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WEAK AND STRONG CONVERGENCE THEOREMS FOR THREE SUZUKI'S GENERALIZED NONEXPANSIVE MAPPINGS

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Abstract. We introduce a new iterative scheme for finding a common fixed point of three Suzuki's generalized nonexpansive mappings in Banach spaces. We establish weak and strong convergence theorems for three Suzuki's generalized nonexpansive mappings. The results obtained extend and improve the recent ones announced by Ali et al. [**1**], Maniu [**4**] and Thakur et al. [**10**].

1. Introduction and Preliminaries

Throughout this paper, K be a nonempty convex subset of a Banach space X and $T : K \to K$ be a mapping. We denote by $F(T)$ the set of fixed points of T. We denote by $F = \bigcap_{i=1}^{3} F(T_i)$ the set of a common fixed points of $T_i : K \to K$, $i = 1, 2, 3$. A mapping T is called contraction if there exists $\theta \in (0, 1)$ such that $\|Tx - Ty\| \leqslant \theta\|x - y\|$ for all $x, y \in K$. A mapping T is called *nonexpansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in X$. T is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and $\|Tx - p\| \leqslant \|x - p\|$ for all $x \in X$ and $p \in F(T)$.

In 2008, Suzuki [**8**] introduced the concept of generalized nonexpansive mappings which is a condition on mappings called (C) condition. Let K be a nonempty convex subset of a Banach space X, a mapping $T : K \to K$ is satisfy condition (C) if for all $x, y \in K$, $\frac{1}{2}\|x - Tx\| \leqslant \|x - y\|$ implies $\|Tx - Ty\| \leqslant \|x - y\|$.

Suzuki [**8**] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness. The mapping satisfying condition (C) is called Suzuki generalized nonexpansive mapping. Recently, fixed point theorems for Suzuki generalized nonexpansive mappings have been studied by a number of authors see e.g., [**1, 4, 10, 12**].

In 2000, Noor [**6**] introduced a three-step iterative scheme and studied the approximate solutions of variational inclusion in Hilbert spaces. Glowinski and Le Tallec [**2**] used three-step iterative schemes to find the approximate solutions of the

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elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation. It has been shown in [**2**] that the three-step iterative scheme gives better numerical results than the Mann-type (one-step) [**5**] and the Ishikawa-type (two-step) [**3**] approximate iterations.

Thakur et al. [**9**] introduced the following iterative scheme for nonexpansive mappings in uniformly convex Banach space. $x_1 \in K$, arbitrary, chosen and $\{x_n\}$ generated by:

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n T x_n \\ y_n &= (1-b_n)z_n + b_n T z_n \\ x_{n+1} &= (1-a_n)T x_n + a_n T y_n, \quad \forall n \geqslant 1, \end{aligned} \tag{1.1}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ in $(0,1)$. They proved that this scheme converges to a fixed point of contraction mapping, faster than all the known iterative schemes. Maniu [**4**] studied the above iteration process defined by (1.1) and discussed some qualitative aspects regarding this process, such as stability and data dependency, under the assumption that F(T) is a singleton. Furthermore, Thakur et al. [**11**] introduced the following iterative scheme for nonexpansive mappings in uniformly convex Banach space. $x_1 \in K$, arbitrary, chosen and $\{x_n\}$ generated by

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n T x_n \\ y_n &= (1-b_n)z_n + b_n T z_n \\ x_{n+1} &= (1-a_n)T z_n + a_n T y_n, \quad \forall n \geqslant 1, \end{aligned} \tag{1.2}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ in $(0,1)$.

They proved that this scheme converges to a fixed point of contraction mapping, faster than all the known iterative schemes. Ali et al. [**1**] studied the above iteration process defined by (1.2). They proved some weak and strong convergence theorems of the above iterative method for Suzuki's generalized nonexpansive mappings in uniformly convex Banach spaces.

Inspired and motivated by these facts, we introduce the following iterative scheme for three Suzuki's generalized nonexpansive mappings in uniformly convex Banach spaces.

