in [**2**]. The basis of this study lies in redefinition of the porosity notion from a point in $[0,\infty)$ to infinity in natural numbers [**3**].

Porosity notion appeared in the papers of Denjoy [**7**,**8**] and Khintchine [**18**] and, Dolzenko [**9**]. It has many applications such as in theory of free boundarie [**16**], generalized subharmonic functions [**11**], complex dynamics [**22**], quasisymmetric maps [**25**], infinitesimal geometry [**5**] and some other areas of mathematics.

Let us remember the definitions of right upper porosity for subsets of real numbers at zero. Let $E\subset\mathbb{R}^+$, then the right upper porosity of E at 0 is defined as

$$p^+(E)=\limsup_{h\to 0^+}\frac{\lambda(E,h)}{h}$$

where $\lambda(E,h)$ is the length of the largest open subinterval of $(0,h)$ that contains no point of E [**24**].

The notion of right lower porosity of E at 0 is defined similarly.

In [**3**] the definition of porosity which was given for the subsets of real numbers, have been redefined for the subsets of natural numbers by using a special function which is called scaling function.

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Let $\mu\colon \mathbb{N} \to \mathbb{R}^+$ be a strictly decreasing function such that $\lim_{n\to\infty} \mu(n) = 0$ and let A be a subset of $\mathbb{N}$. Now, let us recall from [**3**] that upper and lower porosity of A at infinity as follows

$$\bar{p}_\mu(A) := \limsup_{n\to\infty} \frac{\lambda_\mu(A,n)}{\mu(n)}, \qquad \underline{p}_\mu(A) := \liminf_{n\to\infty} \frac{\lambda_\mu(A,n)}{\mu(n)}$$

where

$$\lambda_\mu(A,n) := \sup\left\{|\mu(n^{(1)}) - \mu(n^{(2)})| : n \leqslant n^{(1)} < n^{(2)},\ (n^{(1)}, n^{(2)}) \cap A = \emptyset\right\}.$$

From the definitions of upper and lower porosity of a subset of $\mathbb{N}$ at infinity, we have the following trivial result [**3**].

Remark 1.1. [**3**] If A is a finite subset of $\mathbb{N}$, that is $|A| < \infty$, then, for every $n \in \mathbb{N}$, $\lambda_\mu(A,n)$ is the length of the largest open subinterval of $(0, \mu(n))$ that contains no point of $\mu(A)$ and has a form $(\mu(n^{(2)}), \mu(n^{(1)}))$ with $\mu(n^{(1)}) < \mu(n^{(2)})$. For the case of finite A we evidently have $\lambda_\mu(A,n) = \mu(n)$ for all sufficiently large n. Consequently the equality $\bar{p}_\mu(A) = \underline{p}_\mu(A) = 1$ holds with every scaling function μ for all $A \subseteq \mathbb{N}$ with $|A| < \infty$.

Thoughtout this paper, we will use only the right upper porosity and the following terminology. A set $A \subseteq \mathbb{N}$ is called

(i) porous at infinity if $\bar{p}_\mu(A) > 0$;
(ii) strongly porous at infinity if $\bar{p}_\mu(A) = 1$;
(iii) nonporous at infinity if $\bar{p}_\mu(A) = 0$.

Let us rcall the definition of porosity convergence:

Definition 1.1. [**2**] Let $\tilde{x} = (x_n)_{n\in\mathbb{N}}$ be a real valued sequence. We say that, $\tilde{x}$ is $\bar{p}_\mu$ convergent to l if for each $\varepsilon > 0$,

$$\bar{p}_\mu(A_\varepsilon) > 0 \quad \text{and} \quad \bar{p}_\mu(A_\varepsilon^c) = 0$$

where $A_\varepsilon := \{n \ :\ |x_n - l| \geqslant \varepsilon\}$ and A_ε^c is the complement of the set A_ε. It is denoted by $\tilde{x} \to l(\bar{p}_\mu)$.

Let us note that the second condition in Definition 1.1 is necessary for only uniqueness of $\bar{p}_\mu$-limit.

