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ON 2-MICROLOCAL MORREY TYPE BESOV AND TRIEBEL–LIZORKIN SPACES WITH VARIABLE EXPONENTS

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ABSTRACT. We introduce 2-microlocal Morrey type Besov and Triebel–Lizorkin spaces with variable exponents and give some characterizations of these spaces by so-called Peetre's maximal functions. The atomic and molecular decompositions of these spaces are obtained. Finally, using molecular decomposition and the property of local means, we get the wavelet characterizations of these spaces.

1. Introduction

The field of variable exponent function spaces has witnessed an explosive growth in recent years. Note that variable exponent Lebesgue spaces first appeared in the literature [**21**] by W. Orlicz in 1931. Inspired by the mentioned references, similar to classical Besov and Triebel–Lizorkin spaces, Xu [**14, 16**] introduced Besov and Triebel–Lizorkin spaces with variable p, but fixed q and s. As a promotion of Triebel–Lizorkin and Besov spaces with variable exponent, some attention has been paid to the study of Morrey type Triebel–Lizorkin and Besov spaces with variable exponent in recent years. In particular, by Peetre's maximal functions, Fu and Xu [**10**] obtained the equivalent norms of Morrey type Triebel–Lizorkin and Besov spaces with variable exponents. Meanwhile, applying those equivalent quasi-norms, Fu and Xu studied the atomic, molecular and wavelet decompositions of these spaces. For more research of Morrey type Triebel–Lizorkin and Besov spaces with variable exponent see [**3**–**5**].

On the other hand, the concept of 2-microlocal type spaces has aroused the interest of some scholars, and initially appeared in the book of Peetre [**12**]. In [**11**], Lévy Véhel and Seuret developed the 2-microlocal formalism, and claimed that 2-microlocal function spaces are a useful tool to measure local regularity of functions. It is worth noting that Almeida and Caetano [**1, 2**] studied various key properties

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for 2-microlocal Besov and Triebel–Lizorkin spaces with all exponents variable, including Sobolev type embeddings, atomic and molecular representations.

The rest of this paper is organized as follows. Based on Morrey type spaces and 2-microlocal spaces, Section 2 gives the definitions of 2-microlocal Morrey type Besov and Triebel–Lizorkin spaces with variable exponents, which are a reasonable promotions of Morrey type and 2-microlocal Besov and Triebel–Lizorkin spaces with variable exponents. In Section 3, the characterizations of these spaces are proved by Peetre's maximal functions. Moreover, an embedding theorem about these spaces is derived to confirm the convergence of the series of molecular decomposition, which is the main result of this paper. Then, by the convergence and the characterization of Peetre's maximal functions, the author obtains the atomic, molecular and wavelet characterizations of these spaces in Sections 4 and 5.

2. Preliminaries and definitions

As usually, we write $B_r(x)$ for the open ball in $\mathbb{R}^n$ centered at $x \in \mathbb{R}^n$ with radius $r > 0$. Use c as a generic positive constant, and denote simply by $A \lesssim B$ if there exists constant $c_1 > 0$ such that $A \leqslant c_1 B$. Further, $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. For a set A, χ_A denotes its characteristic function. The set $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz class of infinitely differentiable rapidly decreasing complex-valued functions. Its topology is generated by the norms $\|\varphi\|_{k,l} = \sup_{x\in\mathbb{R}^n} \langle x\rangle^k \sum_{|\beta|\leqslant l} |D^\beta \varphi(x)|$, where $\langle x\rangle^k = (1+|x|^2)^{k/2}$. By $\mathcal{S}'(\mathbb{R}^n)$ we denote the dual space of $\mathcal{S}(\mathbb{R}^n)$. The Fourier transform of a tempered distribution f is denoted by $\hat{f}$ while its inverse transform is denoted by $\check{f}$.

In this section, we introduce the basic notation in the theory of 2-microlocal Morrey type Besov and Triebel–Lizorkin spaces with variable exponents.

Definition 2.1. Let p be a measurable function on $\mathbb{R}^n$ with range in $[1,\infty)$. $L^{p(\cdot)}(\mathbb{R}^n)$ denotes the set of all measurable functions f on $\mathbb{R}^n$ such that for some $\lambda > 0$,

$$\int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} dx < \infty.$$

The set becomes a Banach function space when equipped with the norm

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Big(\frac{|f(x)|}{\lambda}\Big)^{p(x)} dx \leqslant 1\right\}.$$

These spaces are referred to as variable Lebesgue spaces, since they generalized the standard Lebesgue spaces. Meanwhile,

$$L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) = \{f : f \in L^{p(\cdot)}(\mathbb{K}) \text{ for all compact subsets } \mathbb{K} \subset \mathbb{R}^n \setminus \{0\}\}.$$

Denote $\mathcal{P}(\mathbb{R}^n)$ the set of all measurable functions p on $\mathbb{R}^n$ with range in $[1,\infty)$ such that

$$1 < p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) = p^+ < \infty.$$

In the classical Lebesgue spaces one can work with L^p where $0 < p < 1$. In this paper, we also consider analogous spaces with variable exponents. Define $\mathcal{P}^0(\mathbb{R}^n)$

to be the set of all measurable functions p on $\mathbb{R}^n$ with range in $(0,\infty)$ such that

$$0 < p^- = \operatorname*{ess\,inf}_{x\in\mathbb{R}^n} p(x), \quad \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} p(x) = p^+ < \infty.$$

Given $p(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$, one can define the space $L^{p(\cdot)}(\mathbb{R}^n)$ as above. This is equivalent to defining it to be the set of all functions f such that $|f|^{p_0} \in L^{q(\cdot)}(\mathbb{R}^n)$, where $0 < p_0 < p_-$, and $q(\cdot) = \frac{p(\cdot)}{p_0} \in \mathcal{P}(\mathbb{R}^n)$. Then one can define a quasi-norm on this space by

$$\|f\|_{L^{p(\cdot)}} = \||f|^{p_0}\|_{L^{q(\cdot)}}^{1/p_0}.$$

Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then the standard Hardy–Littlewood maximal operator is defined by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|dy.$$

The key tool we need is the boundedness of the Hardy–Littlewood maximal operator on variable exponent function spaces. Denote $(\mathcal{M}(|f|^t))^{1/t}$ by $\mathcal{M}_t f$ for $0 < t < \infty$. There exist some sufficient conditions on $p(\cdot)$ such that the maximal operator $\mathcal{M}$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$; see, for example, [**6**, **11**, **17**]. By $\mathcal{B}(\mathbb{R}^n)$ we denote the class of all $p \in \mathcal{P}^0(\mathbb{R}^n)$ such that the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Definition 2.2. (i) A continuous function $g\colon \mathbb{R}^n \to \mathbb{R}$ is called locally log-Hölder continuous, abbreviated $g \in \mathcal{C}^{\log}_{\mathrm{loc}}(\mathbb{R}^n)$, if there exists $c_{\log} > 0$ such that for all $x, y \in \mathbb{R}^n$,

$$|g(x) - g(y)| \leqslant \frac{c_{\log}}{\log(e + 1/|x-y|)}.$$

(ii) A function g is called globally log-Hölder continuous, abbreviated $g \in \mathcal{C}^{\log}(\mathbb{R}^n)$, if g is locally log-Hölder continuous and there exists $g_\infty \in \mathbb{R}$ and $C_{\log} > 0$ such that for all $x \in \mathbb{R}^n$,

$$|g(x) - g_\infty| \leqslant \frac{C_{\log}}{\log(e + |x|)}.$$

If $q \in \mathcal{C}^{\log}(\mathbb{R}^n)$, then for every $q_0 < q^-$ we have $q(\cdot)/q_0 \in \mathcal{B}(\mathbb{R}^n)$, see [**7**, Theorem 3.6]. The notation $\mathcal{P}^{\log}(\mathbb{R}^n)$ is used for those variable exponents $p \in \mathcal{P}^0(\mathbb{R}^n)$ with $\frac{1}{p} \in \mathcal{C}^{\log}$. If $p(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, then we have for every $p_0 < p^-$ that $\mathcal{M}$ is bounded on $L^{p(\cdot)/p_0}(\mathbb{R}^n)$ or, equivalently, that $\mathcal{M}_t$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$, where $t = \min(1, p_0)$.

