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A THEORY OF VARIATIONS VIA P-STATISTICAL CONVERGENCE

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Abstract. We introduce some notions of variation using the statistical convergence with respect to power series method. By the use of the notions of variation, we prove criterions that can be used to verify convergence without using limit value. Also, some results that give relations between P-statistical variations are studied.

1. Introduction

The subject of regular variation was initiated by Jovan Karamata in 1930 in his pioneering paper [**21**] (see also [**20**, **22**, **23**]). The theory deals with asymptotic analysis of divergent processes and the results are very important in Tauberian theorems. The Karamata theory was released by Bingham et al. whose book [**4**] is a nice exposition of the subject (see also [**18**,**25**,**29**]). The concept of rapid variation was defined and investigated by de Haan [**19**] in his 1970 thesis. These two concepts motivated significant developments in asymptotic analysis. Another concept is $\mathcal{O}$-regularly varying sequences. This concept with the theory of regularly varying functions that first appeared in [**3**] (see also [**9**]). Bojanić and Seneta [**5**] have showed that Karamata's original theory of regularly varying functions is similar to the theory of regularly varying sequences. They also studied the properties of regularly varying sequences without proofs. Djurčić and Božin [**10**] showed similar connection for the concept $\mathcal{O}$-regularly varying functions and sequences.

The idea of statistical convergence first appeared (under the name "almost convergence") in the first edition (1935) of the famous monograph [**34**] of Zygmund. In an explicit form statistical convergence was introduced, independently, by Fast in [**15**] and Steinhaus in [**31**], in 1951. There are many variants of statistical convergence in the literature. Recently, Ünver and Orhan in [**33**] introduced another interesting convergence method called statistical convergence with respect to a

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power series or shortly P_p-statistical convergence. Ünver and Orhan have showed that statistical convergence and P_p-statistical convergence are different from each other.

On the other side, Di Maio et al. introduced and studied the statistical rapid convergence in [**8**]. It was the first paper dealing with variations by using statistical convergence. More recently, Dutta and Das [**14**] have continued such an investigation of variation via statistical convergence. Our aim here is to define and study P_p-statistical analogues of these concepts. We find out criterions that can be used to verify convergence without using limit value. Also, some results that give relations among P_p-statistical variations are studied.

2. Preliminaries

We shall be concerned initially with certain properties related to the statistical convergence and the convergence in the sense of power series method of a sequence $\mathbf{x} = \{x_n\}$.

NOTE 2.1. In what follows $\mathbf{x} = \{x_n\}$, $\mathbf{y} = \{y_n\}$, $\mathbf{z} = \{z_n\}$ will denote sequences of positive real numbers. Sometimes we simply write $\mathbf{x}$ or $\{x_n\}$ instead of $\mathbf{x} = \{x_n\}$.

As usually, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and real numbers, respectively. By $2\mathbb{N}$ and $2\mathbb{N}+1$ denote the sets of even and odd natural numbers, respectively, and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$.

2.1. Statistical convergence. Let E be a subset of $\mathbb{N}$. The *natural density of* E, denoted by $\delta(E)$, is given by

$$\delta(E) := \lim_{k\to\infty} \frac{1}{k} |\{n \leqslant k : n \in E\}|$$

whenever the limit exists, where $|.|$ denotes the cardinality of a set [**28**].

A sequence $\mathbf{x} = \{x_n\}$ is *statistically convergent* to ℓ provided that for every $\varepsilon > 0$,

$$\lim_{k\to\infty} \frac{1}{k} |\{n \leqslant k : |x_n - \ell| \geqslant \varepsilon\}| = 0$$

that is,

$$E_k(\varepsilon) := \{n \leqslant k : |x_n - \ell| \geqslant \varepsilon\}$$

has its natural density zero. This is denoted by $\text{st-}\lim_n x_n = \ell$ [**7, 15, 27, 31**]. It is worth noting that every convergent sequence (in the usual sense) is statistically convergent to the same limit, while a statistically convergent sequence need not be convergent.

2.2. Power series method. In what follows $\mathbf{p} = \{p_n\}$ will be a given non-negative real sequence such that $p_0 > 0$ and the corresponding power series

$$p(u) := \sum_{n=0}^{\infty} p_n u^n$$

has a radius of convergence R_p with $0 < R_p \leqslant \infty$.

The power series methods, include both Abel and Borel methods, are well known and are more effective than ordinary convergence. We turn now to these methods.

Let $\mathbf{x} = \{x_n\}$ be a sequence of positive real numbers. If the limit

$$\lim_{0<u\to R_p^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n x_n = \ell$$

exists then we say that $\mathbf{x}$ is *convergent in the sense of power series method* [**26**,**30**].

It is worth to point out that the method is regular if and only if $\lim_{0<u\to R_p^-} \frac{p_n u^n}{p(u)} = 0$ for every n (see, e.g. [**6**]).

Remark 2.1. Let us notice first that in the case of $R_p = 1$, it is not difficult see that if $p_n = 1$ and $p_n = \frac{1}{n+1}$, the power series methods coincide with the Abel summability method and the logarithmic summability method, respectively. Furthermore, in the case of $R_p = \infty$ and $p_n = \frac{1}{n!}$, the power series method coincides with the Borel summability method.

In this article the power series method is always assumed to be regular.

2.3. Statistical power series method. Ünver and Orhan [**33**] have recently introduced P_p-statistical convergence.

Definition 2.1. [**33**] Let $\mathbf{p} = \{p_n\}$ be a given sequence. A real sequence $\mathbf{x} = \{x_n\}$ is *P_p-strongly convergent* to ℓ if

$$\lim_{0<u\to R_p^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n |x_n - \ell| = 0.$$

Definition 2.2. [**33**] Let $E \subset \mathbb{N}_0$. If the limit

$$\delta_{P_p}(E) := \lim_{0<u\to R_p^-} \frac{1}{p(u)} \sum_{n\in E} p_n u^n$$

exists, then $\delta_{P_p}(E)$ is said to be the *P_p-density* of E.

Notice that, from the definition, it follows that $0 \leqslant \delta_{P_p}(E) \leqslant 1$ whenever it exists.

Definition 2.3. [**33**] A sequence $\mathbf{x} = \{x_n\}$ is *statistically convergent with respect to power series method* (given by a sequence $\mathbf{p}$), shortly, P_p-statistically convergent, to ℓ provided that for any $\varepsilon > 0$

$$\lim_{0<u\to R_p^-} \frac{1}{p(u)} \sum_{n\in E_\varepsilon} p_n u^n = 0,$$

where $E_\varepsilon = \{n \in \mathbb{N}_0 : |x_n - \ell| \geqslant \varepsilon\}$, that is $\delta_{P_p}(E_\varepsilon) = 0$ for any $\varepsilon > 0$.

In this case we write $\mathrm{st}_{\mathrm{P_p}}\text{-}\lim x_n = \ell$.

The notion of statistically Cauchy sequence has been first introduced by Fridy in [**16**]. In the following definitions we give the notion of P_p-statistically Cauchy sequence, and the notion of P_p-statistical boundedness of sequences.

