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LINEAR COMBINATIONS OF POLYNOMIALS WITH THREE-TERM RECURRENCE

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Abstract. We study the zero distribution of the sum of the first n polynomials satisfying a three-term recurrence whose coefficients are linear polynomials. We also extend this sum to a linear combination, whose coefficients are powers of $az + b$ for $a, b \in \mathbb{R}$, of Chebyshev polynomials. In particular, we find necessary and sufficient conditions on a, b such that this linear combination is hyperbolic.

1. Introduction

The sequence of Chebyshev polynomials of the first kind $\{T_n(z)\}_{n=0}^{\infty}$ defined by the recurrence

$$T_{n+1}(z) = 2zT_n(z) + T_{n-1}(z)$$

with $T_0(z) = 1$ and $T_1(z) = z$ forms a sequence of orthogonal polynomials whose zeros are real (i.e., hyperbolic polynomials). The location of zeros of polynomials satisfying a more general recurrence

$$R_{n+1}(z) = A(z)R_n(z) + B(z)R_{n-1}(z) \tag{1.1}$$

where $A(z), B(z) \in \mathbb{C}[z]$ was given in [**3**]. In [**2**], the author studied the set of zeros of a linear combination of Chebyshev polynomials $\sum_{k=0}^{m} a_k T_{n-k}(z)$, $m \leqslant n$, $a_k \in \mathbb{R}$, and provided a connection between this sequence and the theory of Pisot and Salem numbers in number theory. In the special case when $m = n$ and $a_k = 1$ $\forall k$, the sum of the first n Chebyshev polynomials connects to Direchlet kernel in the Fourier analysis. In Section 2 of this paper, we to study the zeros of this sum (cf. Theorem 2.1) when the sequence of Chebyshev polynomials are replaced by a more general sequence $\{R_n(z)\}$ given in (1.1) where $A(z)$ and $B(z)$ are any linear polynomials with real coefficients.

The sequence of Chebyshev polynomials of the second kind $\{U_n(z)\}$ satisfies the same recurrence as that of the first kind with the initial condition $U_0(z) = 1$ and $U_1(z) = 2z$. This initial condition can be written in the form $U_0(z) = 1$ and

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$U_{-n}(z) = 0$, $\forall n \in \mathbb{N}$. In Section 3 of this paper, we study the zeros of a linear combination of Chebyshev polynomials of the second kind whose coefficients are powers of $az + b$. In particular, we consider

$$Q_n(z) = \sum_{k=0}^{n} (az+b)^k U_{n-k}(z), \qquad a, b \in \mathbb{R}. \tag{1.2}$$

We find the necessary and sufficient conditions on a and b under which the zeros of the resulting polynomials are real (cf. Theorem 3.2).

2. Sum of polynomials with three-term recurrence

For $a_1, b_1, a_2, b_2 \in \mathbb{R}$, $a_2 \neq 0$, we let $R_n(z)$ be the sequence of polynomials satisfying the recurrence

$$R_{n+1}(z) = (a_1 z + b_1) R_n(z) + (a_2 z + b_2) R_{n-1}(z)$$

with $R_0(z) = 1$ and $R_{-n}(z) = 0$, $\forall n \in \mathbb{N}$. Equivalently the sequence $\{R_n(z)\}_{n=0}^{\infty}$ is generated by

$$\sum_{n=0}^{\infty} R_n(z) t^n = \frac{1}{1 - (a_1 z + b_1)t - (a_2 z + b_2)t^2}.$$

In this section, we study the necessary and sufficient conditions on a_1, b_1, a_2, and b_2 under which all the zeros of the polynomial $\sum_{k=0}^{n} R_{n-k}(z)$ are real. Those polynomials form a sequence whose generating function is

$$\begin{aligned}\sum_{n=0}^{\infty} \sum_{k=0}^{n} R_k(z) t^n &= \sum_{k=0}^{\infty} t^k \sum_{n=k}^{\infty} R_{n-k}(z) t^{n-k} \\ &= \frac{1}{(1-t)(1 - (a_1 z + b_1)t - (a_2 z + b_2)t^2)}.\end{aligned}$$

With the substitutions t by $-t$, a_2 by $-a_2$, and b_2 by $-b_2$, and then substitute $a_2 z + b_2$ by z, we reduce the generating function to the form

$$\frac{1}{(t+1)((az+b)t^2 + zt + 1)}.$$

Note that all the substitutions above preserve the reality of the zeros of the generated sequence of polynomials. We state the main theorem of this section.