In [**2**], it is particularly shown that $\bar{p}_\mu$-convergence is a regular summability method for real (or complex) valued sequences.

Our aim is to establish the relationship between porosity convergence and pretangent space of the set $\mu(A_\varepsilon) \cup \{0\} \subset [0, \infty)$.

The concept of pretangent space was defined by Dovgoshey and Martio in [**12**, **13**] for the first time. After this basic studies, tangent spaces are the focus of research [**1**, **6**, **10**].

Now, let us recall construction of pretangent spaces to E in the particular case when $E \subset \mathbb{R}^+$. Let $\tilde{r} = (r_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} r_n = 0$. The sequence $\tilde{r}$ will be called a normalizing sequence. We define the set

$$\tilde{E} := \left\{\tilde{x} = (x_n) : x_n \in E,\ \forall n \in \mathbb{N} \text{ and } \lim_{n\to\infty} x_n = 0\right\}.$$

Definition 1.2. [**3**] Two sequences $\tilde{x} = (x_n)_{n\in\mathbb{N}} \in \tilde{E}$ and $\tilde{y} = (y_n)_{n\in\mathbb{N}} \in \tilde{E}$ are mutually stable w.r.t. $\tilde{r}$ if the following limit

$$|\tilde{x} - \tilde{y}|_{\tilde{r}} := \lim_{n\to\infty} \frac{|x_n - y_n|}{r_n} \tag{1.1}$$

exists and is finite.

A family $\tilde{F} \subseteq \tilde{E}$ is called self-stable (w.r.t. $\tilde{r}$) if each pair of sequences $\tilde{x}, \tilde{y} \in \tilde{F}$ are mutually stable. A family $\tilde{F} \subseteq \tilde{X}$ is called maximal self-stable if $\tilde{F}$ is self-stable and for an arbitrary $\tilde{z} \in \tilde{E}$ either $\tilde{z} \in \tilde{F}$ or there is a sequence $\tilde{x} \in \tilde{F}$ such that $\tilde{x}$ and $\tilde{z}$ are not mutually stable.

Proposition 1.1. [**12**,**13**] *Let $E \subseteq \mathbb{R}^+$ be a pointed set with the marked point $0 \in E$. Then, for every normalizing sequence $\tilde{r} = (r_n)_{n\in\mathbb{N}}$, there exists a maximal self-stable family $\tilde{E}_{0,\tilde{r}}$ such that $\tilde{0} := (0, \dots, 0, 0, \dots) \in \tilde{E}_{0,\tilde{r}}$.*

Consider a function $|.,.|_{\tilde{r}} : \tilde{E}_{0,\tilde{r}} \times \tilde{E}_{0,\tilde{r}} \to [0,\infty)$ such that $|\tilde{x},\tilde{y}|_{\tilde{r}} = |\tilde{x} - \tilde{y}|_{\tilde{r}}$ is defined by (1.1). Obviously, it is nonnegative, symmetric and satisfies the triangle inequality $|\tilde{x} - \tilde{y}|_{\tilde{r}} \leqslant |\tilde{x} - \tilde{z}|_{\tilde{r}} + |\tilde{z} - \tilde{y}|_{\tilde{r}}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{E}_{0,\tilde{r}}$. Therefore, $(\tilde{E}_{0,\tilde{r}}, |.,.|_{\tilde{r}})$ is a pseudometric space.

Definition 1.3. [**12**,**13**] Let $\tilde{E}_{0,\tilde{r}}$ be a maximal self-stable family. A pretangent space to $E \subseteq \mathbb{R}^+$ (at the point $0 \in E$ w.r.t. $\tilde{r}$) is the metric identification of a pseudometric space $(\tilde{E}_{0,\tilde{r}}, |.,.|_{\tilde{r}})$.