Definition 2.4. A sequence $\mathbf{x} = \{x_n\}$ is a P_p*-statistically Cauchy sequence* provided that for every $\varepsilon > 0$ there exists a number N such that

$$\lim_{0<u\to R_p^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi(\{n \in \mathbb{N}_0 : |x_n - x_N| \geqslant \varepsilon\}) = 0.$$

Definition 2.5. A sequence $\mathbf{x} = \{x_n\}$ is P_p*-statistically bounded* if for some $M > 0$ it holds $\delta_{P_p}(\{n : |x_n| > M\}) = 0$.

The notions of statistical limit superior and statistical limit inferior have been introduced by Fridy and Orhan [**17**]. Now, in view of this study, we can introduce the concepts of P_p-statistical limit superior and P_p-statistical limit inferior. The P_p*-statistical limit superior* of a sequence $\mathbf{x} = \{x_n\}$, denoted by $\mathrm{st}_{\mathrm{P_p}} - \limsup x_n$, is defined by

$$\mathrm{st}_{\mathrm{P_p}} - \limsup x_n = \begin{cases} \sup B_x, & \text{if } B_x \neq \emptyset, \\ -\infty, & \text{if } B_x = \emptyset, \end{cases}$$

where $B_x := \{b \in \mathbb{R} : \delta_{P_p}(\{n : x_n > b\}) > 0 \text{ or does not exist in } \mathbb{R}\}$. Similarly, the P_p*-statistical limit inferior* of $\{x_n\}$ denoted by $\mathrm{st}_{\mathrm{P_p}} - \liminf x_n$, is defined by

$$\mathrm{st}_{\mathrm{P_p}} - \liminf x_n = \begin{cases} \inf C_x, & \text{if } C_x \neq \emptyset, \\ +\infty, & \text{if } C_x = \emptyset, \end{cases}$$

where $C_x := \{c \in \mathbb{R} : \delta_{P_p}(\{n : x_n < c\}) > 0 \text{ or does not exist in } \mathbb{R}\}$.

Clearly,

$$\mathrm{st}_{\mathrm{P_p}} - \liminf x_n \leqslant \mathrm{st}_{\mathrm{P_p}} - \limsup x_n.$$

Also, $\mathbf{x} = \{x_n\}$ is P_p-statistically bounded and $\mathrm{st}_{\mathrm{P_p}} - \lim x_n = \ell$ if and only if $\mathrm{st}_{\mathrm{P_p}} - \liminf x_n = \mathrm{st}_{\mathrm{P_p}} - \limsup x_n = \ell$.

Theorem 2.1. *If $\beta = \mathrm{st}_{\mathrm{P_p}} - \limsup x_n$ is finite, then for every $\varepsilon > 0$*

$$\delta_{P_p}(\{n : x_n > \beta - \varepsilon\}) \neq 0 \text{ and } \delta_{P_p;}(\{n : x_n > \beta + \varepsilon\}) = 0. \tag{2.1}$$

Conversely, if (2.1) *hold for every positive ε, then $\beta = \mathrm{st}_{\mathrm{P_p}} - \limsup x_n$.*

Theorem 2.2. *If $\mathrm{st}_{\mathrm{P_p}} - \lim x_n = \ell_1$, $\mathrm{st}_{\mathrm{P_p}} - \lim y_n = \ell_2$, then:*

(a) $\mathrm{st}_{\mathrm{P_p}} - \lim(x_n + y_n) = \ell_1 + \ell_2$.
(b) $\mathrm{st}_{\mathrm{P_p}} - \lim(x_n y_n) = \ell_1 \ell_2$.

2.4. Regular and rapid variations. Now, we pause to collect some basic concepts and notations about variations.

Here and throughout the paper L is always assumed to be a positive function and (as we have already mentioned) $\mathbf{x} = \{x_n\}$ is always assumed to be a positive real sequence.

Definition 2.6. [**4**] Let L be a measurable function, defined on $[a, \infty)$, $a > 0$. Then we say that L is *regularly varying* provided that for each $\zeta > 0$

$$\lim_{x\to\infty} \frac{L(\zeta x)}{L(x)} = h(\zeta) < \infty.$$

If $(\zeta) = 1$ for each $\zeta > 0$, then L is called *slowly varying.*

It is known that the function $h(\zeta)$ is a power ζ^{ρ} for some $\rho \in \mathbb{R}$; the number ρ is called the *index of variability* of L.

DEFINITION 2.7. [**4**] A sequence $\mathbf{x} = \{x_n\}$ is *regularly varying* provided that

$$\lim_{n\to\infty} \frac{x_{[\zeta n]}}{x_n} = k(\zeta) < \infty \text{ for all } \zeta > 0.$$

If $k(\zeta) = 1$ for each $\zeta > 0$, then $\mathbf{x}$ is called *slowly varying*.

It is known (see [**4**]) that $k(\zeta) = \zeta^{\rho}$ for some $\rho \in \mathbb{R}$; ρ is called the *index of variability* of $\mathbf{x}$.

By $\mathsf{RV_s}$ (resp. $\mathsf{SV_s}$) we denote the class of regularly varying (resp. slowly varying) sequences, and $\mathsf{RV}_{\mathsf{s},\rho}$ stands for the class of regularly varying sequences of index of variability ρ.

DEFINITION 2.8. [**13, 32**] A sequence $\mathbf{x} = \{x_n\}$ is in the class $\mathsf{Tr(RV_s)}$ of *translationally regularly varying sequences* provided that for each $\zeta \geqslant 1$

$$\lim_{n\to\infty} \frac{x_{[n+\zeta]}}{x_n} = r(\zeta) < \infty.$$

THEOREM 2.3. [**13, 24**] *If* $\mathbf{x} = \{x_n\} \in \mathsf{Tr(RV_s)}$, *then* $r(\zeta) = e^{\rho[\zeta]}$ *for some* $\rho \in \mathbb{R}$.

The number ρ in the previous theorem is called the *index of variability* of $\mathbf{x}$. The symbol $\mathsf{Tr}(\mathsf{RV}_{\mathsf{s},\rho})$ stands for the class all sequences in $\mathsf{Tr(RV_s)}$ of index of variability ρ.

DEFINITION 2.9. [**3**] A measurable function L defined on $[a, \infty)$, $a > 0$, is said to be $\mathcal{O}$*-regularly varying* provided that for each $\zeta > 0$

$$\limsup_{x\to\infty} \frac{L(\zeta x)}{L(x)} = v(\zeta) < \infty.$$

The function $v(\zeta)$, $\zeta > 0$, is said to be the *index function* of L.

DEFINITION 2.10. [**3**] A sequence $\mathbf{x} = \{x_n\}$ is said to be $\mathcal{O}$*-regularly varying* provided that, for each $\zeta > 0$

$$\limsup_{n\to\infty} \frac{x_{[\zeta n]}}{x_n} = t(\zeta) < \infty.$$

It is evident from the definitions that every regularly varying sequence is $\mathcal{O}$-regularly varying, but the converse may not be true (see [**9**]).

Denote by $\mathsf{ORV_s}$ the class of all $\mathcal{O}$-regularly varying sequences.

The main features and various aspects of $\mathcal{O}$)-regularly varying functions and sequences can be found in [**1, 2, 9, 10**].

A measurable function L defined on $[a, \infty)$, $a > 0$, is *rapidly varying of index of variability* ∞ provided that for each $\zeta > 1$ it satisfies $\lim_{x\to\infty} \frac{L(\zeta x)}{L(x)} = \infty$.