Theorem 2.1. *Let $a, b \in \mathbb{R}$. The zeros of all the polynomials $P_n(z)$ generated by*

$$\sum_{n=0}^{\infty} P_n(z) t^n = \frac{1}{(t+1)((az+b)t^2 + zt + 1)} \tag{2.1}$$

are real if and only if $b \geqslant 1 + 2|a|$. Under this condition the zeros of $P_n(z)$ lie on

$$\left(2a - 2\sqrt{a^2+b}, 2a + 2\sqrt{a^2+b}\right) \tag{2.2}$$

and are dense there as $n \to \infty$.

PROOF. *Suficience* We assume $b \geqslant 1 + 2|a|$. To prove that the zeros of $P_n(z)$ lie on (2.2), we count the number of real zeros of $P_n(z)$ on this interval and show that this number is at least the degree of this polynomial which is given by the lemma below.

LEMMA 2.1. *For each $n \in \mathbb{N}$, the degree of $P_n(z)$ is at most n.*

PROOF. We collect the coefficients in t of the denominator of the right-hand side of (2.1) and obtain the recurrence

$$P_n(z) = -(z+1)P_{n-1}(z) - ((a+1)z + b)P_{n-2}(z) - (az+b)P_{n-3}(z)$$

where $P_0(z) = 1$ and $P_{-n}(z) = 0$, $\forall n \in \mathbb{N}$. The lemma follows from induction. □

To count the number of real zeros of $P(z)$, we construct two auxiliary real-valued functions $z(\theta)$ and $\tau(\theta)$ on $\theta \in (0, \pi)$. The first function is defined as

$$z(\theta) = 2a\cos^2\theta - 2\cos\theta\sqrt{a^2\cos^2\theta + b}. \tag{2.3}$$

By the quadratic formula, $z(\theta)$ satisfies

$$z(\theta)^2 - 4az(\theta)\cos^2\theta - 4b\cos^2\theta = 0. \tag{2.4}$$

We will show later that there are n values of $\theta \in (0, \pi)$, each of which yields a zero of $P_n(z)$ on (2.2) via $z(\theta)$. The lemma below ensures a bijective correspondence between θ and $z(\theta)$.

LEMMA 2.2. *The function $z(\theta)$ is increasing on $(0, \pi)$ and it maps this interval onto the interval*

$$\left(2a - 2\sqrt{a^2+b}, 2a + 2\sqrt{a^2+b}\right).$$

PROOF. To show $z(\theta)$ is increasing, we compute its derivative

$$\frac{dz}{d\theta} = -4a\cos\theta\sin\theta + \frac{4a^2\cos^2\theta\sin\theta + 2b\sin\theta}{\sqrt{a^2\cos^2\theta + b}}$$

and see that it suffices to show

$$2a^2\cos^2\theta + b > 2|a\cos\theta|\sqrt{a^2\cos^2\theta + b}.$$

The left-hand side is positive and the squares of both sides reduce the inequality to $b^2 > 0$, which shows that $z(\theta)$ is increasing. We complete the lemma by computing the limits $\lim_{\theta\to 0} z(\theta) = 2a - 2\sqrt{a^2+b}$ and $\lim_{\theta\to\pi} z(\theta) = 2a + 2\sqrt{a^2+b}$. □

To define the second function $\tau(\theta)$, we need the following lemma.

LEMMA 2.3. *For any $\theta \in (0, \pi)$, we have $az(\theta) + b > 0$.*

PROOF. From Lemma 2.2, it suffices to show that $b + 2a^2 > 2|a|\sqrt{a^2+b}$. Since we know the left-hand side is positive by $b \geqslant 1 + 2|a|$, we obtain the inequality above by squaring both sides. □

From Lemma 2.3, we define the functions

$$\tau(\theta) = \frac{1}{\sqrt{az(\theta) + b}}, \quad t_1(\theta) = \tau(\theta)e^{-i\theta}, \quad t_2(\theta) = \tau(\theta)e^{i\theta}$$

on $\theta \in (0, \pi)$.

LEMMA 2.4. *For any $\theta \in (0,\pi)$, the two zeros of*

$$(az(\theta)+b)t^2 + z(\theta)t + 1 \tag{2.5}$$

are $t_1(\theta)$ and $t_2(\theta)$.

PROOF. We verify that $\tau(\theta)e^{\pm i\theta}$ satisfy Vieta's formulas. Indeed, we have

$$t_1(\theta)t_2(\theta) = \tau(\theta)^2 = \frac{1}{az(\theta)+b}, \tag{2.6}$$

$$t_1(\theta)+t_2(\theta) = 2\tau(\theta)\cos\theta = \frac{2\cos\theta}{\sqrt{az(\theta)+b}}.$$

From (2.3), we note that $z(\theta)\cos\theta < 0$ since $b > 0$. As a consequence, we obtain

$$\frac{2\cos\theta}{\sqrt{az(\theta)+b}} = \frac{-z(\theta)}{az(\theta)+b} \tag{2.7}$$

by squaring both sides and applying (2.4). □

The lemma below shows that for each $\theta \in (0,\pi)$, the two zeros of (2.5) lie inside the unit ball.