Definition 2.3. Let $p(\cdot), q(\cdot) \in \mathcal{P}^0$ with $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ for all $x \in \mathbb{R}^n$. The space $M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)$ is defined to be the set of all measurable functions $f \in L^{q(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{n(\frac{1}{p(x)} - \frac{1}{q(x)})} \|f\|_{L^{q(x)}(B_r(x))} < \infty.$$

Definition 2.4. Let $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha \geqslant 0$ and $\alpha_1 \leqslant \alpha_2$. We say that a sequence of positive measurable functions $\omega = (\omega_j)_j \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ if

(i) there exists $c > 0$ such that $0 < \omega_j(x) \leqslant c\,\omega_j(y)(1 + 2^j|x - y|)^\alpha$ for all $j \in \mathbb{N}_0$ and $x, y \in \mathbb{R}^n$.

(ii) there holds $2^{\alpha_1}\omega_j(x) \leqslant \omega_{j+1}(x) \leqslant 2^{\alpha_2}\omega_j(x)$ for all $j \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$. Such a sequence will be called an admissible weight sequence. When we write $\omega \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$ without any restrictions it means that $\alpha \geqslant 0$ and $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ (with $\alpha_1 \leqslant \alpha_2$) are arbitrary but fixed numbers.

We now recall the Fourier analytical approach to function spaces of Morrey type Besov and Triebel–Lizorkin. Let $\varphi_0, \varphi \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0 \geqslant 0$ and satisfy the following conditions:

$$\varphi_0(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2. \end{cases}$$

Set $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$ with $x \in \mathbb{R}^n$. For $j \in \mathbb{Z}$, we also put $\varphi_j(x) = \varphi(2^{-j}x)$ and $\Phi_j = \check{\varphi}_j$, then we call $\{\varphi_j\}_{j\in\mathbb{N}_0}$ is a resolution of unity, it follows that

$$\sum_{j=0}^{\infty} \varphi_j(x) = 1.$$

Remark 2.1. Such a resolution of unity can easily be constructed. Consider the following example. Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ with $\varphi_0(x) = 1$ for $|x| \leqslant 1$ and supp $\varphi_0 \subseteq \{x \in \mathbb{R}^n : |x| \leqslant 2\}$. For $j \geqslant 1$ we define $\varphi_j(x) = \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x)$. It is obvious that $\varphi = \{\varphi_j\}_{j\in\mathbb{N}_0}$ is a resolution of unity.

Now we introduce 2-microlocal Morrey type Besov and Triebel–Lizorkin spaces with variable exponents as follows.

Definition 2.5. Let $\{\varphi_j\}_{j\in\mathbb{N}_0}$ be a resolution of unity and $\omega = (\omega_j)_j \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < \beta \leqslant \infty$, $p(\cdot), q(\cdot) \in \mathcal{P}^0$ with $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ for all $x \in \mathbb{R}^n$.

(i) Let $MB^{\omega,\beta}_{p(\cdot),q(\cdot)}$ denote 2-microlocal Morrey type Besov space with variable exponents by

$$MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \big\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} < \infty \big\},$$

where

$$\|f\|_{MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = \bigg(\sum_{j=0}^{\infty} \big\| \omega_j(\varphi_j * \hat{f})^\vee \big\|^\beta_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)} \bigg)^{1/\beta}.$$

(ii) Let $MF^{\omega,\beta}_{p(\cdot),q(\cdot)}$ denote 2-microlocal Morrey type Triebel–Lizorkin spaces with variable exponents by

$$MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n) = \big\{ f \in \mathcal{S}'(\mathbb{R}^n) :\ \|f\|_{MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} < \infty \big\},$$

where

$$\|f\|_{MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = \bigg\| \bigg(\sum_{j=0}^{\infty} |\omega_j(\varphi_j * \hat{f})^\vee|^\beta \bigg)^{1/\beta} \bigg\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)}.$$

In the next section, we will show that Definition 2.5 is independent of the choice of functions φ. Namely, we will characterize these spaces in terms of Peetre's maximal functions and local mean.

3. Local mean characterization

Let us recall the classical Peetre maximal operator, introduced in [**13**]. In the following we define the system of maximal functions.

DEFINITION 3.1. Given a sequence of functions $\{\Psi_j\}_j \in \mathcal{S}(\mathbb{R}^n)$, let $f \in \mathcal{S}'(\mathbb{R}^n)$ and a positive number $a > 0$, the Peetre's maximal functions are defined as

$$(\Psi_j^* f)_a(x) = \sup_{y\in\mathbb{R}^n} \frac{|\Psi_j * f(y)|}{1+|2^j(x-y)|^a}, \quad x \in \mathbb{R}^n,\ j \in \mathbb{N}_0.$$

To prove the main theorems, we need some technical lemmas as follows. The next lemma is a discrete convolution inequality which we will need later on.

LEMMA 3.1. [**10**, Lemma 2.4] *Let $0 < \beta \leqslant \infty$ and $\delta > 0$, and $p(\cdot), q(\cdot) \in \mathcal{P}^0(\mathbb{R}^n)$ with $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ for all $x \in \mathbb{R}^n$. For any sequence $\{g_j\}_{j=0}^\infty$ of nonnegative measurable functions on $\mathbb{R}^n$, let $G_j = \sum_{k=0}^\infty 2^{-|k-j|\delta} g_k$. Then*

$$\|\{G_j\}_{j=0}^\infty\|_{M_{q(\cdot)}^{p(\cdot)}(l_\beta)} \leqslant C_1 \|\{g_j\}_{j=0}^\infty\|_{M_{q(\cdot)}^{p(\cdot)}(l_\beta)},$$
$$\|\{G_j\}_{j=0}^\infty\|_{l_\beta(M_{q(\cdot)}^{p(\cdot)})} \leqslant C_2 \|\{g_j\}_{j=0}^\infty\|_{l_\beta(M_{q(\cdot)}^{p(\cdot)})},$$

where $C_1 = C_1(\beta,\delta)$ and $C_2 = C_2(p(\cdot), q(\cdot), \beta, \delta)$ are positive constants.

LEMMA 3.2 (See [**19**]). *Let $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $1 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ for all $x \in \mathbb{R}^n$. Then the Hardy–Littlewood maximal operator $\mathcal{M}$ is bounded on $M_{q(\cdot)}^{p(\cdot)}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f \in M_{q(\cdot)}^{p(\cdot)}(\mathbb{R}^n)$, $\|\mathcal{M}f\|_{M_{q(\cdot)}^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\|f\|_{M_{q(\cdot)}^{p(\cdot)}(\mathbb{R}^n)}$.*

LEMMA 3.3. [**10**, Theorem 2.2], [**15**] *Let $1 < \beta \leqslant \infty$ and $q(\cdot) \in \mathcal{C}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ with $1 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ for all $x \in \mathbb{R}^n$, Then there exists a positive constant C such that for all sequences $\{f_j\}_{j=0}^\infty$ of locally integrable functions on $\mathbb{R}^n$,*

$$\bigg\|\bigg(\sum_{j=0}^\infty |\mathcal{M}f_j|^\beta\bigg)^{1/\beta}\bigg\|_{M_{q(\cdot)}^{p(\cdot)}(\mathbb{R}^n)} \leqslant C\bigg\|\bigg(\sum_{j=0}^\infty |f_j|^\beta\bigg)^{1/\beta}\bigg\|_{M_{q(\cdot)}^{p(\cdot)}(\mathbb{R}^n)}.$$

The following two theorems are the main results of this section. It is not difficult to see that Theorem 3.1 shows that the definition of the 2-microlocal Morrey type Besov and Triebel–Lizorkin spaces of variable integrability is independent of the resolution of unity.