L is said to be *rapidly varying of index of variability* $-\infty$ provided that for each $\zeta > 1$ it satisfies $\lim_{x\to\infty} \frac{L(\zeta x)}{L(x)} = 0$.

Definition 2.11. [**11, 12**] A sequence $\mathbf{x} = \{x_n\}$ belongs to the class $\mathsf{R}_{\mathsf{s},\infty}$ of *rapidly varying sequences of index of variability* ∞ provided that for each $\zeta > 1$

$$\lim_{n\to\infty} \frac{x_{[\zeta n]}}{x_n} = \infty,$$

or, equivalently

$$\lim_{n\to\infty} \frac{x_{[\zeta n]}}{x_n} = 0 \text{ for } 0 < \zeta < 1.$$

A sequence $\mathbf{x} = \{x_n\}$ is *rapidly varying of index of variability* $-\infty$ provided that for each $\zeta > 1$ the following condition is satisfied

$$\lim_{n\to\infty} \frac{x_{[\zeta n]}}{x_n} = 0.$$

Denote by $\mathsf{R}_{\mathsf{s},-\infty}$ the class of rapidly varying sequences of index of variability $-\infty$.

3. P_p-statistical regular variations

In this section we define and study P_p-statistical regular variations for sequences considering P_p-statistical analogues of regular variation, translational regular variation, $\mathcal{O}$-regular variation.

3.1. P_p-statistical regular variation.

Definition 3.1. We say that a sequence $\mathbf{x} = \{x_n\}$ is in the class $\mathsf{P_pSRV_s}$ of *P_p-statistically regularly varying sequences* provided that

$$\mathrm{st}_{\mathrm{P_p}} - \lim \frac{x_{[\zeta n]}}{x_n} = k_{ps}(\zeta) < \infty \quad \text{for all } \zeta > 0.$$

From the definition, it is easy to verify that every regularly varying sequence is P_p-statistically regularly varying, but the converse may not be true as the following example shows.

Example 3.1. Consider the sequence $\mathbf{x} = \{x_n\}$ defined by

$$x_n = \begin{cases} e^{\sqrt{n}}, & n \in 2\mathbb{N}, \\ 1, & n \in 2\mathbb{N}+1. \end{cases}$$

Also, let the power series method is given with the sequence

$$p_n = \begin{cases} 0, & n \in 2\mathbb{N}, \\ 1, & n \in 2\mathbb{N}+1. \end{cases}$$

First observe that $\delta_{P_p}(2\mathbb{N}+1) = 1$ (and $\delta_{P_p}(2\mathbb{N}) = 0$), and that for the sequence $\mathbf{p}$, $R_p = 1$.

We have $\mathrm{st}_{\mathrm{P_p}} - \lim x_n = 1$.

Further, we have that for any $\varepsilon > 0$

$$\lim_{u\to 1^-} \frac{1}{p(u)} \sum_{\{n\in\mathbb{N}_0 : |\frac{x_{[\zeta n]}}{x_n} - l| \geqslant \varepsilon\}} p_n u^n = 0.$$

This means that $\mathbf{x}$ is a P_p-statistically regularly varying sequence. However, since $\lim_{n\to\infty} \frac{x_{[\zeta n]}}{x_n}$ does not exist, $\mathbf{x}$ is not a regularly varying sequence.

Definition 3.2. It is said that a sequence $\mathbf{x}=\{x_n\}$ is a *P_p-statistically regularly varying Cauchy sequence* provided that for every $\varepsilon>0$ there exists a number N (depending on ε) such that for each $\zeta>0$

$$\lim_{0<u\to R_p^-}\frac{1}{p(u)}\sum_{n=0}^{\infty}p_nu^n\chi\Big(\Big\{n\in\mathbb{N}_0:\Big|\frac{x_{[\zeta n]}}{x_n}-\frac{x_{[\zeta N]}}{x_N}\Big|\geqslant\varepsilon\Big\}\Big)=0.$$

Our first result concerns regular variation and can be used to verify convergence without using limit value.

Theorem 3.1. *For a sequence $\mathbf{x}$ the following conditions are equivalent:*

(a) $\mathbf{x}$ *is a P_p-statistically regularly varying sequence;*
(b) $\mathbf{x}$ *is a P_p-statistically regularly varying Cauchy sequence;*
(c) $\mathbf{x}$ *is a sequence for which there is a regularly varying sequence $\mathbf{y}$ such that $\delta_{P_p}(\{n\in\mathbb{N}_0:\frac{x_{[\zeta n]}}{x_n}\neq\frac{y_{[\zeta n]}}{y_n}\})=0$ for each $\zeta>0$.*

Proof. Everywhere in the proof of this theorem we suppose that $\zeta>0$ is arbitrary and fixed.

(a) $\Rightarrow$ (b) Let $\mathrm{st}_{\mathrm{P_p}}-\lim\frac{x_{[\zeta n]}}{x_n}=k_{ps}(\zeta)$ and $\varepsilon>0$. Then

$$\delta_{P_p}\Big(\Big\{n\in\mathbb{N}_0:\Big|\frac{x_{[\zeta n]}}{x_n}-k_{ps}(\zeta)\Big|<\frac{\varepsilon}{2}\Big\}\Big)=1.$$

Choose $N=N(\varepsilon)$ so that $|\frac{x_{[\zeta N]}}{x_N}-k_{ps}(\zeta)|<\frac{\varepsilon}{2}$. The triangle inequality now yields

$$\delta_{P_p}\Big(\Big\{n\in\mathbb{N}_0:\Big|\frac{x_{[\zeta n]}}{x_n}-\frac{x_{[\zeta N]}}{x_N}\Big|<\varepsilon\Big\}\Big)=1.$$

Since ε was arbitrary, $\mathbf{x}$ is a P_p-statistically regularly varying Cauchy sequence.

(b) $\Rightarrow$ (c) Choose N so that the closed interval $I=[\frac{x_{[\zeta N]}}{x_N}-1,\frac{x_{[\zeta N]}}{x_N}+1]$ contains $\frac{x_{[\zeta n]}}{x_n}$ for n belonging to the set of all k satisfying $\delta_{P_p}(\{k\in\mathbb{N}_0:\frac{x_{[\zeta k]}}{x_k}\in I\})=1$. Also, choose M so that the interval $I'=[\frac{x_{[\zeta M]}}{x_M}-\frac{1}{2},\frac{x_{[\zeta M]}}{x_M}+\frac{1}{2}]$ contains $\frac{x_{[\zeta n]}}{x_n}$ for n belonging to the set of all k satisfying $\delta_{P_p}(\{k\in\mathbb{N}_0:\frac{x_{[\zeta k]}}{x_k}\in I'\})=1$. We claim that $I_1=I\cap I'$ contains $\frac{x_{[\zeta n]}}{x_n}$ for n belonging to the set of all k for which $\delta_{P_p}(\{k\in\mathbb{N}_0:\frac{x_{[\zeta k]}}{x_k}\in I_1\})=1$.