LEMMA 2.5. *For any $\theta \in (0,\pi)$, we have $|\tau(\theta)| < 1$.*

PROOF. From (2.6), (2.7), and (2.3), it suffices to show

$$\sqrt{a^2\cos^2\theta + b} > 1 + a\cos\theta.$$

If the right-hand side is negative, the inequality is trivial. If not, we square both sides and the inequality follows from $b \geqslant 1 + 2|a| > 1 + 2a\cos\theta$. □

For each $\theta \in (0,\pi)$, the Cauchy differentiation formula gives

$$\begin{aligned} P_n(z(\theta)) &= \frac{1}{2\pi i}\oint_{|t|=\epsilon} \frac{1}{(t+1)((az(\theta)+b)t^2 + z(\theta)t+1)t^{n+1}}dt \\ &= \frac{1}{2\pi i}\oint_{|t|=\epsilon} \frac{1}{(az(\theta)+b)(t+1)(t-t_1(\theta))(t-t_2(\theta))t^{n+1}}dt. \end{aligned}$$

We recall that $az(\theta)+b \neq 0$ by Lemma 2.3. If we integrate the integrand over the circle Re^{it}, $0 \leqslant t \leqslant 2\pi$, and let $R \to \infty$, then the integral approaches 0. Thus the sum of $P_n(z(\theta))$ and the residue of the integrand at the three simple poles -1, $t_1(\theta)$ and $t_2(\theta)$ is 0. We compute these residues and deduce that $-(az(\theta)+b)P_n(z(\theta))$ equals to

$$\begin{aligned} &\frac{(-1)^{n+1}}{(1+t_1(\theta))(1+t_2(\theta))} + \frac{1}{t_1(\theta)^{n+1}(1+t_1(\theta))(t_1(\theta)-t_2(\theta))} \\ &\qquad + \frac{1}{t_2^{n+1}(\theta)(1+t_2(\theta))(t_2(\theta)-t_1(\theta))}. \end{aligned}$$

We multiply this expression by $(1+t_1(\theta))(1+t_2(\theta))\tau(\theta)^{n+1}$, which is nonzero $\forall\theta \in (0,\pi)$, and conclude θ is a zero of $P_n(z(\theta))$ if and only if it is a zero of

$$(-1)^{n+1}\tau(\theta)^{n+1} + \frac{1+\tau(\theta)e^{i\theta}}{(\tau(\theta)e^{-i\theta} - \tau(\theta)e^{i\theta})e^{-i(n+1)\theta}} + \frac{1+\tau(\theta)e^{-i\theta}}{(\tau(\theta)e^{i\theta} - \tau(\theta)e^{-i\theta})e^{i(n+1)\theta}}$$

or equivalently a zero of

$$(-1)^{n+1}\tau(\theta)^{n+1}-\frac{\sin((n+1)\theta)/\tau(\theta)+\sin((n+2)\theta)}{\sin\theta}.$$

With the trigonometric identity $\sin(n+2)\theta=\sin((n+1)\theta)\cos\theta+\cos((n+1)\theta)\sin\theta$, we write the expression above as

$$(-1)^{n+1}\tau(\theta)^{n+1}-\cos((n+1)\theta)-\frac{\sin((n+1)\theta)(\cos\theta+1/\tau(\theta))}{\sin\theta}. \tag{2.8}$$

We note that if $\theta=\frac{k\pi}{n+1}$, $1\leqslant k\leqslant n$, then the sign of (2.8) is $(-1)^{k+1}$ since $\tau(\theta)<1$ by Lemma 2.5. By the intermediate value theorem, (2.8) has at least $n-1$ solution on $(\pi/(n+1), n\pi/(n+1))$. We also note that as $\theta\to 0$, the sign of (2.8) is negative since $\sin((n+1)\theta)/\sin\theta$ approaches $n+1$ and $\tau(\theta)<1$. Thus (2.8) has another zero on $(0,\pi/(n+1))$. From Lemma 2.2, each zero in θ of (2.8) gives exactly one zero in z of $P_n(z)$ on $\left(2a-2\sqrt{a^2+b}, 2a+2\sqrt{a^2+b}\right)$. Thus all the zeros of $P_n(z)$ lie on the interval above by the fundamental theorem of algebra and Lemma 2.1. The density of the zeros of $P_n(z)$ as $n\to\infty$ on this interval follows directly from the density of the solutions of (2.8) and the continuity of $z(\theta)$.