THEOREM 3.1. *Let $\omega = (\omega_k)_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < \beta \leqslant \infty$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}^0$ with $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ for all $x \in \mathbb{R}^n$. Let $a \in \mathbb{R}$, $R \in \mathbb{N}_0$ with $R > \alpha_2$. Further, let ψ_0, ψ_1 belong to $\mathcal{S}(\mathbb{R}^n)$ with $D^\beta\psi_1(0) = 0$, for $0 \leqslant |\beta| < R$, and*

$$|\psi_0(x)| > 0 \quad on \quad \{x \in \mathbb{R}^n : |x| < \varepsilon\},$$
$$|\psi_1(x)| > 0 \quad on \quad \{x \in \mathbb{R}^n : \varepsilon/2 < |x| < 2\varepsilon\}$$

for some $\varepsilon > 0$.

(i) *If there exists* $0 < q_0 < q^-$ *with* $q(\cdot)/q_0 \in \mathcal{B}(\mathbb{R}^n)$, *then for* $a > \frac{n}{q_0} + \alpha$

$$\|f\|_{MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \sim \|(\Psi_k^* f)_a \omega_k\|_{l_\beta(M^{p(\cdot)}_{q(\cdot)})} \sim \|(\Psi_k * f)\omega_k\|_{l_\beta(M^{p(\cdot)}_{q(\cdot)})}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

(ii) *If there exists* $q_0 < \min(q^-, \beta)$ *with* $q(\cdot)/q_0 \in \mathcal{B}(\mathbb{R}^n)$, *then for* $a > \frac{n}{q_0} + \alpha$

$$\|f\|_{MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \sim \|(\Psi_k^* f)_a \omega_k\|_{M^{p(\cdot)}_{q(\cdot)}(l_\beta)} \sim \|(\Psi_k * f)\omega_k\|_{M^{p(\cdot)}_{q(\cdot)}(l_\beta)}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. The idea of the proof is from Rychkov in [**20**]. The whole proof is divided into two steps, which together give the proof of Theorem 3.1.

We start with two given functions $\psi_0, \psi_1 \in \mathcal{S}(\mathbb{R}^n)$ and define $\psi_j(x) = \psi_1(2^{-j+1}x)$, for $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$. Furthermore, for all $j \in \mathbb{N}_0$, we write $\Psi_j = \hat{\psi_j}$ and in an analogous manner we define Φ_j from two starting functions $\phi_0, \phi_1 \in \mathcal{S}(\mathbb{R}^n)$.

Step 1. Firstly, proceeding as in the proof of [**7**, Theorem 3.6], it can easily be seen that for $\delta > 0$,

$$(\Phi_\nu^* f)_a(x)\omega_\nu(x) \leqslant c \sum_{k=0}^{\infty} 2^{-|k-\nu|\delta} (\Phi_k^* f)_a(x)\omega_k(x), \quad x \in \mathbb{R}^n.$$

Thus according to Lemma 3.1, we prove that for all $f \in \mathcal{S}'(\mathbb{R}^n)$ the following estimates are true:

$$\|(\Psi_k^* f)_a \omega_k\|_{l_\beta(M^{p(\cdot)}_{q(\cdot)})} \leqslant c\|(\Phi_k^* f)_a \omega_k\|_{l_\beta(M^{p(\cdot)}_{q(\cdot)})},$$
$$\|(\Psi_k^* f)_a \omega_k\|_{M^{p(\cdot)}_{q(\cdot)}(l_\beta)} \leqslant c\|(\Phi_k^* f)_a \omega_k\|_{M^{p(\cdot)}_{q(\cdot)}(l_\beta)}.$$

Step 2. According to the proof of [**7**, Theorem 3.8] and Lemma 3.1, replace $l_{\beta/r}(L_{q(\cdot)/r})$ and $L_{q(\cdot)/r}(l_{\beta/r})$ to $l_{\beta/r}(M^{p(\cdot)/r}_{q(\cdot)/r})$ and $M^{p(\cdot)/r}_{q(\cdot)/r}(l_{\beta/r})$, we have

$$\|((\Psi_\nu^* f)_a \omega_\nu)^r\|_{l_{\beta/r}(M^{p(\cdot)/r}_{q(\cdot)/r})} \leqslant c\|\mathcal{M}(|\Psi_{k+\nu} * f|^r(\omega_{k+\nu})^r)(x)\|_{l_{\beta/r}(M^{p(\cdot)/r}_{q(\cdot)/r})},$$
$$\|((\Psi_\nu^* f)_a \omega_\nu)^r\|_{M^{p(\cdot)/r}_{q(\cdot)/r}(l_{\beta/r})} \leqslant c\|\mathcal{M}(|\Psi_{k+\nu} * f|^r(\omega_{k+\nu})^r)(x)\|_{M^{p(\cdot)/r}_{q(\cdot)/r}(l_{\beta/r})}.$$

Now we choose $\frac{n}{a-\alpha} < r < p_0$, clearly, $p(\cdot)/r \in \mathcal{B}(\mathbb{R}^n)$. By Lemma 3.2 and Lemma 3.3, we have

$$\|(\Psi_k^* f)_a \omega_k\|_{l_\beta(M^{p(\cdot)}_{q(\cdot)})} \leqslant c\|(\Psi_k * f)\omega_k\|_{l_\beta(M^{p(\cdot)}_{q(\cdot)})}$$
$$\|(\Psi_k^* f)_a \omega_k\|_{M^{p(\cdot)}_{q(\cdot)}(l_\beta)} \leqslant c\|(\Psi_k * f)\omega_k\|_{M^{p(\cdot)}_{q(\cdot)}(l_\beta)}$$

holds for all $f \in \mathcal{S}'(\mathbb{R}^n)$. □

Now, we discuss usual local mean characterization of 2-microlocal Morrey type Besov and Triebel–Lizorkin spaces with variable exponents. Let $B_1(0)$ be the unit ball and $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ with support in $B_1(0)$, and $(\hat{k_0})(0) \neq 0$, $(\hat{k^0})(0) \neq 0$. For $N \in \mathbb{N}_0$ we define the iterated Laplacian

$$k(y) := \Delta^N k^0(y) = \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}\right)^N k^0(y), \quad y \in \mathbb{R}^n.$$

For a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ the corresponding local mean is defined for $x \in \mathbb{R}^n$ and $t > 0$ by (at least formally)

$$k(t,f)(x) = \int_{\mathbb{R}^n} k(y)f(x+ty)dy = t^{-n}\int_{\mathbb{R}^n} k(\frac{y-x}{t})f(y)dy.$$

THEOREM 3.2. *Let $\omega = (\omega_k)_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < \beta \leqslant \infty$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}^0$ with $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ for all $x \in \mathbb{R}^n$. Further more, let $N \in \mathbb{N}_0$ with $2N > \alpha_2$ and let $k_0, k^0 \in \mathcal{S}(\mathbb{R}^n)$ and the function k be defined as above.*

(i) *If there exists a $q_0 < q^-$ with $q(\cdot)/q_0 \in \mathcal{B}(\mathbb{R}^n)$, then*

$$\|k_0(1,f)\omega_0\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)} + \left(\sum_{j=1}^{\infty}\|k(2^{-j},f)\omega_j\|^{\beta}_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)}\right)^{1/\beta}$$

is an equivalent norm on $MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

(ii) *If there exists a $q_0 \leqslant \min(q^-,\beta)$ with $q(\cdot)/q_0 \in \mathcal{B}(\mathbb{R}^n)$, then*

$$\|k_0(1,f)\omega_0\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)} + \left\|\left(\sum_{j=1}^{\infty}|k(2^{-j},f)(\cdot)\omega_j(\cdot)|^{\beta}\right)^{1/\beta}\right\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)}$$

is an equivalent norm on $MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ for all $f \in \mathcal{S}'(\mathbb{R}^n)$.

Combing the proof of [**7**, Theorem 2.4] and Theorem 3.1, it is evident to see that Theorem 3.2 holds.

4. Atomic and molecular decompositions

The main goal of this section is to prove atomic and molecular decompositions for $MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. First we introduce some basic concepts. We use the notation Q_{jm} with $j \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, for the closed cube with sides parallel to the coordinate axes, centered at $2^{-j}m$ and with side length 2^{-j}. By χ_{jm} we denote the corresponding characteristic function. The notation dQ_{jm}, $d > 0$ will stand for the closed cube concentric with Q_{jm} and of side length $d2^{-j}$. Then, we recall the definitions of atoms and molecules as follow.