Because the notion of pretangent space is important for the paper, we shall describe the metric identification construction (see, for example, [**17**]). Define a binary relation $\sim$ on $\tilde{E}_{0,\tilde{r}}$ by $\tilde{x} \sim \tilde{y}$ if and only if $|\tilde{x} - \tilde{y}|_{\tilde{r}} = 0$. It is clear that $\sim$ is an equivalence relation. Let us denote by $\Omega^E_{0,\tilde{r}}$ the set of equivalence classes in $\tilde{E}_{0,\tilde{r}}$ under $\sim$. For an arbitrary $\alpha, \beta \in \Omega^E_{0,\tilde{r}}$, we set

$$\rho(\alpha,\beta) := |\tilde{x} - \tilde{y}|_{\tilde{r}}, \quad \tilde{x} \in \alpha,\ \tilde{y} \in \beta.$$

The function ρ is a well-defined metric on $\Omega^E_{0,\tilde{r}}$. By definition, $(\Omega^E_{0,\tilde{r}}, \rho)$ is the metric identification of $(\tilde{E}_{0,\tilde{r}}, |.,.|_{\tilde{r}})$.

Lemma 1.1. *The equality*

$$\bar{\Omega}^{\mathbb{R}^+}_{0,\tilde{r}} = \mathbb{R}^+ \tag{1.2}$$

holds for any normalizing sequence $\tilde{r}$.

Proof. Let us note that $0 \in \bar{\Omega}^{\mathbb{R}^+}_{0,\tilde{r}}$ to prove (1.2). If $\tilde{x} := (hr_n)_{n\in\mathbb{N}}$ for any $h \in (0,\infty)$, then we obviously have $\lim_{n\to\infty} hr_n/r_n = h$. By [**3**, Corollary 2.5] we obtain $\tilde{x} \in \tilde{\mathbb{R}}^+_{0,\tilde{r}}$ is a maximal self-stable family corresponding to $\Omega^{\mathbb{R}^+}_{0,\tilde{r}}$. The statements $h \in \bar{\Omega}^{\mathbb{R}^+}_{0,\tilde{r}}$ is fulfilled by [**3**, Proposition 2.6]. Consequently, (1.2) holds. □

Definition 1.4. Let A and B be any subsets of $\mathbb{R}^+$. We shall write $A \preceq B$, if for every sequence $(a_n)_{n\in\mathbb{N}} \in \tilde{A} \setminus \{\tilde{0}\}$, there is a sequence $(t_n)_{n\in\mathbb{N}} \in \tilde{B} \setminus \{\tilde{0}\}$, such that $\lim_{n\to\infty} a_n/t_n = 1$ holds [**3**].

2. Main results

Let $\tilde{x}=(x_n)$ be a real valued sequence and $\mu\colon \mathbb{N}\to\mathbb{R}^+$ be a scaling function. Consider the sets $A_\varepsilon^\mu := \mu(A_\varepsilon)\cup\{0\}\subset[0,\infty)$ and $A_\varepsilon^{c\mu} := \mu(A_\varepsilon^c)\cup\{0\}\subset[0,\infty)$ where $A_\varepsilon := \{k : |x_k-l|\geqslant\varepsilon\}$ for any $\varepsilon>0$.

THEOREM 2.1. *The following statements are equivalent.*

(i) *The sequence $\tilde{x}=(x_n)$ is not $\bar{p}_\mu$-convergent to l. i.e.,*

$$x\nrightarrow l(\bar{p}_\mu),\quad n\to\infty. \tag{2.1}$$

(ii) *The equality*

$$\bar{\Omega}_{0,\tilde{r}}^{A_\varepsilon^\mu}=\mathbb{R}^+ \tag{2.2}$$

holds for every normalizing sequence $\tilde{r}$.

(iii) *There exists a subsequence $\tilde{r}'$ of normalizing sequence $\tilde{r}$ such that the pretangent space $\bar{\Omega}_{0,\tilde{r}'}^{A_\varepsilon^\mu}$ includes a dense subset of $(0,1)$.*

PROOF. (i) $\Rightarrow$ (ii) Assume that $x\nrightarrow l(\bar{p}_\mu)$, $n\to\infty$. Then, the set A_ε is a nonporous subset of $\mathbb{N}$ at infinity for every ε. So, the set A_ε has infinitely many elements, and it can be represented as $A_\varepsilon=\{n_1,n_2,\dots,n_k,n_{k+1},\dots\}$ where (n_k) is strictly increasing sequence of natural numbers.