Necessity In this necessary direction, we will show that if either (1) $b\leqslant -1$ or (2) $-1<b<1+2|a|$, then not all polynomials $P_n(z)$ are hyperbolic. By [**1**, Theorem 1.5], it suffices to find $z^*\in\mathbb{C}\smallsetminus\mathbb{R}$ such that the zeros of

$$(t+1)((az^*+b)t^2+z^*t+1) \tag{2.9}$$

are distinct and the two smallest (in modulus) zeros of this polynomial have the same modulus. Note that every small neighborhood of such z^* will contain a zero of $P_n(z)$ for all large n and consequently $P_n(z)$ is not hyperbolic for all large n. For more details on this application of the theorem, see [**4**].

For the first case $b\leqslant -1$, we let θ^* be any angle with $a^2\cos^2\theta^*<-b$ and let τ^* be any zero of $b\tau^2-2a\tau\cos\theta^*-1$. Note that $\tau^*\notin\mathbb{R}$ since $a^2\cos^2\theta^*+b<0$ and consequently $\tau^{*2}\notin\mathbb{R}$ by the definition of τ^*. With the note that $2a\tau^*\cos\theta^*+1$ is nonreal (and thus nonzero), we choose

$$z^*=\frac{-2b\tau^*\cos\theta^*}{2a\tau^*\cos\theta^*+1}$$

which is nonreal since $1/z^*\notin\mathbb{R}$. From the definitions of τ^*, θ^*, and z^* above, the two solutions of $(az^*+b)t^2+z^*t+1=0$ are $\tau^*e^{\pm i\theta^*}$ since they satisfy Vieta's formulas

$$\tau^{*2}=\frac{1}{az^*+b},\quad 2\tau^*\cos\theta^*=-\frac{z^*}{az^*+b}.$$

Since τ^* and $\overline{\tau^*}$ are solutions of $b\tau^2-2a\tau\cos\theta^*-1$, we have $\tau^*\overline{\tau^*}=|\tau^*|^2=-1/b\leqslant 1$. Thus the two smallest (in modulus) zeros of (2.9) equal in modulus and we complete the case $b\leqslant -1$.

We now consider the case $-1<b<1+2|a|$. We will find $z^*\notin\mathbb{R}$ so that the smaller (in modulus) zero of $(az^*+b)t^2+z^*t+1$ lie on the unit circle. The inequality $|2|a|-b||<1$ implies that $1+2|a|>|b|>|b|\cdot|2|a|-b|$ and consequently

$$1-b^2+2|a|+2b|a|>0.$$

We conclude there is $\theta^* \in (0, \pi)$ sufficiently close to 0 when $a \geqslant 0$ or close to π when $a < 0$ such that $b^2 - 2ab\cos\theta^* < 1 + 2a\cos\theta^*$. With this choice of θ^*, we have

$$\frac{|be^{i\theta^*} - a|}{|ae^{i\theta^*} + 1|} = \frac{b^2 + a^2 - 2ab\cos\theta}{a^2 + 1 + 2a\cos\theta} < 1. \tag{2.10}$$

We define

$$z^* = \frac{-1 - be^{2i\theta^*}}{ae^{2i\theta^*} + e^{i\theta^*}}$$

and write

$$(az^* + b)t^2 + z^*t + 1 \tag{2.11}$$

as $z^*(at^2 + t) + bt^2 + 1$ to conclude that $e^{i\theta^*}$ is a zero of this polynomial. Since the product of the two zeros of this polynomial is $1/(az^* + b)$, we claim that the other zero of this polynomial is more than 1 in modulus by showing that

$$\frac{1}{|az^* + b|} > 1.$$

Indeed, from the definition of z^*, this inequality is equivalent to (2.10). We note that $z^* \notin \mathbb{R}$ since a solution of (2.11) is $e^{i\theta^*} \notin \mathbb{R}$ and the other solution is more than 1 in modulus. □

3. Linear combination of Chebyshev polynomials

The goal of this section is to study necessary and sufficient conditions under which the zeros of (1.2) are real. The sequence $\{Q_n(z)\}$ in (1.2) is generated by

$$\begin{aligned}\sum_{n=0}^{\infty} Q_n(z)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}(az+b)^k U_{n-k}(z)t^n \\ &= \sum_{k=0}^{\infty}(az+b)^k t^k \sum_{n=k}^{\infty} U_{n-k}(z)t^{n-k} \\ &= \frac{1}{(1+(az+b)t)(1-2zt+t^2)}.\end{aligned}$$

With the substitution z by $-z/2$ and then $-a/2$ by a, it suffice to study the hyperbolicity of the sequence generated of polynomials by

$$\frac{1}{(1+(az+b)t)(1+zt+t^2)}.$$

As a small digression of the main goal, we will prove the following theorem which states that the positivity of the t^2-coefficient in the factor $1 + zt + t^2$ is important to ensure the hyperbolicity of the generated sequence of polynomials.