Let K be a nonempty convex subset of a Banach space X and $T_i : K \to K$, $i = 1, 2, 3$ be mappings. Then for arbitrary $x_1 \in K$, the scheme is defined as follows:

$$\begin{aligned} z_n &= (1-c_n)T_1 x_n + c_n x_n \\ y_n &= (1-b_n)T_2 x_n + b_n T_2 z_n \\ x_{n+1} &= (1-a_n)T_3 x_n + a_n T_3 y_n, \end{aligned} \tag{1.3}$$

$\forall n \geqslant 1$, where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $(0,1)$.

Our aim is to introduce and study convergence problem of three-step iterative sequence (1.3) for three Suzuki's generalized nonexpansive mappings in uniformly convex Banach spaces. The results presented in this paper generalize and extend some recent ones [**1, 4, 10**].

The following definitions will be needed in proving our main results.

A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta(\varepsilon) = \inf\Big\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1, \|x-y\| = \varepsilon\Big\} > 0.$$

for all $0 < \varepsilon \leqslant 2$ (i.e., $\delta(\varepsilon)$ is a function $(0,2] \to (0,1)$).

Recall that a Banach space X is said to satisfy *Opial's condition* [**7**], if for each sequence $\{x_n\}$ in X, the condition $x_n \to x$ weakly as $n \to \infty$ and for all $y \in X$ with $y \neq x$ imply that $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$. We now list some properties of mapping that satisfy condition (C). In what follows, we shall make use of the following lemmas.

LEMMA 1.1. [**8**] *Let T be a mapping on a subset K of a Banach space X with Opial's condition. Assume that T satisfies condition* (C). *If $\{x_n\}$ converges weakly to p and $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, then $Tp = p$. That is, $I - T$ is demiclosed at zero.*

LEMMA 1.2. [**8**] *Let T be a mapping on a weakly compact convex subset K of uniformly convex Banach space X. Assume that T satisfies condition (C), then T has a fixed point.*

Let $\{x_n\}$ be a bounded sequence in a Banach space X. For $x \in X$, we set $r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|$. The asymptotic radius of $\{x_n\}$ relative to K is defined by $r(K, \{x_n\}) = \inf\{r(x, \{x_n\}) : x \in K\}$. The asymptotic center of $\{x_n\}$ relative to K is the set $A(K, \{x_n\}) = \{x \in K : r(x, \{x_n\}) = r(K, \{x_n\})\}$. It is known that, in uniformly convex Banach space, $A(K, \{x_n\})$ consists of exactly one-point.

LEMMA 1.3. (1) *If T is nonexpansive then T satisfies condition* (C) [**8**, Proposition 1].

(2) *If T satisfies condition* (C) *and has a fixed point, then T is a quasi-nonexpansive mapping* [**8**, Proposition 2].

(3) *If T satisfies condition* (C), *then $\|x - Ty\| \leqslant 3\|Tx - x\| + \|x - y\|$ for all $x, y \in K$* [**8**, Lemma 7].

LEMMA 1.4. [**13**, Theorem 2] *Let $k > 1$ and $r > 0$ be two fixed real numbers. Then a Banach space X is uniformly convex if and only if there is a continuous strictly increasing convex function $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^k \leqslant \lambda\|x\|^k + (1-\lambda)\|y\|^k + \omega_k(\lambda)g(\|x-y\|)$$

for all $x, y \in B_r := \{x \in X : \|x\| \leqslant r\}$ and $\lambda \in [0,1]$ where $\omega_k(\lambda) = \lambda^k(1-\lambda) + \lambda(1-\lambda)^k$.

2. Main results

In this section, we prove three-step iterative scheme (1.3) to converge to a common fixed point for Suzuki's generalized nonexpansive mappings in a uniformly convex Banach space.