Since

$$\Big\{n\in\mathbb{N}_0:\frac{x_{[\zeta n]}}{x_n}\notin I_1\Big\}=\Big\{n\in\mathbb{N}_0:\frac{x_{[\zeta n]}}{x_n}\notin I\Big\}\cup\Big\{n\in\mathbb{N}_0:\frac{x_{[\zeta n]}}{x_n}\notin I'\Big\},$$

we have

$$\delta_{P_p}\Big(\Big\{n\in0\mathbb{N}_0:\frac{x_{[\zeta]}}{x_n}\notin I_1\Big\}\Big)\leqslant\delta_{P_p}\Big(\Big\{n\in\mathbb{N}_0:\frac{x_{[\zeta n]}}{x_n}\notin I\Big\}\Big)+\delta_{P_p}\Big(\Big\{n\in\mathbb{N}_0:\frac{x_{[\zeta n]}}{x_n}\notin I'\Big\}\Big)=0.$$

Therefore, I_1 is a closed interval of length less than or equal to 1 which contains $\frac{x_{[\zeta n]}}{x_n}$ for n belonging to the set of all k with $\delta_{P_p}(\{k\in\mathbb{N}_0:\frac{x_{[\zeta k]}}{x_k}\in I_1\})=1$.

We proceed by choosing N'^1 so that $I''=[\frac{x_{[\zeta N']}}{x_{N'}}-\frac{1}{4},\frac{x_{[\zeta N']}}{x_{N'}}+\frac{1}{4}]$ contains $\frac{x_{[\zeta n]}}{x_n}$ belonging to the set of all k satisfying $\delta_{P_p}(\{k\in\mathbb{N}_0:\frac{x_{[\zeta k]}}{x_k}\in I''\})=1$ and by a similar argument $I_2 = I_1 \cap I''$ contains $\frac{x_{[\zeta n]}}{x_n}$ for n from the set of all k satisfying $\delta_{P_p}(\{k \in \mathbb{N}_0 : \frac{x_{[\zeta k]}}{x_k} \in I_2\}) = 1$, and I_2 has length not greater than $\frac{1}{2}$.

Continuing in this way, we inductively can construct a sequence $\{I_m\}_{m=1}^{\infty}$ of closed intervals having the properties:

(i) for each m, $I_{m+1} \subseteq I_m$;
(ii) the length of I_m is less than or equal to 2^{1-m};
(iii) I_m contains $\frac{x_{[\zeta n]}}{x_n}$ for n belonging to the set of all k satisfying
$$\delta_{P_p}\Big(\Big\{k \in \mathbb{N}_0 : \frac{x_{[\zeta k]}}{x_k} \in I_m\Big\}\Big) = 1.$$

By the well-known Nested Intervals Theorem there exists a real number, say, μ such that $\{\mu\} = \bigcap_{m=1}^{\infty} I_m$. By construction of I_m's we can choose an increasing sequence $\{P_m\}_{m=1}^{\infty}$ such that

$$\frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\Big(\Big\{n \in \mathbb{N}_0 : n > P_m, \frac{x_{[\zeta n]}}{x_n} \notin I_m\Big\}\Big) < \frac{1}{m}. \tag{3.1}$$

Define a subsequence $\mathbf{z}$ of $\mathbf{x}$ consisting of all terms x_n such that
$$n > P_0 \text{ and } P_m < n \leqslant P_{m+1} \text{ then } \frac{x_{[\zeta n]}}{x_n} \notin I_m.$$

Let $\mathbf{y} = \{y_n\}$ defined by
$$\frac{y_{[\zeta n]}}{y_n} = \begin{cases} \mu, & \text{if } x_n \text{ is a term of } \mathbf{z}, \\ \frac{x_{[\zeta n]}}{x_n}, & \text{otherwise.} \end{cases}$$

Then for each $\zeta > 0$, $\lim_{n\to\infty} \frac{y_{[\zeta n]}}{y_n} = \mu$. For if $P_m < n$ and $0 < \frac{1}{m} < \varepsilon$, then either x_n is a term of $\mathbf{z}$, namely, $\frac{y_{[\zeta n]}}{y_n} = \mu$, or $\frac{y_{[\zeta n]}}{y_n} = \frac{x_{[\zeta n]}}{x_n} \in I_m$ and $|\frac{y_{[\zeta n]}}{y_n} - \mu|$ is not greater than the length of I_m.

Next, we assert that $\frac{x_{[\zeta n]}}{x_n} = \frac{y_{[\zeta n]}}{y_n}$ for n that belongs to the set of all k satisfying $\delta_{P_p}(\{k \in \mathbb{N}_0 : \frac{x_{[\zeta k]}}{x_k} = \frac{y_{[\zeta k]}}{y_k}\}) = 1$. We note at this point that for $P_m < n \leqslant P_{m+1}$, we have
$$\Big\{n \in \mathbb{N}_0 : \frac{x_{[\zeta n]}}{x_n} \neq \frac{y_{[\zeta n]}}{y_n}\Big\} \subseteq \Big\{n \in \mathbb{N}_0 : \frac{x_{[\zeta n]}}{x_n} \notin I_m\Big\}.$$

This implies
$$\frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\Big(\Big\{n \in \mathbb{N}_0 : \frac{x_{[\zeta n]}}{x_n} \neq \frac{y_{[\zeta n]}}{y_n}\Big\}\Big)$$
$$\leqslant \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\Big(\Big\{n \in \mathbb{N}_0 : \frac{x_{[\zeta n]}}{x_n} \notin I_m\Big\}\Big) < \frac{1}{m} \quad \text{(by (3.1))}.$$

Taking $0 < u \to R_p^-$, we get
$$\lim_{0<u\to R_p^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\Big(\Big\{n \in \mathbb{N}_0 : \frac{x_{[\zeta n]}}{x_n} \neq \frac{y_{[\zeta n]}}{y_n}\Big\}\Big) = 0,$$

i.e., $\frac{x_{[\zeta n]}}{x_n} = \frac{y_{[\zeta n]}}{y_n}$ for n in the set of all k such that $\delta_{P_p}\big(\{k \in \mathbb{N}_0 : \frac{x_{[\zeta k]}}{x_k} = \frac{y_{[\zeta k]}}{y_k}\}\big) = 1$.

(c) $\Rightarrow$ (a) Let (c) be satisfied, i.e.

$$\frac{x_{[\zeta n]}}{x_n} = \frac{y_{[\zeta n]}}{y_n}$$

for n in the set of all k satisfying

$$\delta_{P_p}\Big(\Big\{n \in \mathbb{N}_0 : \frac{x_{[\zeta k]}}{x_k} = \frac{y_{[\zeta k]2}}{y_k}\Big\}\Big) = 1 \text{ and } \lim_{n\to\infty} \frac{y_{[\zeta k]}}{y_k} = k_{ps}(\zeta).$$

Let $\varepsilon > 0$. Then

$$\Big\{n \in \mathbb{N}_0 : \Big|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\Big| \geqslant \varepsilon\Big\} \subseteq \Big\{n \in \mathbb{N}_0 : \frac{x_{[\zeta n]}}{x_n} \neq \frac{y_{[\zeta n]}}{y_n}\Big\} \cup \Big\{n \in \mathbb{N}_0 : \Big|\frac{y_{[\zeta n]}}{y_n} - k_{ps}(\zeta)\Big| > \varepsilon\Big\}$$

Therefore, $\delta_{P_p}\big(\big\{n \in \mathbb{N}_0 : \big|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\big| \geqslant \varepsilon\big\}\big) = 0$, which means that $\mathbf{x}$ is a P_p-statistically regularly varying sequence. □

By Theorem 3.1 and observation after Definition 2.7, it is straightforward to get the following result.