Theorem 3.1. *Suppose $a, b, c \in \mathbb{R}$ where $c \neq 0$. If $c \leqslant 0$, then not all the polynomials $P_n(z)$ generated by*

$$\frac{1}{((az+b)t+1)(ct^2+zt+1)}.$$

are hyperbolic.

We note that if $c = 0$, the sequence of generated polynomials satisfies a three-term recurrence and their zeros have been studied in [**3**]. Under the condition $c > 0$, with the substitution $t \to t/\sqrt{c}$, we can assume $c = 1$. The following theorem settles the necessary and sufficient conditions for the hyperbolicity of (1.2).

THEOREM 3.2. *Suppose $a, b \in \mathbb{R}$. The zeros of all the polynomials $P_n(z)$ generated by*

$$\sum_{n=0}^{\infty} P_n(z)t^n = \frac{1}{((az+b)t+1)(t^2+zt+1)}. \tag{3.1}$$

are real if and only if $|b| \leqslant 1 - 2|a|$. Moreover when $|b| \leqslant 1 - 2|a|$, the zeros of $P_n(z)$ lie on $(-2, 2)$ and are dense there as $n \to \infty$.

PROOF OF THEOREM 3.1. In the case $c < 0$, with the substitution $t \to t/\sqrt{|c|}$, it suffices to show that for any $a, b \in \mathbb{R}$, not all the polynomials generated by

$$\frac{1}{((az+b)t+1)(-t^2+zt+1)}$$

are hyperbolic. Recall a consequence of [**1**, Theorem 1.5] that we will need to find $z^* \notin \mathbb{R}$ so that the two smallest zeros of

$$((az^*+b)t+1)(-t^2+z^*t+1)$$

equal in modulus.

In the case $|b| < 1$, we choose $z^* = iy^*$ where

$$0 < y^* < \min\left(\frac{\sqrt{1-b^2}}{|a|}, 2\right)$$

if $a \neq 0$ and $0 < y^* < 2$ if $a = 0$. The two zeros of $-t^2 + z^*t + 1$,

$$\frac{iy^* \pm \sqrt{4 - y^{*2}}}{2}$$

lie on the unit circle and thus their modulus is less than

$$\frac{1}{|az^*+b|} = \frac{1}{\sqrt{a^2y^{*2}+b^2}}.$$

For the remainder of Section 3.1, we assume $|b| \geqslant 1$. To make a suitable choice for z^*, we consider the following lemma.

LEMMA 3.1. *With the principal cut, there exists $\theta^* \neq k\pi$, $k \in \mathbb{Z}$, such that*

$$|b| + \sqrt{b^2 + 4a^2 - 4ae^{i\theta^*}} \geqslant |2a - 2e^{i\theta^*}|.$$

PROOF. We note that $b^2 + 4a^2 \geqslant 4|a|$ since $4|a|(1 - |a|) \leqslant 1 \leqslant |b|$. Thus with the principle cut, the function

$$f(z) := \frac{|b| + \sqrt{b^2 + 4a^2 - 4az}}{2a - 2z}$$

is meromorphic on the open unit ball with the possible pole at $z = a$ if $|a| < 1$. To prove this lemma, we will find $z \notin \mathbb{R}$ and $|z| = 1$ such that $|f(z)| \geqslant 1$.

We note that if $|a| \geqslant 1$, then $f(z)$ is analytic on the unit ball and

$$|f(0)| = \frac{|b| + \sqrt{b^2 + 4a^2}}{2|a|} > 1.$$

Thus by the maximum modulus principle $|f(z)| > 1$ for some $|z| = 1$. We can choose such $z \notin \mathbb{R}$ by the continuity of $f(z)$.