LEMMA 2.1. *Let K be a nonempty closed convex subset of a uniformly convex Banach space X, $T_i : K \to K$, $i = 1, 2, 3$ be mappings satisfying condition* (C) *with $F \neq \emptyset$. For arbitrarily chosen $x_1 \in K$, $\{x_n\}$ be a sequence generated by* (1.3)*, then we have, for p is a common fixed point of T_i, $i = 1, 2, 3$, $\lim_{n\to\infty} \|x_n - p\|$ exits.*

PROOF. For any $p \in F$, and $z \in K$, since for $T_i : K \to K$, $i = 1, 2, 3$, satisfy condition (C), $\frac{1}{2}\|p - T_i p\| = 0 \leqslant \|p - z\|$ implies that $\|T_i p - T_i z\| \leqslant \|p - z\|$. Using (1.3) and by Lemma 1.3(2), we have

$$\begin{aligned} \|z_n - p\| &= \|(1 - c_n)T_1 x_n + c_n x_n - p\| \\ &\leqslant \|(1 - c_n)(T_1 x_n - p) + c_n(x_n - p)\| \\ &\leqslant (1 - c_n)\|T_1 x_n - p\| + c_n\|x_n - p\| \leqslant \|x_n - p\|. \end{aligned} \tag{2.1}$$

Using (2.1) together with Lemma 1.3 (2), we get

$$\begin{aligned} \|y_n - p\| &= \|(1 - b_n)T_2 x_n + b_n T_2 z_n - p\| \\ &\leqslant \|(1 - b_n)(T_2 x_n - p) + b_n(T_2 z_n - p)\| \\ &\leqslant (1 - b_n)\|T_2 x_n - p\| + b_n\|T_2 z_n - p\| \leqslant \|x_n - p\|. \end{aligned} \tag{2.2}$$

By using (2.2) together with Lemma 1.3 (2), we get

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - a_n)T_3 x_n + a_n T_3 y_n - p\| \\ &\leqslant \|(1 - a_n)(T_3 x_n - p) + a_n(T_3 y_n - p)\| \\ &\leqslant (1 - a_n)(\|T_3 x_n - p\|) + a_n(\|T_3 y_n - p\|) \leqslant \|x_n - p\| \end{aligned}$$

Thus we have $\|x_{n+1} - p\| \leqslant \|x_n - p\|$. This implies that $\{\|x_n - p\|\}$ is bounded and non-increasing for p is a common fixed point of T_i, $i = 1, 2, 3$. It follows that $\lim_{n\to\infty} \|x_n - p\|$ exits. □

THEOREM 2.1. *K be a nonempty closed convex subset of a uniformly convex Banach space X, $T_i : K \to K$, $i = 1, 2, 3$ be mappings satisfying condition* (C)*, p is a common fixed point of T_i, $i = 1, 2, 3$ and let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be real sequences in $(0, 1)$, for all $n \geqslant 1$. Let $\{x_n\}$ be a sequence in K defined by* (1.3)*, and parameters satisfy the following conditions:*

(1) *If $0 < \liminf_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} a_n < 1$,*
(2) *If $0 < \liminf_{n\to\infty} a_n$ and $0 \leqslant \liminf_{n\to\infty} b_n \leqslant \limsup_{n\to\infty} b_n < 1$,*
(3) *If $0 < \liminf_{n\to\infty} a_n b_n$, and $0 < \liminf_{n\to\infty} c_n \leqslant \limsup_{n\to\infty} c_n < 1$.*

Then $F \neq \emptyset$ if and only if $\{x_n\}$ is bounded and we assume that for all $x, y \in K$, $T_i : K \to K$, $i = 2, 3$ are mappings satisfying condition (C)*,*

$$\lim_{n\to\infty} \|T_1 x_n - x_n\| = 0, \ \lim_{n\to\infty} \|T_2 z_n - T_2 x_n\| = 0, \ \lim_{n\to\infty} \|T_3 y_n - T_3 x_n\| = 0,$$
$$\lim_{n\to\infty} \|z_n - x_n\| = 0, \ \lim_{n\to\infty} \|y_n - x_n\| = 0,$$
$$\lim_{n\to\infty} \|T_2 x_n - x_n\| = 0, \ \lim_{n\to\infty} \|T_3 x_n - x_n\| = 0.$$