DEFINITION 4.1. Let K, L be a nonnegative integer. A function a in $C^K(\mathbb{R}^n)$ is called a (K,L)-atom for a cube $Q_{\nu k}$ if it satisfies the following conditions:

$$\operatorname{supp} a \subseteq \gamma Q_{\nu k}, \tag{4.1}$$

$$|D^{\beta}a(x)| \leqslant 2^{|\beta|\nu}, \quad \text{for} \quad 0 \leqslant |\beta| \leqslant K \tag{4.2}$$

and if

$$\int_{\mathbb{R}^n} x^{\beta}a(x)dx = 0 \quad \text{for} \quad 0 \leqslant |\beta| \leqslant L \text{ and } \nu \geqslant 1. \tag{4.3}$$

If the atom a located at $Q_{\nu k}$, that means if it fulfills (4.1), then we will denote it by $a_{\nu k}$. For $\nu = 0$ or $L = 0$ there are no moment conditions (4.3) required.

Definition 4.2. Let $K, L \in \mathbb{N}_0$ and $M > 0$. A function μ in $C^K(\mathbb{R}^n)$ is called a (K, L, M)-molecule for a cube $Q_{\nu k}$, if for some $\nu \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$

$$|D^\beta \mu(x)| \leqslant 2^{|\beta|\nu}(1 + 2^\nu|x - 2^{-\nu}m|)^{-M}, \quad \text{for} \quad 0 \leqslant |\beta| \leqslant K, \tag{4.4}$$

$$\int_{\mathbb{R}^n} x^\beta \mu(x)dx = 0 \quad \text{if} \quad 0 \leqslant |\beta| \leqslant L \text{ and } \nu \geqslant 1. \tag{4.5}$$

For $\nu = 0$ or $L = 0$ there are no moment conditions (4.5) required. If a molecule is concentrated in $Q_{\nu k}$, that means it satisfies (4.4), then it is denoted by $\mu_{\nu k}$. If $a_{\nu k}$ is a (K, L)-atom, then it is a (K, L, M)-molecule for every $M > 0$.

Definition 4.3. Let $\omega = (\omega_k)_{k\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < \beta \leqslant \infty$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}^0$ with $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ for all $x \in \mathbb{R}^n$. Then for all complex valued sequences $\lambda = \{\lambda_{\nu m} \in \mathbb{C} : \nu \in \mathbb{N}_0, m \in \mathbb{Z}^n\}$, we define

$$b^{\omega,\beta}_{p(\cdot),q(\cdot)} = \{\lambda : \|\lambda\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}} < \infty\},$$

where

$$\|\lambda\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}} = \Bigg(\sum_{\nu=0}^{\infty}\Bigg\|\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}(\cdot)\Bigg\|^\beta_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)}\Bigg)^{1/\beta}.$$

Furthermore, we define

$$f^{\omega,\beta}_{p(\cdot),q(\cdot)} = \{\lambda : \|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}} < \infty\},$$

where

$$\|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}} = \Bigg\|\Bigg(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^\beta\omega_\nu^\beta(2^{-\nu}m)\chi_{\nu m}(\cdot)\Bigg)^{1/\beta}\Bigg\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)}.$$

Now, to prove the next theorem, we first give an embedding theorem about the above sequences spaces.

Theorem 4.1. *Let* $\omega \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}^0$ *with* $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ *for all* $x \in \mathbb{R}^n$ *and* $0 < \beta \leqslant \infty$, *then*

$$b^{\omega,\min(p^-,q^-,\beta)}_{p(\cdot),q(\cdot)} \hookrightarrow f^{\omega,\beta}_{p(\cdot),q(\cdot)} \hookrightarrow b^{\omega,\max(p^+,q^+,\beta)}_{p(\cdot),q(\cdot)}.$$

Proof. Let $f_\nu(x) = \sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}(x)$, $r = \min(p^-, q^-, \beta)$ and $s = \max(p^+, q^+, \beta)$. First we prove the left-hand side, we know

$$\|\|f_\nu\|_{l_\beta}\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)} \leqslant \|\|f_\nu\|_{l_r}\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)} = \Bigg\|\Bigg(\sum_{j=0}^{\infty}f_\nu^r\Bigg)^{1/r}\Bigg\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)}.$$

Using the triangle inequality, we have

$$\|\|f_\nu\|_{l_\beta}\|_{M^{p(\cdot)}_{q(\cdot)}} \leqslant \Bigg\|\sum_{j=0}^{\infty}f_\nu^r\Bigg\|^{1/r}_{M^{p(\cdot)/r}_{q(\cdot)/r}} \leqslant \Bigg(\sum_{j=0}^{\infty}\|f_\nu^r\|_{M^{p(\cdot)/r}_{q(\cdot)/r}}\Bigg)^{1/r} = \Bigg(\sum_{j=0}^{\infty}\|f_\nu\|^r_{M^{p(\cdot)}_{q(\cdot)}}\Bigg)^{1/r}.$$

On the right-hand side, using the reverse triangle inequality which holds since $\frac{p(\cdot)}{s} \leqslant 1$ and $\frac{q(\cdot)}{s} \leqslant 1$, we obtain

$$\|\|f_\nu\|_{M^{p(\cdot)}_{q(\cdot)}}\|_{l_s} = \left(\sum_{j=0}^{\infty} \|f_\nu^s\|^{\frac{1}{s}\cdot s}_{M^{p(\cdot)/s}_{q(\cdot)/s}} \right)^{1/s} \leqslant \left\| \sum_{j=0}^{\infty} f_\nu^s \right\|^{1/s}_{M^{p(\cdot)/s}_{q(\cdot)/s}} = \left\| \left(\sum_{j=0}^{\infty} f_\nu^s \right)^{1/s} \right\|_{M^{p(\cdot)}_{q(\cdot)}}. \quad \square$$

Before proving the decompositions of atomic and molecular, we need to show that the sum $\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(x)$ converges in $\mathcal{S}'(\mathbb{R}^n)$. Using the above embedding theorem, we begin to prove the convergence of the sum in the following theorem.

THEOREM 4.2. *Let* $\omega = (\omega_k)_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < \beta \leqslant \infty$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}^0$ *with* $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ *for all* $x \in \mathbb{R}^n$.

$$L > -\alpha_1, \quad K \textit{ arbitrary and } M \textit{ large enough.}$$

Let $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ *are* $[K, L, M]$*-molecules concentrated in* $Q_{\nu m}$, *if* $\lambda \in b^{\omega,\beta}_{p(\cdot),q(\cdot)}$ *or* $\lambda \in f^{\omega,\beta}_{p(\cdot),q(\cdot)}$, *then the sum* $\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(x)$ *converges in* $\mathcal{S}'(\mathbb{R}^n)$.

PROOF. For $\omega_\nu \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$ and $\lambda \in b^{\omega,\beta}_{p(\cdot),q(\cdot)}$ or $\lambda \in f^{\omega,\beta}_{p(\cdot),q(\cdot)}$. By the proof of [**8**, Lemma 3.11], we choose $t < \min(p^-, q^-, \beta, 1)$.