On the other hand if $|a| < 1$, then the Cauchy integral formula implies that

$$\oint_{|z|=1} |f(z)||dz| \geqslant \left| \oint_{|z|=1} f(z)dz \right| = 2\pi|b| \geqslant 2\pi.$$

Consequently $|f(z)| > 1$ for some $|z| = 1$ or $|f(z)| = 1$ for all $|z| = 1$ and the lemma follows. □

We now define

$$z^* = \frac{-2ab + be^{i\theta^*} + \operatorname{sign}(b)e^{i\theta^*}\sqrt{b^2 + 4a^2 - 4ae^{i\theta^*}}}{2a^2 - 2ae^{i\theta^*}}$$

where θ^* is given in Lemma 3.1. With this definition, z^* is a solution of

$$(a^2 - ae^{i\theta^*})z^2 + (2ab - be^{i\theta^*})z + b^2 - e^{2i\theta^*} = 0$$

from which we deduce that

$$-\frac{e^{i\theta^*}}{az^* + b} = -\frac{2a - 2e^{i\theta^*}}{-b + \operatorname{sign}(b)\sqrt{b^2 + 4a^2 - 4ae^{i\theta^*}}} \tag{3.2}$$

is a zero in t of

$$-t^2 + z^*t + 1.$$

The modulus of (3.2) is the same as the modulus of the zero in t of $(az^* + b)t + 1$ which is at most 1 by the definition of θ^*. This modulus is larger than the modulus of the other zero of $-t^2 + z^*t + 1$ since the product of two zeros of this polynomial is -1. We finish the proof of Theorem 3.1 by noting that $z^* \notin \mathbb{R}$ since the two zeros of $-t^2 + z^*t + 1$ are neither real nor complex conjugate. □

Proof of Theorem 3.2. *Sufficience* Let $\{P_n(z)\}$ be the sequence of polynomials defined in (3.1) where $|b| \leqslant 1 - 2|a|$. The proof of the following lemma is the same as that of Lemma 4 in [**4**]. For brevity, we omit the proof in this paper.

Lemma 3.2. *For each $b \in [-1, 1]$, let S_b be a dense subset of*

$$\left[\frac{|b| - 1}{2}, \frac{1 - |b|}{2}\right] \tag{3.3}$$

and $n \in \mathbb{N}$ be fixed. If for any $a \in S_b$, the zeros of $P_n(z)$ lie on $(-2, 2)$, then the same conclusion holds for any a in (3.3) .

Suppose $|b| \leqslant 1-2|a|$. From Lemma 3.2, it suffices to consider $a \neq 0$. We define the monotone function $z(\theta) = -2\cos\theta$ on $(0,\pi)$ and note that for each $\theta \in (0,\pi)$ the two zeros of $t^2 + z(\theta)t + 1$ are $e^{\pm i\theta}$. We consider the function

$$t_0(\theta) = \frac{-1}{az(\theta)+b}, \qquad \theta \in (0,\pi),$$

which has a vertical asymptote at $\theta = \cos^{-1}(b/2a)$ if $|b| < 2|a|$. For any $\theta \in (0,\pi)$ such that $2a\cos\theta \neq b$, the Cauchy differentiation formula gives

$$P_n(z(\theta)) = \frac{1}{az(\theta)+b} \oint_{|t|=\epsilon} \frac{dt}{(t-t_0(\theta))(t-e^{i\theta})(t-e^{-i\theta})t^{n+1}}.$$

After computing the residue of the integrand at the three nonzero simple poles $t_0(\theta)$, $e^{\pm i\theta}$, and letting the radius of the integral approach infinity, we apply similar computations in (2.8) to conclude that $\theta \in (0,\pi)$, $2a\cos\theta \neq b$, is a zero of $P_n(z(\theta))$ if and only if it is a zero of

$$\frac{-1}{t_0(\theta)^{n+1}} + \cos((n+1)\theta) + \frac{(\cos\theta - t_0(\theta))\sin((n+1)\theta)}{\sin\theta}. \tag{3.4}$$

From Lemma 3.2, it suffices to consider $|b| \neq 2|a|$. We note that the limits of (3.4) as $\theta \to 0$ and $\theta \to \pi$ are

$$n+2+\frac{n+1}{b-2a} + (-1)^n(b-2a)^{n+1}, \tag{3.5}$$

$$(-1)^{n+1}(n+2) + (-1)^n\left(\frac{n+1}{b+2a} + (b+2a)^{n+1}\right) \tag{3.6}$$

respectively.

In the case $|b| > 2|a|$, (3.4) is a continuous function of θ on $(0,\pi)$ and its sign at $\theta = k\pi/(n+1)$, for $1 \leqslant k \leqslant n$, is $(-1)^k$ since

$$|t_0(\theta)| > \frac{1}{2|a|+|b|} \geqslant 1.$$

By the intermediate value theorem, we obtain at least $n-1$ zeros of (3.4) on $(\pi/(n+1), n\pi/(n+1))$. If $b > 0$, then (3.5) is positive since $0 < b-2a \leqslant 1$ and we obtain at least another zero of (3.4) on $(0, \pi/(n+1))$. On the other hand, if $b < 0$, then the inequalities

$$-1 < b+2a < 0$$

imply that the sign of (3.6) is $(-1)^{n+1}$ and we have at least another zero of (3.4) on $(n\pi/(n+1), \pi)$. We conclude that when $|b| > 2|a|$, (3.4) has at least n zeros on $(0,\pi)$, each of which yields a zero of $P_n(z)$ on the interval $(-2,2)$ by the map $z(\theta)$. Thus all the zeros of $P_n(z)$ lie on $(-2,2)$ by the fundamental theorem of algebra.