PROOF. By Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - p\|$ exits for any $p \in F$. Then the sequence $\{x_n - p\}$ is bounded. Assume that $T_i : K \to K$, $i = 1, 2, 3$ are mappings satisfying condition (C) and $T_i : K \to K$, $i = 1, 2, 3$ has a common fixed point p. By Lemma 1.3(2), since $T_i : K \to K$, $i = 1, 2, 3$ are quasi-nonexpansive mappings then the sequences $\{T_1x_n - p\}$, $\{T_2x_n - p\}$, $\{T_3y_n - p\}$, $\{T_2z_n - p\}$ and $\{T_3x_n - p\}$ are also bounded. Therefore, there exists $R > 0$ such that $\{x_n - p\}$, $\{T_1x_n - p\}, \{T_2x_n - p\}, \{T_3y_n - p\}, \{T_2z_n - p\}$ and $\{T_3x_n - p\} \subset B_R$. By (1.3) and Lemma 1.4, we have

$$\begin{aligned}
\|z_n - p\|^2 &= \|(1 - c_n)T_1x_n + c_nx_n - p)\|^2\\
&\leqslant \|(1 - c_n)(T_1x_n - p) + c_n(x_n - p) - p)\|^2\\
&\leqslant (1 - c_n)\|T_1x_n - p\|^2 + c_n\|x_n - p\|^2 - c_n(1 - c_n)\big(g(\|x_n - T_1x_n\|)\big)\\
&\leqslant (1 - c_n)\|x_n - p\|^2 + c_n\|x_n - p\|^2 - c_n(1 - c_n)\big(g(\|x_n - T_1x_n\|)\big)\\
&= \|x_n - p\|^2 - c_n(1 - c_n)\big(g(\|x_n - T_1x_n\|)\big)
\end{aligned}$$

Thus we have $\|z_n - p\|^2 \leqslant \|x_n - p\|^2 - c_n(1 - c_n)\big(g(\|x_n - T_1x_n\|)\big)$. Now by (1.3) and Lemma 1.4, we have

$$\begin{aligned}
\|y_n - p\|^2 &= \|(1 - b_n)T_2x_n + b_nT_2z_n - p\|^2\\
&\leqslant \|(1 - b_n)(T_2x_n - p) + b_n(T_2z_n - p)\|^2\\
&\leqslant (1 - b_n)\|x_n - p\|^2 + b_n\|z_n - p\|^2\\
&\quad - b_n(1 - b_n)\big(g(\|T_2z_n - T_2x_n\|)\big) - b_nc_n(1 - c_n)\big(g(\|x_n - T_1x_n\|)\big)\\
&\leqslant (1 - b_n)\|x_n - p\|^2 + b_n\|x_n - p\|^2\\
&\quad - b_n(1 - b_n)\big(g(\|T_2z_n - T_2x_n\|)\big) - b_nc_n(1 - c_n)\big(g(\|x_n - T_1x_n\|)\big)\\
\|y_n - p\|^2 &\leqslant \|x_n - p\|^2 - b_n(1 - b_n)\big(g(\|T_2z_n - T_2x_n\|)\big)\\
&\quad - b_nc_n(1 - c_n)\big(g(\|T_1x_n - x_n\|)\big)
\end{aligned}$$

Moreover, by (1.3) and Lemma 1.4, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(1 - a_n)T_3x_n + a_nT_3y_n - p\|^2\\
&\leqslant (1 - a_n)\|T_3x_n - p\|^2 + a_n\|T_3y_n - p\|^2\\
&\quad - a_n(1 - a_n)\big(g(\|T_3y_n - T_3x_n\|)\big)\\
&\leqslant (1 - a_n)\|x_n - p\|^2 + a_n\|y_n - p\|^2\\
&\quad - a_n(1 - a_n)\big(g(\|T_3y_n - T_3x_n\|)\big)\\
&\leqslant \|x_n - p\|^2 - a_n(1 - a_n)\big(g(\|T_3y_n - T_3x_n\|)\big)\\
&\quad - a_nb_n(1 - b_n)\big(g(\|T_2z_n - T_2x_n\|)\big)\\
&\quad - a_nb_nc_n(1 - c_n)\big(g(\|x_n - T_1x_n\|)\big)
\end{aligned}$$

Thus we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leqslant \|x_n - p\|^2 - a_n(1 - a_n)\big(g(\|T_3y_n - T_3x_n\|)\big)\\
&\quad - a_nb_n(1 - b_n)\big(g(\|T_2z_n - T_2x_n\|)\big)\\
&\quad - a_nb_nc_n(1 - c_n)\big(g(\|x_n - T_1x_n\|)\big)
\end{aligned}$$