Owing to $\langle y \rangle^k = (1 + |y|^2)^{k/2}$, we have $\langle y \rangle^k \leqslant \langle y - 2^{-\nu}m \rangle^k \langle \xi \rangle^k$. Let $M > 0$, $k > 0$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$. Using the moment conditions of $\mu_{\nu m}$, the properties of the weight sequence and the boundedness of the maximal operator, we obtain

$$\begin{aligned}
&\left| \int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m}(y)\varphi(y)dy \right| \\
&\leqslant \int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}||\mu_{\nu m}(y)| \left| \varphi(y) - \sum_{|\gamma|<L} \frac{D^\gamma \varphi(2^{-\nu}m)}{\gamma!}(y - 2^{-\nu}m)^\gamma \right| \frac{\omega_\nu(y)\langle y \rangle^k}{\omega_\nu(y)\langle y \rangle^k} dy \\
&\lesssim \int_{\mathbb{R}^n} \sum_{m\in\mathbb{Z}^n} \frac{(1 + 2^\nu|y - 2^{-\nu}m|)^{L+k-M}}{2^{\nu(L+\alpha_1)}} |\lambda_{\nu m}| \frac{\omega_\nu(y)}{\langle y \rangle^{k-\gamma}} \sup_{\xi\in\mathbb{R}^n} \langle \xi \rangle^k \sum_{|\eta|=L} \frac{D^\eta \varphi(\xi)}{\eta!} dy \\
&\lesssim \frac{\|\varphi\|_{k,L}}{2^{\nu(L+\alpha_1)}} \left\| \sum_{m\in\mathbb{Z}^n} (1 + 2^\nu|y - 2^{-\nu}m|)^{L+\gamma+k-M} |\lambda_{\nu m}|\omega_\nu(2^{-\nu}m) \right\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \frac{\|\varphi\|_{k,L}}{2^{\nu(L+\alpha_1)}} \|\varphi\|_{k,L} \left\| \mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}(\cdot) \Big) \right\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)} \\
&\lesssim \frac{\|\varphi\|_{k,L}}{2^{\nu(L+\alpha_1)}} \left\| \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}(\cdot) \Big)^t \right\|^{1/t}_{M^{p(\cdot)/t}_{q(\cdot)/t}(\mathbb{R}^n)} \\
&\lesssim 2^{-\nu(L+\alpha_1)} \|\lambda^t\|^{1/t}_{b^{\omega,\infty}_{\frac{p(\cdot)}{t},\frac{q(\cdot)}{t}}}.
\end{aligned}$$

Let $s=\max(p^+,q^+,\beta)$ and $f_\nu(x)=\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}(x)$, according to Theorem 4.1, we get the following facts

$$\|\|f_\nu^t\|_{M^{p(\cdot)/t}_{q(\cdot)/t}}\|_{l_\infty}^{1/t}\leqslant \|\|f_\nu^t\|_{M^{p(\cdot)/t}_{q(\cdot)/t}}\|_{l_{\beta/t}}^{1/t}=\bigg(\sum_{j=0}^\infty \|f_\nu\|^{t\cdot\frac{\beta}{t}}_{M^{p(\cdot)}_{q(\cdot)}}\bigg)^{\frac{t}{\beta}\cdot\frac{1}{t}}=\|\lambda\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}},$$

$$\|\|f_\nu^t\|_{M^{p(\cdot)/t}_{q(\cdot)/t}}\|_{l_\infty}^{1/t}=\|\|f_\nu\|^t_{M^{p(\cdot)}_{q(\cdot)}}\|_{l_\infty}^{1/t}\leqslant \|\|f_\nu\|^t_{M^{p(\cdot)}_{q(\cdot)}}\|_{l_{s/t}}^{1/t}=\|\lambda\|^{t\cdot\frac{1}{t}}_{b^{\omega,s}_{p(\cdot),q(\cdot)}}\leqslant\|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}.$$

Thus, combining the above estimates, we obtain

$$\bigg|\bigg\langle\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m},\varphi\bigg\rangle\bigg|\leqslant c2^{-\nu(L+\alpha_1)}\|\lambda\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}},$$
$$\bigg|\bigg\langle\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m},\varphi\bigg\rangle\bigg|\leqslant c2^{-\nu(L+\alpha_1)}\|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}.$$

Since $L>-\alpha_1$ and $\lambda\in b^{\omega,\beta}_{p(\cdot),q(\cdot)}$ or $\lambda\in f^{\omega,\beta}_{p(\cdot),q(\cdot)}$, the sum $\sum_{\nu=0}^\infty\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}\mu_{\nu m}(x)$ converges in $\mathcal{S}'(\mathbb{R}^n)$. □

In 2010, Kempka [**8**] established the atomic and molecular decompositions for 2-microlocal Besov and Triebel–Lizorkin spaces with variable exponents. Using the argument in [**8**], we can obtain the atomic and molecular decompositions of 2-microlocal Morrey type Besov and Triebel–Lizorkin spaces with variable exponents. We leave the details of their proofs.

Corollary 4.1. *Let $\omega=(\omega_k)_{k\in\mathbb{N}_0}\in\mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0<\beta\leqslant\infty$. Further, let $K,L\in\mathbb{N}_0$, $q(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot)\in\mathcal{P}^0$ with $0<q^-\leqslant q(x)\leqslant p(x)\leqslant p^+<\infty$ for all $x\in\mathbb{R}^n$.*

(i) *If $K>\alpha_2$ and $L>\sigma_q-\alpha_1$, then for each $f\in MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, there exist $[K,L]$-atoms $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ centered at $Q_{\nu m}$ and $\lambda\in b^{\omega,\beta}_{p(\cdot),q(\cdot)}$ such that*

$$f=\sum_{\nu=0}^\infty\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}\ \text{converging in}\ \mathcal{S}'(\mathbb{R}^n),$$

holds. Moreover $\|\lambda\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}}\leqslant c\|f\|_{MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$. where the constant $c>0$ is universal for all $f\in MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(ii) *If $K>\alpha_2$ and $L>\sigma_{q,\beta}-\alpha_1$, then for each $f\in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, there exist $[K,L]$-atoms $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ centered at $Q_{\nu m}$ and $\lambda\in f^{\omega,\beta}_{p(\cdot),q(\cdot)}$ such that*

$$f=\sum_{\nu=0}^\infty\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}\ \text{converging in}\ \mathcal{S}'(\mathbb{R}^n)$$

holds. Moreover $\|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}\leqslant c\|f\|_{MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$ where the constant $c>0$ is universal for all $f\in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

COROLLARY 4.2. *Let* $\omega = (\omega_k)_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < \beta \leqslant \infty$. *Further, let* $K, L \in \mathbb{N}_0$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}^0$ *with* $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ *for all* $x \in \mathbb{R}^n$.

(i) *Let* $K > \alpha_2$, $L > \sigma_p - \alpha_1$ *and* $M > 0$ *large enough. If* $\lambda \in b^{\omega,\beta}_{p(\cdot),q(\cdot)}$ *and* $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ *are* $[K, L, M]$*-molecules, then for each* $f \in MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *such that*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m} \quad \text{converging in } \mathcal{S}'(\mathbb{R}^n)$$

holds. Moreover $|f\|_{MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leqslant c\|\lambda\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}}$. *where the constant* $c > 0$ *is universal for all* $f \in MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(ii) *If* $K > \alpha_2$, $L > \sigma_{p,q} - \alpha_1$ *and* $M > 0$ *large enough. If* $\lambda \in f^{\omega,\beta}_{p(\cdot),q(\cdot)}$ *and* $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ *are* $[K, L, M]$*-molecules, then for each* $f \in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *such that*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m} \quad \text{converging in } \mathcal{S}'(\mathbb{R}^n)$$

holds. Moreover $\|f\|_{MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leqslant c\|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}$, *where the constant* $c > 0$ *is universal for all* $f \in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

Since every $[K, L]$-atoms is a $[K, L, M]$-molecule for every $M > 0$, we obtain the following corollary.

COROLLARY 4.3. *Let* $\omega = (\omega_k)_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < \beta \leqslant \infty$. *Further, let* $K, L \in \mathbb{N}_0$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}^0$ *with* $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ *for all* $x \in \mathbb{R}^n$.

(i) *Let* $K > \alpha_2$ *and* $L > \sigma_p - \alpha_1$.

(a) *If* $\lambda \in b^{\omega,\beta}_{p(\cdot),q(\cdot)}$ *and* $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ *are* $[K, L]$*-atoms, then*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m} \text{ converging in } \mathcal{S}'(\mathbb{R}^n)$$

belongs to the spaces $MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *and there exists a constant* $c > 0$ *with* $|f\|_{MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leqslant c\|\lambda\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}}$, *where the constant* $c > 0$ *is universal for all* λ *and* $a_{\nu m}$.