We now consider the case $|b| < 2|a|$. As a function of θ on $(0,\pi)$, (3.4) has a vertical asymptote at $\theta = \cos^{-1}(b/2a)$ since $t_0(\theta)$ does. By Lemma 3.2, we can assume

$$\cos^{-1}\frac{b}{2a} \neq \frac{k\pi}{n+1}, \qquad 1 \leqslant k \leqslant n.$$

Thus for some $0 \leqslant k_0 \leqslant n$, the open interval

$$\left(\frac{k_0}{n+1}\pi, \frac{k_0+1}{n+1}\pi\right) \tag{3.7}$$

contains $\cos^{-1}(b/2a)$. We note that this interval may or may not contain a zero of (3.4). In the case $a < 0$, we observe that (3.5) is positive and the sign of (3.6) is $(-1)^{n+1}$. Thus there are at least n zeros of (3.4) on the n intervals

$$(k\pi/(n+1), (k+1)\pi/(n+1)), \text{ for } 0 \leqslant k \leqslant n \text{ and } k \neq k_0$$

and we conclude that all the zeros of $P_n(z)$ lie on $(-2, 2)$ by the same argument as in the previous case. On the other hand, if $a > 0$, then the limits (3.4) as θ approaches the left and right of $\cos^{-1}(b/2a)$ are

$$\lim_{\theta \to \cos^{-1}(b/2a)^-} \frac{\sin((n+1)\theta)}{b - 2a\cos(\theta)} = (-1)^{k_0+1}\infty$$

and

$$\lim_{\theta \to \cos^{-1}(b/2a)^+} \frac{\sin((n+1)\theta)}{b - 2a\cos(\theta)} = (-1)^{k_0}\infty,$$

respectively. If $k_0 \neq 0$ and $k_0 \neq n$, then we conclude that (3.7) contains at least two zeros of (3.4). Thus we obtain at least n zeros of this expression on the $n-1$ intervals $(k\pi/(n+1), (k+1)\pi/(n+1))$, for $1 \leqslant k < n$. In the case $k_0 = 0$ or $k_0 = n$, (3.7) contains at least one zero of (3.4) and thus there are at least n zeros of (3.4) on the n intervals $(k\pi/(n+1), (k+1)\pi/(n+1))$, for $1 \leqslant k < n$ and $k = k_0$.

Necessity Here we assume $|b| + 2|a| > 1$ and show that not all zeros of $P_n(z)$ defined in (3.1) are real when n is large. From [**1**, Theorem 1.5], it suffices to find $z \notin \mathbb{R}$ so that $|t_0| = |t_1| \leqslant |t_2|$ where

$$t_0 := -\frac{1}{az+b} \tag{3.8}$$

and t_1 and t_2 are the two zeros of $1 + zt + t^2$. To motivate the choice of z, we provide heuristic arguments by noticing that $t_1 t_2 = 1$ and letting

$$t_1 = t_0 e^{i\theta} = -\frac{e^{i\theta}}{az+b} \tag{3.9}$$

$$t_2 = -e^{-i\theta}(az+b). \tag{3.10}$$

The equation $1 + zt_2 + t_2^2 = 0$ yields

$$(az+b)^2 - ze^{i\theta}(az+b) + e^{2i\theta} = 0$$

or equivalently

$$(a^2 - ae^{i\theta})z^2 + (2ab - be^{i\theta})z + b^2 + e^{2i\theta} = 0. \tag{3.11}$$

With a choice of branch cut which will be specified later, the equation above has two solutions

$$z = \frac{-2ab + be^{i\theta} \pm e^{i\theta}\sqrt{b^2 - 4a^2 + 4ae^{i\theta}}}{2a^2 - 2ae^{i\theta}}$$

and the corresponding values for $az + b$ are

$$az+b=\frac{-be^{i\theta}\pm e^{i\theta}\sqrt{b^2-4a^2+4ae^{i\theta}}}{2a-2e^{i\theta}}. \tag{3.12}$$

For a formal proof of the necessary condition, we consider the following cases.