From the last inequality, we have

$$a_n(1-a_n)\big(g(\|T_3y_n - T_3x_n\|)\big) \leqslant \big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\big), \tag{2.3}$$

$$a_nb_n(1-b_n)\big(g(\|T_2z_n - T_2x_n\|)\big) \leqslant \big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\big), \tag{2.4}$$

$$a_nb_nc_n(1-c_n)\big(g(\|x_n - T_1x_n\|)\big) \leqslant \big(\|x_n - q\|^2 - \|x_{n+1} - p\|^2\big). \tag{2.5}$$

By condition $0 < \liminf_{n\to\infty} a_n \leqslant \limsup_{n\to\infty} a_n < 1$, there exists a positive integer n_0 and $\delta, \delta' \in (0,1)$ such that $0 < \delta < a_n$ and $a_n < \delta' < 1$ for all $n \geqslant n_0$. Then it follows from (2.3) that

$$(\delta(1-\delta'))\lim_{n\to\infty} g(\|T_3y_n - T_3x_n\|) \leqslant \big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\big),$$

for all $n \geqslant n_0$. Thus, for $m \geqslant n_0$, we write

$$\begin{aligned}\sum_{n=n_0}^{m} g(\|T_3y_n - T_3x_n\|) &\leqslant \frac{1}{(\delta(1-\delta'))}\sum_{n=n_0}^{m}\big(\|x_n - p\|^2 - \|x_{n+1} - p\|^2\big)\\ &\leqslant \frac{1}{(\delta(1-\delta'))}\big(\|x_{n_0} - p\|^2\big).\end{aligned}$$

Letting $m \to \infty$, we have $\sum_{n=n_0}^{m} g(\|T_3y_n - T_3x_n\|) < \infty$ so that

$$\lim_{n\to\infty} g(\|T_3y_n - T_3x_n\|) = 0.$$

From g is continuous strictly increasing with $g(0) = 0$ then we have

$$\lim_{n\to\infty} \|T_3y_n - T_3x_n\| = 0. \tag{2.6}$$

By using a similar method for inequalities (2.4) and (2.5) we have

$$\lim_{n\to\infty} \|T_2z_n - T_2x_n\| = 0, \qquad \lim_{n\to\infty} \|T_1x_n - x_n\| = 0. \tag{2.7}$$

Now also, we have

$$\|z_n - x_n\| \leqslant \|(1-c_n)T_1x_n + c_nx_n - x_n\| \leqslant (1-c_n)\|T_1x_n - x_n\| \to 0, \ \ n \to \infty. \tag{2.8}$$

From condition (C) we have

$$\lim_{n\to\infty} \|T_2z_n - z_n\| = 0. \tag{2.9}$$

From (2.7) and (2.9) it follows that

$$\|T_2x_n - z_n\| \leqslant \|T_2x_n - T_2z_n\| + \|T_2z_n - z_n\| \to 0, \ \ n \to \infty. \tag{2.10}$$

Hence, from (2.8) and (2.10) we have

$$\|T_2x_n - x_n\| \leqslant \|T_2x_n - z_n\| + \|z_n - x_n\| \to 0, \ \ n \to \infty.$$

Moreover, it follows that

$$\begin{aligned}\|y_n - x_n\| &\leqslant \|(1-b_n)T_2x_n + b_nT_2z_n - x_n\|\\ &\leqslant \|T_2x_n - x_n\| + b_n\|T_2z_n - T_2x_n\| \to 0, \ \ n \to \infty.\end{aligned} \tag{2.11}$$

Also, from condition (C) we have

$$\lim_{n\to\infty} \|T_3y_n - y_n\| = 0. \tag{2.12}$$

From (2.6) and (2.12) it follows that

$$\|T_3x_n - y_n\| \leqslant \|T_3x_n - T_3y_n\| + \|T_3y_n - y_n\| \to 0, \ \ n \to \infty. \tag{2.13}$$