Corollary 3.1. *If a sequence* $\mathbf{x}$ *is such that* $\mathsf{st}_{\mathsf{P_p}} - \lim \frac{x_{[\zeta n]}}{x_n} = k_{ps}(\zeta)$, $\zeta > 0$, *then* $\mathbf{x}$ *has a subsequence* $\mathbf{y} = \{y_n\}$ *such that* $\lim_{n\to\infty} \frac{y_{[\zeta n]}}{y_n} = k_{ps}(\zeta)$ *and* $k_{ps}(\zeta) = \zeta^\rho$, *for some* $\rho \in \mathbb{R}$.

Now, we give a decomposition theorem for P_p-statistically regularly varying sequences.

Theorem 3.2. *The following conditions for a sequence* $\mathbf{x}$ *are equivalent:*

(a) $\mathrm{st}_{\mathrm{P_p}} - \lim \frac{x_{[\zeta n]}}{x_n} = k_{ps}(\zeta) < \infty$ *for each* $\zeta > 0$;

(b) *There are sequences* $\mathbf{y} = \{y_n\}$ *and* $\mathbf{z} = \{z_n\}$ *such that* $\frac{x_{[\zeta n]}}{x_n} = \frac{y_{[\zeta n]}}{y_n} + \frac{z_{[\zeta n]}}{z_n}$, $\lim_{n\to\infty} \frac{y_{[\zeta n]}}{y_n} = k_{ps}(\zeta)$ *and* $\delta_{P_p}\big(\operatorname{supp} \frac{z_{[\zeta n]}}{z_n}\big) = 0$, *where* $\operatorname{supp}(\mathbf{z}) = \big\{n \in \mathbb{N}_0 : \frac{z_{[\zeta n]}}{z_n} \neq 0\big\}$.

Proof. Throughout the proof we assume that $\zeta > 0$ is arbitrary (and fixed).

(a) $\Rightarrow$ (b) Since (a) is satisfied, there exists a set $K = \{n_0 < n_1 < n_2 < \cdots < n_k < \dots\} \subset \mathbb{N}$ with $\delta_{P_p}(K) = 1$, such that $\lim_{n\to\infty} \frac{x_{[\zeta n]}}{x_n} = k_{ps}(\zeta)$. Now define the sequence $\mathbf{y} = \{y_n\}$ as

$$\frac{y_{[\zeta n]}}{y_n} = \begin{cases} \frac{x_{[\zeta n]}}{x_n}, & n \in K, \\ k_{ps}(\zeta), & n \in \mathbb{N}_0 \setminus K. \end{cases} \tag{3.2}$$

It can be easily seen that $\lim_{n\to\infty} \frac{y_{[\zeta n]}}{y_n} = k_{ps}(\zeta)$. Further, put $\frac{z_{[\zeta n]}}{z_n} = \frac{x_{[\zeta n]}}{x_n} - \frac{y_{[\zeta n]}}{y_n}$, $n \in \mathbb{N}_0$. It is evident from $\{n \in \mathbb{N}_0 : \frac{x_{[\zeta n]}}{x_n} \neq \frac{y_{[\zeta n]}}{y_n}\} \subset \mathbb{N}_0 \setminus K$ and $\delta_{P_p}(\mathbb{N}_0 \setminus K) = 0$, we have $\delta_{P_p}(\{n \in \mathbb{N}_0 : \frac{z_{[\zeta n]}}{z_n} \neq 0\}) = 0$. It follows that $\delta_{P_p}\big(\operatorname{supp} \frac{z_{[\zeta n]}}{z_n}\big) = 0$ and $\frac{x_{[\zeta n]}}{x_n} = \frac{y_{[\zeta n]}}{y_n} + \frac{z_{[\zeta n]}}{z_n}$.

(b) $\Rightarrow$ (a) Now suppose that there exist two sequences $\mathbf{y} = \{y_n\}$ and $\mathbf{z} = \{z_n\}$ such that $\frac{x_{[\zeta n]}}{x_n} = \frac{y_{[\zeta n]}}{y_n} + \frac{z_{[\zeta n]}}{z_n}$, $\lim_{n\to\infty} \frac{y_{[\zeta n]}}{y_n} = k_{ps}(\zeta)$ and $\delta_{P_p}\big(\operatorname{supp} \frac{z_{[\zeta]}}{z_n}\big) = 0$, where $\operatorname{supp}(\mathbf{z}) = \{n \in \mathbb{N}_0 : \frac{z_{[\zeta n]}}{z_n} \neq 0\}$. We will prove that $\mathrm{st}_{\mathrm{P_p}} - \lim \frac{x_{[\zeta n]}}{x_n} = k_{ps}(\zeta)$.

Define $K=\{n_k\}$ to be a subset of $\mathbb{N}_0$ such that $K=\{n\in\mathbb{N}_0 : \frac{z_{[\zeta n]}}{z_n}=0\}$. Since $\delta_{P_p}\big(\operatorname{supp}\frac{z_{[\zeta n]}}{z_n}\big)=0$, we have $\delta_{P_p}(K)=1$, hence $\frac{x_{[\zeta n]}}{x_n}=\frac{y_{[\zeta n]}}{y_n}$ if $n\in K$. Thus, we conclude that there exists a set $K=\{n_0<n_1<n_2<\dots\}$ with $\delta_{P_p}(K)=1$ such that $\lim_{k\to\infty}\frac{x_{[\zeta n_k]}}{x_{n_k}}=k_{ps}(\zeta)$. Hence, we get the result. □

COROLLARY 3.2. *Let* $\mathbf{x}=\{x_n\}$ *be a sequence of positive real numbers. Then,* $\mathrm{st}_{\mathrm{P_p}}-\lim\frac{x_{[\zeta n]}}{x_n}=k_{ps}(\zeta)$, $\zeta>0$, *if and only if there exist* $\mathbf{y}=\{y_n\}$ *and* $\mathbf{z}=\{z_n\}$ *such that* $\frac{x_{[\zeta n]}}{x_n}=\frac{y_{[\zeta n]}}{y_n}+\frac{z_{[\zeta n]}}{z_n}$, $\lim_{n\to\infty}\frac{y_{[\zeta n]}}{y_n}=k_{ps}(\zeta)$ *and* $\mathrm{st}_{\mathrm{P_p}}-\lim\frac{z_{[\zeta n]}}{z_n}=0$.

PROOF. Let $\frac{z_{[\zeta n]}}{z_n}=\frac{x_{[\zeta n]}}{x_n}-\frac{y_{[\zeta n]}}{y_n}$, where $\{y_n\}$ is the sequence defined by (3.2). Then $\lim_{n\to\infty}\frac{y_{[\zeta n]}}{y_n}=k_{ps}(\zeta)$ and Theorem 2.2, we conclude that $st_{P_p}-\lim\frac{z_{[\zeta n]}}{z_n}=0$. Let $\frac{x_{[\zeta n]}}{x_n}=\frac{y_{[\zeta n]}}{y_n}+\frac{z_{[\zeta n]}}{z_n}$, where $\lim_{n\to\infty}\frac{y_{[\zeta n]}}{y_n}=k_{ps}(\zeta)$ and $\mathrm{st}_{\mathrm{P_p}}-\lim\frac{z_{[\zeta n]}}{z_n}=0$. Since $\mathrm{st}_{\mathrm{P_p}}-\lim\frac{y_{[\zeta n]}}{y_n}=k_{ps}(\zeta)$, then by Theorem 2.2 we get $st_P-\lim\frac{x_{[\zeta n]}}{x_n}=k_{ps}(\zeta)$. □

By Theorem 2.1, we get the following result.