Since $\bar{p}_\mu(A_\varepsilon)=0$, then from [**3**, Proposition 3.5] ve have that

$$\lim_{k\to\infty}\frac{\mu(n_{k+1})}{\mu(n_k)}=1. \tag{2.3}$$

Let $\tilde{t}=\{t_m\}_{m\in\mathbb{N}}$ be a sequence of positive reals such that $\lim_{m\to\infty}t_m=0$. For every $m\in\mathbb{N}$, define the number $k(m)$ as follows $k(m):=\min\{k\in\mathbb{N}:\mu(n_k)\leqslant t_m\}$. Then, the double inequalities

$$\mu(n_{k(m)})\leqslant t_m<\mu(n_{k(m)-1}) \tag{2.4}$$

hold for all sufficiently large m. It follows from (2.3) and (2.4) that

$$1\leqslant\liminf_{m\to\infty}\frac{t_m}{\mu(n_{k(m)})}\leqslant\limsup_{m\to\infty}\frac{t_m}{\mu(n_{k(m)})}\leqslant\limsup_{m\to\infty}\frac{\mu(n_{k(m)-1})}{\mu(n_{k(m)})}=1$$

holds. Hence, we conclude that $\lim_{m\to\infty}\frac{t_m}{\mu(n_{k(m)})}=1$ holds. Since $(t_m)\subset\mathbb{R}^+$ and $\mu(n_{k(m))})\subset\mu(A_\varepsilon)\cup\{0\}$, then we have

$$\mathbb{R}^+\preceq\mu(A_\varepsilon)\preceq A_\varepsilon^\mu\preceq\mathbb{R}^+. \tag{2.5}$$

Let $\tilde{r}=(r_n)_{n\in\mathbb{N}}$ be any normalizing sequence. By considering (2.5) with [**3**, Proposition 2.9] we have $\bar{\Omega}_{0,\tilde{r}}^{A_\varepsilon^\mu}=\bar{\Omega}_{0,\tilde{r}}^{\mathbb{R}^+}$. Consequently, from Lemma 1.1, (2.2) holds.

(ii) $\Rightarrow$ (iii) is trivial. Let prove (iii) $\Rightarrow$ (i). Now assume that (iii) holds. Using [**3**, Theorem 3.6] we obtain that

$$\bar{p}(A_\varepsilon^\mu)=0. \tag{2.6}$$

Since $\bar{p}(A_\varepsilon^\mu)=\bar{p}(\mu(A_\varepsilon))$, then equality (2.6) implies that $\bar{p}(\mu(A_\varepsilon))=0$. By the equality of $\bar{p}(\mu(E))=\bar{p}_\mu(E)$ for $E\subseteq\mathbb{N}$, we have $\bar{p}(\mu(A_\varepsilon))=\bar{p}_\mu(A_\varepsilon)$. Consequently (2.1) holds. $\square$

THEOREM 2.2. *The following statements are equivalent.*

(i) *The sequence* $\tilde{x} = (x_n)$ *is* $(\bar{p}_\mu)-$*convergent to* l*. i.e.,* $x_n \to l(\bar{p}_\mu)$*.*

(ii) *There is a normalizing sequence* $\tilde{r}$ *and an interval* $(a, b) \subseteq (0, 1)$ *with* $|a - b| > 0$ *such that the equalities* $\bar{\Omega}_{0,\tilde{r}'}^{A_\varepsilon^\mu} \cap (a, b) = \emptyset$ *and* $\bar{\Omega}_{0,\tilde{r}'}^{A_\varepsilon^{c\mu}} = \mathbb{R}^+$ *holds for every* $\tilde{r}'$ *and* $\varepsilon > 0$*.*

PROOF. Let us assume that $x_n \to l(\bar{p}_\mu)$ holds. So, $\bar{p}_\mu(A_\varepsilon) > 0$ and $\bar{p}_\mu(A_\varepsilon^c) = 0$ hold for any $\varepsilon > 0$. From [**3**, Theorem 3.4] we have $\bar{p}(\mu(A_\varepsilon)) > 0$ and $\bar{p}(\mu(A_\varepsilon^c)) = 0$. Also, it is clear that $\bar{p}(A_\varepsilon^\mu) > 0$ and $\bar{p}(A_\varepsilon^{c\mu}) = 0$ hold. If we use [**3**, Theorems 2.1 and 2.12], then we obtain that (i) $\Leftrightarrow$ (ii). □