Hence, from (2.11) and (2.13) we have

$$\|T_3x_n - x_n\| \leqslant \|T_3x_n - y_n\| + \|y_n - x_n\| \to 0, \ \ n \to \infty.$$

Thus we obtain

$$\lim_{n\to\infty} \|T_3x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|T_2x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|T_1x_n - x_n\| = 0.$$

Conversely, suppose that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|T_3x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|T_2x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|T_1x_n - x_n\| = 0.$$

Let $p \in A(K, \{x_n\})$. By Lemma 1.2, we have, for $T_i : K \to K$, $i = 1, 2, 3$,

$$\begin{aligned} r(T_ip, \{x_n\}) = \limsup_{n\to\infty} \|x_n - T_ip\| &\leqslant \limsup_{n\to\infty} (3\|T_ix_n - x_n\| + \|x_n - p\|) \\ &\leqslant \limsup_{n\to\infty} \|x_n - p\| = r(p, \{x_n\}) \end{aligned}$$

This implies that for $i = 1, 2, 3$, $T_ip \in A(K, \{x_n\})$. Since X is a uniformly Banach space, $A(K, \{x_n\})$ is singleton, hence $i = 1, 2, 3$, $T_ip = p$. □

In the next result, we prove our strong convergence theorem as follows.

THEOREM 2.2. *Let X be a real uniformly convex Banach space and K be a nonempty copact convex subset of X and $T_i : K \to K$, $i = 1, 2, 3$ be mappings satisfying condition* (C). *Assume that $p \in F$ is a common fixed point of T_i, $i = 1, 2, 3$ and let $\{x_n\}$ be a sequence in K defined by* (1.3) *where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in* $(0, 1)$ *for all $n \geqslant 1$, and satisfy the conditions of Theorem 2.1. Then $\{x_n\}$ converges strongly to a common fixed point of T_i, $i = 1, 2, 3$.*

PROOF. By Lemma 1.2, $F \neq \emptyset$, so by Theorem 2.1, we have

$$\lim_{n\to\infty} \|T_3x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|T_2x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|T_1x_n - x_n\| = 0.$$

Since K is compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p$ as $k \to \infty$ for $p \in K$. By Lemma 1.3(3), we have

$$\|x_{n_k} - T_1p\| \leqslant 3\|T_1x_{n_k} - x_{n_k}\| + \|x_{n_k} - p\| \text{ for all } n \geqslant 1.$$

Letting $k \to \infty$, we get $T_1p = p$, $p \in F$. By using a similar method, $p = T_2p$ and then we have $p = T_3p$. Since by Lemma 2.1, $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in F$, so $\{x_n\}$ converges strongly to a common fixed point of T_i, $i = 1, 2, 3$. □

Finally, we prove the weak convergence of iterative scheme (1.3) for Suzuki's generalized nonexpansive mappings in a uniformly convex Banach space satisfying Opial's condition.

THEOREM 2.3. *Let X be a real uniformly convex Banach space satisfying Opial's condition and K be a nonempty closed convex subset of X. and $T_i : K \to K$, $i = 1, 2, 3$ be mappings satisfying condition* (C). *Assume that $p \in F$ is a common fixed point of T_i, $i = 1, 2, 3$ and let $\{x_n\}$ be a sequence in K defined by* (1.3) *where*

$\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are real sequences in $(0,1)$ and satisfy the conditions of Theorem 2.2. Then $\{x_n\}$ converges weakly to a common fixed point of T_i, $i = 1,2,3$.

Proof. Since $F \neq \emptyset$, it follows from Theorem 2.2 that $\{x_n\}$ is bounded and

$$\lim_{n\to\infty} \|T_3x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|T_2x_n - x_n\| = 0, \quad \lim_{n\to\infty} \|T_1x_n - x_n\| = 0.$$