THEOREM 3.3. *If* $\eta(\zeta)=\mathrm{st}_{\mathrm{P_p}}-\limsup\frac{x_{[\zeta n]}}{x_n}$ *is finite for each* $\zeta>0$, *then for every* $\varepsilon>0$

$$\delta_{P_p}\Big(\Big\{n:\frac{x_{[\zeta n]}}{x_n}>\eta(\zeta)-\varepsilon\Big\}\Big)\neq 0 \quad \text{and} \quad \delta_{P_p}\Big(\Big\{n:\frac{x_{[\zeta n]}}{x_n}>\eta(\zeta)+\varepsilon\Big\}\Big)=0. \tag{3.3}$$

Conversely, if (3.3) *holds for every positive* ε, *then* $\eta(\zeta)=\mathrm{st}_{\mathrm{P_p}}-\limsup\frac{x_{[\zeta n]}}{x_n}$ *for each* $\zeta>0$.

THEOREM 3.4. *If* $\gamma(\zeta)=\mathrm{st}_{\mathrm{P_p}}-\liminf\frac{x_{[\zeta n]}}{x_n}$ *is finite for each* $\zeta>0$, *then for every* $\varepsilon>0$

$$\delta_{P_p}\Big(\Big\{n:\frac{x_[\zeta n]}{x_n}<\gamma(\zeta)+\varepsilon\Big\}\Big)\neq 0 \quad \text{and} \quad \delta_{P_p}\Big(\Big\{n:\frac{x_{[\zeta n]}}{x_n}<\gamma(\zeta)-\varepsilon\Big\}\Big)=0. \tag{3.4}$$

Conversely, if (3.4) *holds for every positive* ε, *then* $\gamma(\zeta)=\mathrm{st}_{\mathrm{P_p}}-\liminf\frac{x_{[\zeta n]}}{x_n}$ *for each* $\zeta>0$.

DEFINITION 3.3. A sequence $\mathbf{x}=\{x_n\}$ is P_p*-statistically* RV*-bounded* provided that for each $\zeta>0$ there is a number B such that $\delta_{P_p}\big(\{n:\frac{x_{[\zeta n]}}{x_n}>B\}\big)=0$.

THEOREM 3.5. *The* P_p*-statistically* RV*-bounded sequence* $\mathbf{x}$ *is* P_p*-statistically regularly varying if and only if*

$$\mathrm{st}_{\mathrm{P_p}}-\liminf\frac{x_{[\zeta n]}}{x_n}=\mathrm{st}_{\mathrm{P_p}}-\limsup\frac{x_{[\zeta n]}}{x_n} \quad \text{for } \zeta>0.$$

PROOF. Let $\eta(\zeta)=\mathrm{st}_{\mathrm{P_p}}-\limsup\frac{x_{[\zeta n]}}{x_n}$ for each $\zeta>0$ and $\gamma(\zeta)=\mathrm{st}_{\mathrm{P_p}}-\liminf\frac{x_{[\zeta n]}}{x_n}$ for each $\zeta>0$. Assume that $\mathrm{st}_{\mathrm{P_p}}-\lim\frac{x_{[\zeta n]}}{x_n}=k_{ps}(\zeta)$ and $\varepsilon>0$. Then $\delta_{P_p}\big(\{n\in\mathbb{N}_0:\big|\frac{x_{[\zeta n]}}{x_n}-k_{ps}(\zeta)\big|\geqslant\varepsilon\}\big)=0$, so $\delta_{P_p}\big(\{n:\frac{x_{[\zeta n]}}{x_n}>k_{ps}(\zeta)+\varepsilon\}\big)=0$, which implies $\eta(\zeta)\leqslant k_{ps}(\zeta)$, $\zeta>0$. We also have $\delta_{P_p}\big(\{n:\frac{x_{[\zeta n]}}{x_n}<k_{ps}(\zeta)-\varepsilon\}\big)=0$, which implies that $k_{ps}(\zeta)\leqslant\gamma(\zeta)$, $\zeta>0$. Therefore $\eta(\zeta)\leqslant\gamma(\zeta)$, $\zeta>0$. Using the fact that $\gamma(\zeta)\leqslant\eta(\zeta)$ always holds, we can conclude that $\eta(\zeta)=\gamma(\zeta)$ for each $\zeta>0$.

Next, suppose that $\eta(\zeta) = \gamma(\zeta)$, $\zeta > 0$, and define $k_{ps}(\zeta) = \gamma(\zeta)$, $\zeta > 0$. If $\varepsilon > 0$, then from (3.3) and (3.4) of Theorem 3.3 and Theorem 3.4, respectively, we have

$$\delta_{P_p}\Big(\Big\{n : \frac{x_{[\zeta n]}}{x_n} > k_{ps}(\zeta) + \frac{\varepsilon}{2}\Big\}\Big) = 0 \text{ and } \delta_{P_p}\Big(\Big\{n : \frac{x_{[\zeta n]}}{x_n} < k_{ps}(\zeta) - \frac{\varepsilon}{2}\Big\}\Big) = 0.$$

Hence, $\mathrm{st}_{\mathrm{P_p}} - \lim \frac{x_{[\zeta n]}}{x_n} = k_{ps}(\zeta)$. □

3.2. *P_p*-statistical translational regular variation. In this subsection we define P_p-statistically translationally regularly varying sequences, give some examples and prove results relate to this kind of sequences.

Definition 3.4. We say that a sequence $\mathbf{x} = \{x_n\}$ belongs to the class $\mathsf{P_pSTr(RV_s)}$ of *P_p-statistically translationally regularly varying sequences* provided that for each $\zeta \geqslant 1$

$$\mathrm{st}_{\mathrm{P_p}} - \lim \frac{x_{[n+\zeta]}}{x_n} = r_{ps}(\zeta) < \infty.$$

It is clear from the definition that if a sequence is in the class $\mathsf{Tr(RV_s)}$, then it belongs to the class $\mathsf{P_pSTr(RV_s)}$. However, by the following example, the converse is not true in general.

Example 3.2. Consider the sequence $\mathbf{x} = \{x_n\}$ defined by

$$x_n = \begin{cases} n!, & n \in 2\mathbb{N}, \\ 1, & n \in 2\mathbb{N}+1. \end{cases}$$

Take the sequence $\mathbf{p} = \{p_n\}$ defined by

$$p_n = \begin{cases} 0, & n \in 2\mathbb{N}, \\ 1, & n \in 2\mathbb{N}+1. \end{cases}$$

Clearly, $R_p = 1$, and $r_{ps}(\zeta) = 1$.