(b) *For each* $f \in MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, *there exist* $\lambda \in b^{\omega,\beta}_{p(\cdot),q(\cdot)}$ *and* $[K, L]$*-atoms* $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ *centered at* $Q_{\nu m}$ *such that there exists a representation*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}a_{\nu m} \text{ converging in } \mathcal{S}'(\mathbb{R}^n)$$

holds. Moreover $\|\lambda\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}} \leqslant c\|f\|_{MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$*, where the constant* $c > 0$ *is universal for all* $f \in MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*.*

(ii) *Let* $K > \alpha_2$ *and* $L > \sigma_{p,q} - \alpha_1$*.*

(a) *If* $\lambda \in f^{\omega,\beta}_{p(\cdot),q(\cdot)}$ *and* $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ *are* $[K, L]$*-atoms, then for each* $f \in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *such that*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}\mu_{\nu m} \text{ converging in } \mathcal{S}'(\mathbb{R}^n)$$

holds. Moreover

$$|f\|_{MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} \leqslant c\|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}},$$

where the constant $c > 0$ *is universal for all* λ *and* $a_{\nu m}$*.*

(b) *For each* $f \in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ *there exist* $\lambda \in f^{\omega,\beta}_{p(\cdot),q(\cdot)}$ *and* $[K, L]$*-atoms* $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0, m\in\mathbb{Z}^n}$ *centered at* $Q_{\nu m}$ *such that*

$$f = \sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} \lambda_{\nu m}a_{\nu m} \text{ converging in } \mathcal{S}'(\mathbb{R}^n)$$

holds. Moreover $\|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}} \leqslant c\|f\|_{MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}$*, where the constant* $c > 0$ *is universal for all* $f \in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$*.*

5. Wavelet decompositions

In this section we present wavelets characterization for $MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$. First, we recall the local means with kernel k

$$k(t, f)(x) = \int_{\mathbb{R}^n} k(y)f(x + ty)dy = t^{-n}\int_{\mathbb{R}^n} k\Big(\frac{y - x}{t}\Big)f(y)dy.$$

Let

$$k(2^{-j}, f)(2^{-j}l) = 2^{jn}\int_{\mathbb{R}^n} k(2^j f - l)f(y)dy = \int_{\mathbb{R}^n} k_{jl}(y)f(y)dy = k_{jl}(f).$$

where $t = 2^{-j}$, $x = 2^{-j}l$, $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$. The usual properties on k get shifted to the kernels k_{jl} as follows.

Definition 5.1. Let $A, B \in \mathbb{N}_0$ and $c > 0$. Further, let $k_{jl} \in C^A(\mathbb{R}^n)$ with $j \in \mathbb{N}_0$ and $l \in \mathbb{Z}^n$ be functions in $\mathbb{R}^n$ with

$$|D^\beta k_{jl}(x)| \leqslant c2^{j|\beta|+jn}(1 + 2^j|x - 2^{-j}l|)^{-C}, \quad |\beta| \leqslant A$$

for all $x \in \mathbb{R}^n$, $j \in \mathbb{N}_0$, $l \in \mathbb{Z}^n$, and

$$\int_{\mathbb{R}^n} x^\beta k_{jl}(x)dx = 0, \quad |\beta| \leqslant B$$

for $j \geqslant 1$ and $l \in \mathbb{Z}^n$.

From the above definition it is clear that $\{2^{-jn}k_{jl}\}_{j\in\mathbb{N}_0,l\in\mathbb{Z}^n}$ are $[A,B,C]$ molecules. Below we have to adapt our sequence spaces $\tilde{b}^{\omega,\beta}_{p(\cdot),q(\cdot)}$ and $\tilde{f}^{\omega,\beta}_{p(\cdot),q(\cdot)}$ to the new situation as follows.

Definition 5.2. Let $\omega=(\omega_k)_{k\in\mathbb{N}_0}\in\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0<\beta\leqslant\infty$, $q(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot)\in\mathcal{P}^0$ with $0<q^-\leqslant q(x)\leqslant p(x)\leqslant p^+<\infty$ for all $x\in\mathbb{R}^n$. Then

$$\tilde{b}^{\omega,\beta}_{p(\cdot),q(\cdot)}=\{\lambda=\{\lambda^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^{\nu},m\in\mathbb{Z}^n}\subset\mathbb{C}:\|\lambda\|_{\tilde{b}^{\omega,\beta}_{p(\cdot),q(\cdot)}}<\infty\},$$

where

$$\|\lambda\|_{\tilde{b}^{\omega,\beta}_{p(\cdot),q(\cdot)}}=\bigg(\sum_{\nu=0}^{\infty}\sum_{G\in G^{\nu}}\Big\|\sum_{m\in\mathbb{Z}^n}|\lambda^{\nu}_{Gm}|\omega_{\nu}(2^{-\nu}m)\chi_{\nu m}\Big\|^{\beta}_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)}\bigg)^{1/\beta}.$$

Furthermore, we define

$$\tilde{f}^{\omega,\beta}_{p(\cdot),q(\cdot)}=\{\lambda=\{\lambda^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^{\nu},m\in\mathbb{Z}^n}\subset\mathbb{C}:\|\lambda\|_{\tilde{f}^{\omega,\beta}_{p(\cdot),q(\cdot)}}<\infty\},$$

where

$$\|\lambda\|_{\tilde{f}^{\omega,\beta}_{p(\cdot),q(\cdot)}}=\bigg\|\bigg(\sum_{\nu=0}^{\infty}\sum_{G\in G^{\nu}}\sum_{m\in\mathbb{Z}^n}|\lambda^{\nu}_{Gm}|^{\beta}\omega^{\beta}_{\nu}(2^{-\nu}m)\chi_{\nu m}\bigg)^{1/\beta}\bigg\|_{M^{p(\cdot)}_{q(\cdot)}(\mathbb{R}^n)}.$$

Now, we recall some results from wavelet theory. The following Proposition 5.1 is taken over from [**9**, Theorem 1.61].

Proposition 5.1. (i) *There are a real scaling function $\psi_F\in\mathcal{S}(\mathbb{R})$ and a real associated wavelet $\psi_M\in\mathcal{S}(\mathbb{R})$ such that their Fourier transforms have compact supports, $\{\hat{\psi}_F\}(0)=(2\pi)^{-1/2}$ and*

$$\operatorname{supp}\{\hat{\psi}_M\}\subseteq\Big[-\frac{8}{3}\pi,-\frac{2}{3}\pi\Big]\cup\Big[\frac{2}{3}\pi,\frac{8}{3}\pi\Big].$$

(ii) *For any $K\in\mathbb{N}$ there exist a real compactly supported scaling function $\psi_F\in C^k(\mathbb{R})$ and a real compactly supported associated wavelet $\psi_M\in C^k(\mathbb{R})$ such that $\{\hat{\psi}_F\}(0)=(2\pi)^{-1/2}$ and*

$$\int_{\mathbb{R}}x^m\psi_M(x)dx=0\ \ \textit{for all}\ \ m\in\{0,\dots,k-1\}.$$

The wavelets in the first part (i) are called Meyer wavelets, the wavelets from the second part (ii) are called Daubechies wavelets.

Let ψ_M,ψ_F be the Meyer or Daubechies wavelets described above. Define

$$G^0=\{F,M\}^n\quad\text{and}\quad G^{\nu}=\{F,M\}^{n*}\text{ if }\nu\geqslant1,$$

where the $*$ indicates, that at least one G_i of $G=(G_1,\cdots,G_n)\in\{F,M\}^{n*}$ must be an M. The cardinal number of $\{F,M\}^{n*}$ is 2^n-1. For $x\in\mathbb{R}^n$, let

$$\Psi^{\nu}_{Gm}(x)=2^{\nu\frac{n}{2}}\prod_{r=1}^{n}\psi_{G_r}(2^{\nu}x_r-m_r),$$

where $G\in G^{\nu}$, $m\in\mathbb{Z}^n$ and $\nu\in\mathbb{N}_0$. Then $\{\Psi^{\nu}_{Gm}:\nu\in\mathbb{N}_0,G\in G^{\nu},m\in\mathbb{Z}^n\}$ is an orthonormal basis in $L^2(\mathbb{R}^n)$.

We assume that $\{\mu_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L,M]$ molecules and $\{k_{jl}\}_{j\in\mathbb{N}_0,l\in\mathbb{Z}^n}$ are the above given functions from Definition 5.1. Before coming to the theorem we recall two fundamental lemmas. First, we have to give estimates of the quantity $|\langle\mu_{\nu m},k_{jl}\rangle|$.