Since X is a uniformly convex Banach space, X is reflexive so by Eberlin's theorem, there exists $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some weakly $q_1 \in X$. Since K is closed and convex, by Mazur's theorem $q_1 \in K$. By Lemma 1.1, $q_1 \in F$. Now we show that $\{x_n\}$ converges weakly to $q_1 \in F$. We assume that q_1 and q_2 are weak limits of the subsequences $\{x_{n_k}\}$ of $\{x_n\}$, respectively, and $q_1 \neq q_2$. By Lemma 1.1, $q_2 \in F$. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - q_1\|$ and $\lim_{n\to\infty} \|x_n - q_2\|$ exist. By Opial's condition, we have

$$\begin{aligned}\lim_{n\to\infty} \|x_n - q_1\| &= \lim_{j\to\infty} \|x_{n_j} - q_1\| < \lim_{j\to\infty} \|x_{n_j} - q_2\| = \lim_{n\to\infty} \|x_n - q_2\| \\ &= \lim_{k\to\infty} \|x_{n_k} - q_2\| < \lim_{k\to\infty} \|x_{n_k} - q_1\| = \lim_{n\to\infty} \|x_n - q_1\|\end{aligned}$$

which is a contraction. So, $q_1 = q_2$. Therefore $\{x_n\}$ converges weakly to a common fixed point of $T_i : K \to K$, $i = 1,2,3$. □

Now we give the examples of three Suzuki's generalized nonexpansive mappings which are not nonexpansive mappings. Define three mappings

Example 1. [**4**] $T_1 : [0,1] \to [0,1]$ by $Tx = \begin{cases} 1-x, & \text{if } x \in [0, \frac{1}{4}) \\ \frac{x+3}{4}, & \text{if } x \in [\frac{1}{4}, 1] \end{cases}$

Example 2. [**10**] $T_2 : [0,1] \to [0,1]$ by $Tx = \begin{cases} 1-x, & \text{if } x \in [0, \frac{1}{5}) \\ \frac{x+4}{5}, & \text{if } x \in [\frac{1}{5}, 1] \end{cases}$

Example 3. [**12**] $T_3 : [0,1] \to [0,1]$ by $Tx = \begin{cases} 1-x, & \text{if } x \in [0, \frac{1}{8}) \\ \frac{x+7}{8}, & \text{if } x \in [\frac{1}{8}, 1] \end{cases}$

It is easily seen from [**4**],[**10**] and [**12**] that $T_i : K \to K$ $(i = 1,2,3)$ are Suzuki's generalized nonexpansive mappings and their common fixed point is 1. Iterative scheme (1.3) converges to the common fixed point of these Suzuki's generalized nonexpansive mappings.

References

1. J. Ali, F. Ali, F. P. Kumar, *Approximation of Fixed Points For Suzuki's Generalized Non-Expansive Mappings*, Mathematics **7**,522 (2019), 1–11.
2. R. Glowinski, P. Le Tallec, *Augmented Lagrangian and Operator-Splitting Methods in Nonlinear Mechanics*, SIAM, Philadelphia, 1989.
3. I. Ishikawa, *Fixed point by a new iteration method*, Proc. Am. Math. Soc. **44** (1974), 147–150.
4. G. Maniu, *On a three-step iteration process for Suzuki mappings with qualitative study*, Numer. Funct. Anal. Optim. **41**(8) (2020), 929–949.
5. W. R. Mann, *Mean value methods in iteration*, Proc. Am. Math. Soc. **4** (1953), 506–510.
6. M. A. Noor, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. **251** (2000), 217–229.
7. Z. Opial, *Weak convergence of successive approximations for nonexpansive mappings*, Bull. Am. Math. Soc. **73** (1967), 591–597.

8. T. Suzuki, *Fixed point theorems and convergence theorems for some generalized nonexpansive mappings*, J. Math. Anal. Appl. **340**(2) (2008), 1088–1095.
9. D. Thakur, B. S. Thakur, M. Postolache, *New iteration scheme for numerical reckoning fixed points of non-expansive mappings*, J. Inequal. Appl. 2014, 328.
10. B. S. Thakur, D. Thakur, M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings*, Appl. Math. Comput. **275** (2016), 147–155.
11. ———, *New iteration scheme for approximating fixed point of non-expansive mappings*, Filomat **30** (2016), 2711–2720.
12. K. Ullah, M. Arshad, *Numerical reckoning fixed points for Suzuki's generalized nonexpansive mappings via new iteration process*, Filomat **32**(1) (2018), 187–196.
13. H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Analysis **16** (1991), 1127–1138.

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