For $\zeta \geqslant 1$ and $\varepsilon > 0$ set

$$E_\varepsilon = \Big\{n \in \mathbb{N}_0 : \Big|\frac{x_{[\zeta+n]}}{x_n} - 1\Big| \geqslant \varepsilon\Big\}.$$

Since for $\zeta \geqslant 1$

$$\lim_{u\to 1^-} \frac{1}{p(u)} \sum_{n\in E_\varepsilon} p_n u^n = 0$$

we obtain

$$\mathrm{st}_{\mathrm{P_p}} - \lim \frac{x_{[n+\zeta]}}{x_n} = 1,$$

that is, the sequence $\mathbf{x}$ is P_p-statistically translationally regularly varying. As for $\zeta \geqslant 1$ $\lim_{n\to\infty} \frac{x_{[n+\zeta]}}{x_n}$ does not exist, the sequence $\mathbf{x}$ is not translationally regularly varying.

Definition 3.5. It is said that a sequence $\mathbf{x} = \{x_n\}$ is a *P_p-statistically translationally regularly varying Cauchy sequence* if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that for each $\zeta > 0$

$$\lim_{0<u\to R^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\Big(\Big\{n \in \mathbb{N}_0 : \Big|\frac{x_{[n+\zeta]}}{x_n} - \frac{x_{[N+\zeta]}}{x_N}\Big| \geqslant \varepsilon\Big\}\Big) = 0.$$

We omit the proof of the following theorem because it is similar to the proof of Theorem 3.1

Theorem 3.6. *For a sequence* $\mathbf{x}$ *the following conditions are equivalent:*

(a) $\mathbf{x}$ *is a* P_p*-statistically translationally regularly varying sequence;*
(b) $\mathbf{x}$ *is a* P_p*-statistically translationally regularly varying Cauchy sequence;*
(c) $\mathbf{x}$ *is a sequence for which there is translationally regularly varying sequence* $\mathbf{y}$ *such that* $\delta_{P_p}\big(\{n \in \mathbb{N}_0 : \frac{x_{[\zeta+n]}}{x_n} \neq \frac{y_{[\zeta+n]}}{y_n}\}\big) = 0$ *for each* $\zeta \geqslant 1$.

It is a simple matter to see that, by Theorem 3.6 and Theorem 2.3, we have the following result.

Corollary 3.3. *If a sequence* $\mathbf{x} = \{x_n\}$ *is* P_p*-statistically translationally regularly varying to* $r_{ps}(\zeta)$, $\zeta \geqslant 1$, *then* $\mathbf{x}$ *has a subsequence* $\mathbf{y} = \{y_n\}$ *which is translationally regularly varying to* $r_{ps}(\zeta)$ *and* $r_{ps}(\zeta) = e^{\rho[\zeta]}$, $\zeta \geqslant 1$, *for some* $\rho \in \mathbb{R}$.

By $\mathsf{P_pSTr(RV_{s,\rho})}$ we denote the class of all P_p-statistically translationally regularly varying sequences of index of variability ρ.

The following theorem is a P_p-statistical generalization of [**13**, 3.6].

Theorem 3.7. *A sequence* $\mathbf{x} = \{x_n\} \in \mathsf{P_pSTr(RV_{s,\rho})}$, $\rho \in \mathbb{R}$, *if and only if*

$$x_n = x_0 \cdot e^{\sum_{i=0}^{n-1} a_i}, \quad n \geqslant 1,$$

where $\{a_n\}$ *is a real sequence such that* $\mathrm{st_{P_p}} - \lim e^{a_n} = e^\rho$ *and* $a_0 > 0$.

Proof. Let $\mathbf{x} = \{x_n\} \in \mathsf{P_pSTr(RV_{s,\rho})}$, $\rho \in \mathbb{R}$. Immediate, by Corollary 3.3, we get

$$\mathrm{st_{P_p}} - \lim \frac{x_{n+1}}{x_n} = r_{ps}(1) = e^\rho < \infty,$$

which means that there is a sequence $\{b_n\}$ of positive real numbers such that

$$\mathrm{st_{P_p}} - \lim b_n = r_{ps}(1) \ \text{ and } \ \frac{x_{n+1}}{x_n} = b_n, \quad n \in \mathbb{N}_0.$$

Also, for $n \geqslant 0$, we have $x_{n+1} = b_n x_n = b_n b_{n-1} \dots b_0 x_0$. Now, putting $a_i = \ln b_i$, $i \in \mathbb{N}_0$, we have $\mathrm{st_{P_p}} - \lim e^{a_n} = \mathrm{st_{P_p}} - \lim b_n = e^\rho$, and for each $n \in \mathbb{N}_0$ we get $x_{n+1} = x_0 \cdot e^{\sum_{i=0}^{n} a_i}$ namely, for each $n \geqslant 1$

$$x_n = x_0 \cdot e^{\sum_{i=0}^{n-1} a_i}, \quad \text{where } \ \mathrm{st_{P_p}} - \lim e^{a_n} = e^\rho.$$

For the converse part, we assume that $x_n = x_0 \cdot e^{\sum_{i=0}^{n-1} a_i}$, $n \geqslant 1$, where $\mathrm{st_{P_p}} - \lim e^{a_n} = e^\rho$ holds. Then, we have

$$r_{ps}(1) = \mathrm{st_{P_p}} - \lim \frac{x_{n+1}}{x_n} = \mathrm{st_{P_p}} - \lim e^{a_n} = e^\rho.$$

Therefore, for each $\zeta \geqslant 1$

$$\mathrm{st_{P_p}} - \lim \frac{x_{[n+\zeta]}}{x_n} = r_{ps}(\zeta) = e^{\rho[\zeta]}.$$

Hence, $\mathbf{x} = \{x_n\} \in \mathsf{P_pSTr(RV_{s,\rho})}$. $\square$

3.3. P_p-statistical $\mathcal{O}$-regular variation. This subsection deals with P_p-statistically $\mathcal{O}$-regular variation for sequences.

DEFINITION 3.6. We say that a sequence $\mathbf{x} = \{x_n\}$ is *P_p-statistically $\mathcal{O}$-regularly varying* provided that for each $\zeta > 0$

$$\mathrm{st}_{\mathrm{P_p}} - \limsup \frac{x_{[\zeta n]}}{x_n} = t_{ps}(\zeta) < \infty.$$

By $\mathsf{P_pSORV_s}$ we denote the class of all P_p-statistically $\mathcal{O}$-regularly varying sequences.

It is easy to verify that

$$\liminf_{n\to\infty} x_n \leqslant \mathrm{st}_{\mathrm{P_p}} - \liminf x_n \leqslant \mathrm{st}_{\mathrm{P_p}} - \limsup x_n \leqslant \limsup_{n\to\infty} x_n.$$

Moreover, every $\mathcal{O}$-regularly varying sequence is also P_p-statistically $\mathcal{O}$-regularly varying. Example 3.1 shows that the converse need not be true.

PROBLEM 3.1. *Find out a suitable definition of P_p-statistically $\mathcal{O}$-regularly varying Cauchy sequences. Would a theorem similar to Theorem* 3.1 *hold in such a case?*

4. P_p-statistical rapid variation

In this section we discuss P_p-statistical rapid variation in the spirit of the previous section.