LEMMA 5.1. [**18**, Appendix B]

(i) *Let $M>a+n$, $L\geqslant A$ and $\nu\geqslant j$, then*

$$|\langle\mu_{\nu m},k_{jl}\rangle|\leqslant c2^{-(\nu-j)(A+n)}(1+2^j|2^{-\nu}m-2^{-j}l|)^{-\min\{M-A-n,\ C\}}.$$

(ii) *Let $C>K+n$, $B\geqslant K$ and $\nu\leqslant j$, then*

$$|\langle\mu_{\nu m},k_{jl}\rangle|\leqslant c2^{-(j-\nu)K}(1+2^\nu|2^{-\nu}m-2^{-j}l|)^{-\min\{M,\ C-K-n\}}.$$

LEMMA 5.2. *Let $0<t<1$ and $R>\frac{n}{t}$, For any $j,\nu\in\mathbb{N}_0$, any $l\in\mathbb{Z}^n$ and any sequence $\{h_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ of complex numbers, we have with $x\in Q_{jl}$*

$$\sum_{m\in\mathbb{Z}^n}|h_{\nu m}|(1+2^j|2^{-j}l-2^{-\nu}m|)^{-R}\leqslant c\max(2^{(\nu-j)\frac{n}{t}},1)\mathcal{M}_t\bigg(\sum_{m\in\mathbb{Z}^n}|h_{\nu m}\chi_{\nu m}|\bigg)(x).$$

Now, we are ready to state the following theorem, which gives us one direction of the wavelet decomposition. We define $k(f)=\{k_{jl}(f):j\in\mathbb{N}_0,l\in\mathbb{Z}^n\}$.

THEOREM 5.1. *Let $\omega=(\omega_k)_{k\in\mathbb{N}_0}\in\mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0<\beta\leqslant\infty$, $q(\cdot)\in\mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot)\in\mathcal{P}^0$ with $0<q^-\leqslant q(x)\leqslant p(x)\leqslant p^+<\infty$ for all $x\in\mathbb{R}^n$. Further, let $\{k_{jl}\}_{j\in\mathbb{N}_0,l\in\mathbb{Z}^n}$ be as in Definition 5.1 with $C>0$ large enough and $A,B\in\mathbb{N}_0$.*

(i) *If $A>\sigma_p-\alpha_1$ and $B>\alpha_2$, then for some $c>0$ and all $f\in MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$,*

$$\|k(f)\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}}\leqslant c\|f\|_{MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

(ii) *If $A>\sigma_{p,q}-\alpha_1$ and $B>\alpha_2$, then for some $c>0$ and all $f\in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$,*

$$\|k(f)\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}\leqslant c\|f\|_{MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

PROOF. We only prove the theorem for the Triebel–Lizorkin spaces. Proof for the Besov spaces follows the same line of arguments. We apply the decomposition by atoms to $f\in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and decompose the sum in

$$f=\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}=\sum_{\nu=0}^{j}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}+\sum_{\nu=j+1}^{\infty}\sum_{m\in\mathbb{Z}^n}\lambda_{\nu m}a_{\nu m}=f_j+f^j,$$

where $\{a_{\nu m}\}_{\nu\in\mathbb{N}_0,m\in\mathbb{Z}^n}$ are $[K,L]$-atoms with $K=B>\alpha_2$ and $L=A>\sigma_{p,q}-\alpha_1$. By Corollary 4.3 it is sufficient to find a $c>0$ with $\|k(f)\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}\leqslant c\|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}$ and derive

$$k_{jl}(f)=\int_{\mathbb{R}^n}k_{jl}(y)f_j(y)dy+\int_{\mathbb{R}^n}k_{jl}(y)f^j(y)dy=k_{jl}(f_j)+k_{jl}(f^j).$$

For $\nu \leqslant j$ and $t < \min(1, q^-, \beta)$, we obtain

$$\begin{aligned}
\omega_j(x)|k_{jl}(f_j)| &\leqslant c\sum_{\nu=0}^{j}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}\langle k_{jl}, a_{\nu m}\rangle|\omega_j(x)\\
&\leqslant c\sum_{\nu=0}^{j}2^{-(j-\nu)(K-\alpha_2)}\sum_{m\in\mathbb{Z}^n}\frac{|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)(1+2^j|}{2^{-\nu}m-2^{-j}l|)^{C-k-n-\alpha}}\\
&\leqslant c\sum_{\nu=0}^{j}2^{-(j-\nu)(K-\alpha_2)}\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\Big)(x).
\end{aligned}$$

Thus, for $x \in Q_{jl}$, we obtain

$$\begin{aligned}
&\|k_{jl}(f_j)\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}\\
&\lesssim \Big\|\Big(\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}|k_{jl}(f_j)\omega_j(\cdot)\chi_{jl}(\cdot)|^\beta\Big)^{1/\beta}\Big\|_{M^{p(\cdot)}_{q(\cdot)}}\\
&\leqslant \Big\|\Big(\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}\Big[\sum_{\nu=0}^{j}2^{(\nu-j)}\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\Big)(\cdot)\Big]^\beta\chi_{jl}(\cdot)\Big)^{1/\beta}\Big\|_{M^{p(\cdot)}_{q(\cdot)}}\\
&\leqslant \Big\|\Big(\sum_{\nu=0}^{\infty}\Big[\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\Big)(\cdot)\Big]^\beta\Big)^{1/\beta}\Big\|_{M^{p(\cdot)}_{q(\cdot)}}\\
&\leqslant \Big\|\Big(\sum_{\nu=0}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|^\beta\omega_\nu^\beta(2^{-\nu}m)\chi_{\nu m}(\cdot)\Big)^{1/\beta}\Big\|_{M^{p(\cdot)}_{q(\cdot)}}\\
&= \|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}.
\end{aligned}$$

For $\nu > j$, we have

$$\begin{aligned}
\omega_j(x)|k_{jl}(f^j)| &\lesssim \sum_{\nu=j+1}^{\infty}\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}\langle k_{jl}, a_{\nu m}\rangle|\omega_j(x)\\
&\leqslant \sum_{\nu=j+1}^{\infty}2^{-(\nu-j)(A+n+\alpha_1)}\sum_{m\in\mathbb{Z}^n}\frac{|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)}{(1+2^j|2^{-\nu}m-2^{-j}l|)^{C-\alpha}}\\
&\leqslant \sum_{\nu=j+1}^{\infty}2^{-(\nu-j)(A+n+\alpha_1-n/t)}\mathcal{M}_t\Big(\sum_{m\in\mathbb{Z}^n}|\lambda_{\nu m}|\omega_\nu(2^{-\nu}m)\chi_{\nu m}\Big)(x).
\end{aligned}$$

Set $\delta = A - \sigma_{p,q} + \alpha_1 > 0$, we obtain

$$\begin{aligned}
&\|k_{jl}(f^j)\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}\\
&= \Big\|\Big(\sum_{j=0}^{\infty}\sum_{l\in\mathbb{Z}^n}|k_{jl}(f^j)\omega_j(\cdot)\chi_{jl}(\cdot)|^\beta\Big)^{1/\beta}\Big\|_{M^{p(\cdot)}_{q(\cdot)}}
\end{aligned}$$

$$\lesssim \Big\| \Big(\sum_{j=0}^{\infty} \sum_{l\in\mathbb{Z}^n} \Big[\sum_{\nu=j+1}^{\infty} 2^{(j-\nu)\delta} \mathcal{M}_t \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| \omega_\nu(2^{-\nu}m)\chi_{\nu m} \Big)(\cdot) \Big]^\beta \chi_{jl}(\cdot) \Big)^{1/\beta} \Big\|_{M^{p(\cdot)}_{q(\cdot)}}$$
$$\leqslant \Big\| \Big(\sum_{\nu=j+1}^{\infty} \Big[\mathcal{M}_t \Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}| \omega_\nu(2^{-\nu}m)\chi_{\nu m} \Big)(\cdot) \Big]^\beta \Big)^{1/\beta} \Big\|_{M^{p(\cdot)}_{q(\cdot)}}$$
$$\leqslant \Big\| \Big(\sum_{\nu=0}^{\infty} \sum_{m\in\mathbb{Z}^n} |\lambda_{\nu m}|^\beta \omega_\nu^\beta(2^{-\nu}m)\chi_{\nu m}(\cdot) \Big)^{1/\beta} \Big\|_{M^{p(\cdot)}_{q(\cdot)}}$$
$$= \|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}}.$$

Finally, we have

$$\|k_{jl}(f)\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}} \leqslant c\|\lambda\|_{f^{\omega,\beta}_{p(\cdot),q(\cdot)}} \leqslant c\|f\|_{MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}.$$

Similarly, we can obtain

$$\|k_{jl}(f)\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}} \leqslant c\|\lambda\|_{b^{\omega,\beta}_{p(\cdot),q(\cdot)}} \leqslant c\|f\|_{MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)}. \qquad \square$$

Note that we can deduce that the wavelet in the decomposition theorem are the real compactly supported Daubechies wavelets from Proposition 5.1.