Case 1: $|a|\leqslant 1$. We have the inequality

$$b^2-4a^2+4|a|-(|b|+2|a|-2)^2=4(1-|a|)(2|a|+|b|-1)\geqslant 0.$$

with equality if and only if $|a|=1$. This implies

$$b^2-4a^2+4|a|\geqslant 0, \tag{3.13}$$

$$\sqrt{b^2-4a^2+4|a|}+|b|\geqslant ||b|+2|a|-2|+|b|\geqslant |2|a|-2| \tag{3.14}$$

with equality if and only if $|a|=1$ and $b=0$. We define $\theta\in(0,\pi)$ sufficiently close to 0 or π such that $e^{i\theta}$ is close to $\operatorname{sign} a$ if $a\neq 0$. If $a=0$, we pick any $\theta\in(0,\pi)$. With this choice of θ and the principal cut, we let

$$z=\begin{cases}\frac{-2ab+be^{i\theta}-\operatorname{sign} b.e^{i\theta}\sqrt{b^2-4a^2+4ae^{i\theta}}}{2a^2-2ae^{i\theta}} & \text{if } ab\neq 0,\\ \frac{ie^{i\theta}}{\sqrt{a^2-ae^{i\theta}}} & \text{if } b=0,\\ \frac{b^2+e^{2i\theta}}{be^{i\theta}} & \text{if } a=0.\end{cases} \tag{3.15}$$

With this choice of z, (3.11) holds and consequently t_1 and t_2 defined in (3.9) and (3.10) are the zeros of $1+zt+t^2$. If $a=0$, then

$$|t_0|=|t_1|<|t_2| \tag{3.16}$$

since $|b|>1$. If $b=0$ then the inequalities $|a|\leqslant 1$ and (3.13) imply that $|a|=1$. As a consequence, (3.16) follows from (3.8), (3.9), (3.10), and (3.15). Finally, if $ab\neq 0$, then from (3.12) and (3.14), we conclude $|az+b|$ approaches

$$\frac{|b|+\sqrt{b^2-4a^2+4|a|}}{2-2|a|}>1$$

as $e^{i\theta}\to\operatorname{sign}(a)$. Thus from (3.9) and (3.10) there is $\theta\in(0,\pi)$ sufficiently close to 0 or π such that $|t_0|=|t_1|<|t_2|$. We also note that $z\notin\mathbb{R}$ since if $z\in\mathbb{R}$, then the fact that $t_1,t_2\notin\mathbb{R}$ by (3.9) and (3.10) implies $t_1=\overline{t_2}$ which contradicts to $|t_1|<|t_2|$.

Case 2: $|a|>1$ and $|b|<1$. By the intermediate value theorem there is $y\in(0,\infty)$ such that

$$2\sqrt{a^2y^2+b^2}-\sqrt{y^2+4}-y=0$$

since the left-hand side is $2|b|-2<0$ when $y=0$ and its limit is ∞ when $y\to\infty$. With the choice $z=iy$, we have

$$|t_0|=\frac{1}{|az+b|}=\frac{1}{\sqrt{a^2y^2+b^2}}$$

and the modulus of the smaller zero of $t^2+iyt+1$ is

$$\frac{\sqrt{y^2+4}-y}{2}=\frac{2}{\sqrt{y^2+4}+y}=|t_0|.$$

Case 3: $|b| \geqslant 1$ and $|a| > 1$. If $2 + |b| > 2|a|$, then with the same choice of θ and z and the same argument as in the first case, this case follows from

$$\left|\sqrt{b^2 - 4a^2 + 4ae^{i\theta}} + |b|\right| > |b| > 2|a| - 2.$$

We now consider $2 + |b| \leqslant 2|a|$. We square both sides of $2|a| - 2 \geqslant |b|$ to obtain

$$b^2 - 4a^2 \leqslant 4 - 8|a| < -4|a|$$

which implies that, with the cut $[0, \infty)$, the function

$$f(z) := \frac{-b + \sqrt{b^2 - 4a^2 + 4az}}{2a - 2z}$$

is analytic on a small region containing the closed unit ball. From the maximum modulus principle and the fact that

$$|f(0)| = \frac{\left|-b + \sqrt{b^2 - 4a^2}\,\right|}{|2a|} = 1,$$

we conclude that there is $\theta \in \mathbb{R}$ so that $|f(e^{i\theta})| > 1$. With this θ, we let

$$z = \frac{-2ab + be^{i\theta} + e^{i\theta}\sqrt{b^2 - 4a^2 + 4ae^{i\theta}}}{2a^2 - 2ae^{i\theta}}$$

and apply (3.8), (3.9), (3.10), and (3.12) to conclude $|t_0| = |t_1| < |t_2|$. The fact that $z \notin \mathbb{R}$ follows from the same argument as in the previous case. $\square$

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