DEFINITION 4.1. It is said that a sequence $\mathbf{x} = \{x_n\}$ is *P_p-statistically rapidly varying of index of variability* ∞ provided that for each $\zeta \in (0, 1)$

$$\mathrm{st}_{\mathrm{P_p}} - \lim \frac{x_{[\zeta n]}}{x_n} = 0.$$

The symbol $\mathsf{P_pSR_{s,\infty}}$ stands for the class of P_p-statistically rapidly varying sequences of index of variability ∞.

EXAMPLE 4.1. There is P_p-statistically rapidly varying sequence of index of variability ∞ which is not rapidly varying (of index of variability ∞).

The power series corresponding to the sequence $\mathbf{p} = \{p_n\}$ defined by

$$p_n = \begin{cases} 0, & n \in 2\mathbb{N}, \\ 1, & n \in 2\mathbb{N}+1. \end{cases}$$

has the radius of convergence $R_p = 1$. We prove that the sequence $\mathbf{x} = \{x_n\}$ defined by

$$x_n = \begin{cases} n, & n \in 2\mathbb{N}, \\ e^n, & n \in 2\mathbb{N}+1 \end{cases}$$

is a required sequence. For $0 < \zeta < 1$ and any $\varepsilon > 0$ we have

$$\lim_{u\to 1^-} \frac{1}{p(u)} \sum_{\{n\in\mathbb{N}_0 : \frac{x_{[\zeta n]}}{x_n} \geqslant \varepsilon\}} p_n u^n = 0,$$

i.e.

$$\mathrm{st}_{\mathrm{P_p}} - \lim \frac{x_{[n\zeta]}}{x_n} = 0, \quad 0 < \zeta < 1.$$

Therefore, the sequence $\mathbf{x}$ is P_p-statistically rapidly varying. Since for $\zeta \in (0,1)$ $\lim_{n\to\infty} \frac{x_{[\zeta n]}}{x_n}$ does not exist, the sequence $\mathbf{x}$ is not rapidly varying.

Definition 4.2. It is said that a sequence $\mathbf{x} = \{x_n\}$ is a *P_p-rapidly varying Cauchy sequence* if for each $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that for each $\zeta \in (0,1)$

$$\lim_{0<u\to R_p^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \chi\Big(\Big\{n \in \mathbb{N}_0 : \Big|\frac{x_{[\zeta n]}}{x_n} - \frac{x_{[\zeta N]}}{x_N}\Big| \geqslant \varepsilon\Big\}\Big) = 0.$$

By a small modification in the proof of Theorem 3.1 one can prove the following result.

Theorem 4.1. *For a sequence $\mathbf{x}$ the following conditions are equivalent:*
- (a) $\mathbf{x}$ *is a P_p-statistically rapidly varying sequence;*
- (b) $\mathbf{x}$ *is a P_p-statistically rapidly varying Cauchy sequence;*
- (c) $\mathbf{x}$ *is a sequence for which there is a rapidly varying sequence* $\mathbf{y}$ *such that* $\delta_{P_p}\big(\{n \in \mathbb{N}_0 : \frac{x_{[\zeta n]}}{x_n} \neq \frac{y_{[\zeta n]}}{y_n}\}\big) = 0$ *for each* $\zeta \in (0,1)$.

5. Relation between P_p-statistical variation and P_p-strong variation

In this final section we introduce the definition of P_p-strongly q-regularly varying sequences and give a relation between P_p-statistically regularly varying sequences and P_p-strongly q-regularly varying sequences.

Definition 5.1. Let q be a positive real number. Then a sequence $\mathbf{x} = \{x_n\}$ is said to be a *P_p-strongly q-regularly varying sequence* if for each $\zeta > 0$ it satisfies the condition

$$\lim_{0<u\to R_p^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \Big|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\Big|^q = 0.$$

We denote the set of all P_p-strongly q-regularly varying sequences by $\mathsf{sP_pRV_{s,q}}$.

Remark 5.1. If $0 < q_1 \leqslant q_2 < \infty$, then $\mathsf{sP_pRV_{s,q_2}} \subseteq \mathsf{sP_pRV_{s,q_1}}$ and $\mathsf{sP_pRV_{s,q}} \cap \mathsf{RVb}_\infty = \mathsf{sP_pRV_{s,1}} \cap \mathsf{RVb}_\infty$, where RVb_∞ is the space of all RV-bounded sequences of positive real numbers.

The main result in this section is the following theorem.

Theorem 5.1. *Let q be a positive real number. Then:*
- (a) *If a sequence $\mathbf{x} = \{x_n\}$ is P_p-strongly q-regularly varying, then it is P_p-statistically regularly varying;*
- (b) $\mathsf{sP_pRV_{s,q}} \cap \mathsf{RVb}_\infty = \mathsf{P_pSRV_s} \cap \mathsf{RVb}_\infty$.

Proof. In the proof of the theorem we suppose that $\zeta > 0$ is arbitrary and fixed.

(a) Let $E_\varepsilon(q) = \left\{n \in \mathbb{N}_0 : \left|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\right|^q \geqslant \varepsilon\right\}$. Since $\mathbf{x}$ is P_p-strongly q-regularly varying, we have

$$\begin{aligned}
0 &= \lim_{0<u\to R_p^-} \frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \left|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\right|^q \\
&= \lim_{0<u\to R_p^-} \frac{1}{p(u)} \left\{ \sum_{n\in E_\varepsilon(q)} p_n u^n \left|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\right|^q + \sum_{n\notin E_\varepsilon(q)} p_n u^n \left|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\right|^q \right\} \\
&\geqslant \lim_{0<u\to R_p^-} \frac{1}{p(u)} \sum_{n\in E_\varepsilon(q)} p_n u^n \varepsilon.
\end{aligned}$$

Therefore, $\mathbf{x}$ is P_p-statistically regularly varying.

(b) Denote $F_\varepsilon(q) = \left\{n \in \mathbb{N}_0 : \left|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\right| \geqslant \left(\frac{\varepsilon}{2}\right)^{\frac{1}{q}}\right\}$. Since $\mathbf{x}$ is a RV-bounded sequence, $M = \left|\frac{x_{[\zeta n]}}{x_n}\right| + \left|k_{ps}(\zeta)\right|$, $\zeta > 0$, is finite.

Since $\mathbf{x}$ is also a P_p-statistically regularly varying sequence, for all $u \in (0, R_p)$ we have

$$\frac{1}{p(u)} \sum_{n\in F_\varepsilon(q)} p_n u^n < \frac{\varepsilon}{2M^q}.$$

Now, for all $u \in (0, R_p)$, for each $\zeta > 0$ we have

$$\begin{aligned}
&\frac{1}{p(u)} \sum_{n=0}^{\infty} p_n u^n \left|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\right|^q \\
&\quad = \frac{1}{p(u)} \left\{ \sum_{n\in F_\varepsilon(q)} p_n u^n \left|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\right|^q + \sum_{n\notin F_\varepsilon(q)} p_n u^n \left|\frac{x_{[\zeta n]}}{x_n} - k_{ps}(\zeta)\right|^q \right\} \\
&\qquad\qquad < \frac{1}{p(u)} p(u) \frac{\varepsilon}{2M^q} M^q + \frac{1}{p(u)} p(u) \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Hence, $\mathbf{x}$ is P_p-strongly q-regularly varying. □

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