COROLLARY 2.1. *Let* $\tilde{x} = (x_n)$ *be a real valued sequence. If* $x_n \to l(\bar{p}_\mu)$*, then tehre exists a normalizing sequence* $\tilde{r}$ *such that* $\mathbb{R}^+ \setminus \bar{\Omega}_{0,\tilde{r}'}^{A_\varepsilon^\mu} \neq \emptyset$ *holds.*

3. Some examples

In this section we give two examples as application of last section. We take here $\mu(n) = \frac{1}{n}$ as a scaling function only for simplicity.

EXAMPLE 3.1. Consider the sequence $\tilde{x} = ((-1)^n)$ for $n \in \mathbb{N}$. It is clear that it is not porosity convergent i.e., $(-1)^n \nrightarrow 1(\bar{p}_\mu)$ and $(-1)^n \nrightarrow -1(\bar{p}_\mu)$. So, from Theorem 2.1 we have

$$\bar{\Omega}_{0,\tilde{r}'}^{A_\varepsilon^\mu} = \mathbb{R}^+ \quad \text{and} \quad \bar{\Omega}_{0,\tilde{r}'}^{{A'}_\varepsilon^\mu} = \mathbb{R}^+ \tag{3.1}$$

for $A_\varepsilon = \{n : |(-1)^n - 1| \geqslant \varepsilon\}$ and $A'_\varepsilon = \{n : |(-1)^n - (-1)| \geqslant \varepsilon\}$, respectively.

Indeed, $A_\varepsilon = \mathbb{N}_O$ and $A'_\varepsilon = \mathbb{N}_E$. So, (3.1) hold. The third condition of Theorem 2.1 is obvious from second.

EXAMPLE 3.2. Consider the sequence $\tilde{x} = (\frac{1}{n})$ for $n \in \mathbb{N}$. It is clear that $x_n \to 0(\bar{p}_\mu)$ because $x_n \to 0$, $n \to \infty$. So, from Theorem 2.2 we can infer that

$$\bar{\Omega}_{0,\tilde{r}'}^{A_\varepsilon^\mu} \cap (a, b) = \emptyset \quad \text{and} \quad \bar{\Omega}_{0,\tilde{r}'}^{A_\varepsilon^{c\mu}} = \mathbb{R}^+ \tag{3.2}$$

for $(a, b) \subseteq (0, 1)$ where $A_\varepsilon = \{n : \frac{1}{n} \geqslant \varepsilon\}$ and A_ε^c is the complement of A_ε.

Indeed, let $\varepsilon = 1/2$. From the definition of porosity convergence the set $A_{1/2} = \{n : \frac{1}{n} \geqslant \frac{1}{2}\}$ is porous at infinity. Also the set $A_{1/2}^c = \{n : \frac{1}{n} < \frac{1}{2}\}$ is nonporous at infinity.

$A_{1/2}^\mu = \mu(A_{1/2}) \cup \{0\}$ is a finite set and 0 is not an accumulation point of this set. So, $\bar{\Omega}_{0,\tilde{r}'}^{A_{1/2}^\mu} = \{0\}$. Therefore, $\bar{\Omega}_{0,\tilde{r}'}^{A_{1/2}^\mu} \cap (a, b) = \emptyset$ for any interval $(a, b) \subseteq (0, 1)$.

$A_{1/2}^{c\mu} = \mu(A_{1/2}^c) \cup \{0\} = \mu(\mathbb{N}) \cup \{0\} = \mathbb{R}^+$. Then $\bar{\Omega}_{0,\tilde{r}'}^{A_{1/2}^{c\mu}} = \bar{\Omega}_{0,\tilde{r}'}^{\mathbb{R}^+} = \mathbb{R}^+$ is obtained.

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