THEOREM 5.2. *Let $\{\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ be the Daubechies wavelets from Proposition* 5.1. *Let $\omega = (\omega_k)_{k\in\mathbb{N}_0} \in \mathcal{W}^\alpha_{\alpha_1,\alpha_2}$, $0 < \beta \leqslant \infty$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}^0$ with $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ for all $x \in \mathbb{R}^n$.*

(i) *For $f \in \mathcal{S}'$ and $k > \max(\sigma_p - \alpha_1, \alpha_2)$. Then $f \in MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, if and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} \lambda^\nu_{Gm} 2^{-\nu\frac{n}{2}} \Psi^\nu_{Gm} \text{ with } \lambda \in \tilde{b}^{\omega,\beta}_{p(\cdot),q(\cdot)} \tag{5.1}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any space $MB^{\zeta,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with $\frac{\zeta_\nu(x)}{\omega_\nu(x)} \to 0$ for $|x| \to \infty$ and all ν and also $\sup_{x\in\mathbb{R}^n} \frac{\zeta_\nu(x)}{\omega_\nu(x)} \to 0$ for $|\nu| \to \infty$. Representation (5.1) *is unique, we have*

$$\lambda^\nu_{Gm} = \lambda^\nu_{Gm}(f) = 2^{\nu\frac{n}{2}} \langle f, \Psi^\nu_{Gm} \rangle$$

and $I\colon f \to \{2^{\nu\frac{n}{2}} \langle f, \Psi^\nu_{Gm}\rangle\}$ is an isomorphic map from $MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ onto $\tilde{b}^{\omega,\beta}_{p(\cdot),q(\cdot)}$. Moreover, if in addition $\beta < \infty$, then $\{\Psi^\nu_{Gm}\}_{\nu\in\mathbb{N}_0,G\in G^\nu,m\in\mathbb{Z}^n}$ is an unconditional basis in $MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(ii) *Let $f \in \mathcal{S}'$ and $k > \max(\sigma_{p,q} - \alpha_1, \alpha_2)$. Then $f \in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, if and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty} \sum_{G\in G^\nu} \sum_{m\in\mathbb{Z}^n} \lambda^\nu_{Gm} 2^{-\nu\frac{n}{2}} \Psi^\nu_{Gm} \text{ with } \lambda \in \tilde{f}^{\omega,\beta}_{p(\cdot),q(\cdot)} \tag{5.2}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any space $MF^{\zeta,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with $\frac{\zeta_\nu(x)}{\omega_\nu(x)} \to 0$ for $|x| \to \infty$ and all ν and also $\sup_{x\in\mathbb{R}^n} \frac{\zeta_\nu(x)}{\omega_\nu(x)} \to 0$ for

$|\nu| \to \infty$. Representation (5.2) is unique, we have

$$\lambda^{\nu}_{Gm} = \lambda^{\nu}_{Gm}(f) = 2^{\nu\frac{n}{2}}\langle f, \Psi^{\nu}_{Gm}\rangle$$

and $I\colon f \to \{2^{\nu\frac{n}{2}}\langle f, \Psi^{\nu}_{Gm}\rangle\}$ is an isomorphic map from $MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ onto $\tilde{f}^{\omega,\beta}_{p(\cdot),q(\cdot)}$. Moreover, if in addition $\beta < \infty$, then $\{\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^{\nu}, m\in\mathbb{Z}^n}$ is an unconditional basis in $MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

By the proof of [**8**, Theorem 4.8], using Theorem 5.1, it is easy to find that the proof is not difficult and so it is omitted.

Now, we present a wavelet decomposition theorem with the help of Meyer wavelets, described in Proposition 5.1.

THEOREM 5.3. *Let $\{\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^{\nu}, m\in\mathbb{Z}^n}$ be the Meyer wavelets according to Proposition 5.1. Let $\omega = (\omega_k)_{k\in\mathbb{N}_0} \in \mathcal{W}^{\alpha}_{\alpha_1,\alpha_2}$, $0 < \beta \leqslant \infty$, $q(\cdot) \in \mathcal{P}^{\log}(\mathbb{R}^n)$, $p(\cdot) \in \mathcal{P}^0$ with $0 < q^- \leqslant q(x) \leqslant p(x) \leqslant p^+ < \infty$ for all $x \in \mathbb{R}^n$ and let $f \in \mathcal{S}'/$*

(i) *we have $f \in MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, if and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty}\sum_{G\in G^{\nu}}\sum_{m\in\mathbb{Z}^n} \lambda^{\nu}_{Gm} 2^{-\nu\frac{n}{2}}\Psi^{\nu}_{Gm} \text{ with } \lambda \in \tilde{b}^{\omega,\beta}_{p(\cdot),q(\cdot)} \tag{5.3}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any space $MB^{\zeta,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with $\frac{\zeta_\nu(x)}{\omega_\nu(x)} \to 0$ for $|x| \to \infty$ and all ν and also $\sup_{x\in\mathbb{R}^n}\frac{\zeta_\nu(x)}{\omega_\nu(x)} \to 0$ for $|\nu| \to \infty$. Representation (5.3) is unique, we have

$$\lambda^{\nu}_{Gm} = \lambda^{\nu}_{Gm}(f) = 2^{\nu\frac{n}{2}}\langle f, \Psi^{\nu}_{Gm}\rangle$$

and $I\colon f \to \{2^{\nu\frac{n}{2}}\langle f, \Psi^{\nu}_{Gm}\rangle\}$ is an isomorphic map from $MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ onto $\tilde{b}^{\omega,\beta}_{p(\cdot),q(\cdot)}$. Moreover, if in addition $\beta < \infty$, then $\{\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^{\nu}, m\in\mathbb{Z}^n}$ is an unconditional basis in $MB^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

(ii) *Let $f \in MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, if and only if, it can be represented as*

$$f = \sum_{\nu=0}^{\infty}\sum_{G\in G^{\nu}}\sum_{m\in\mathbb{Z}^n} \lambda^{\nu}_{Gm} 2^{-\nu\frac{n}{2}}\Psi^{\nu}_{Gm} \text{ with } \lambda \in \tilde{f}^{\omega,\beta}_{p(\cdot),q(\cdot)} \tag{5.4}$$

with unconditional convergence in $\mathcal{S}'(\mathbb{R}^n)$ and in any space $MF^{\zeta,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ with $\frac{\zeta_\nu(x)}{\omega_\nu(x)} \to 0$ for $|x| \to \infty$ and all ν and also $\sup_{x\in\mathbb{R}^n}\frac{\zeta_\nu(x)}{\omega_\nu(x)} \to 0$ for $|\nu| \to \infty$. The representation (5.4) is unique, we have

$$\lambda^{\nu}_{Gm} = \lambda^{\nu}_{Gm}(f) = 2^{\nu\frac{n}{2}}\langle f, \Psi^{\nu}_{Gm}\rangle$$

and $I\colon f \to \{2^{\nu\frac{n}{2}}\langle f, \Psi^{\nu}_{Gm}\rangle\}$ is an isomorphic map from $MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ onto $\tilde{f}^{\omega,\beta}_{p(\cdot),q(\cdot)}$. Moreover, if in addition $\beta < \infty$, then $\{\Psi^{\nu}_{Gm}\}_{\nu\in\mathbb{N}_0, G\in G^{\nu}, m\in\mathbb{Z}^n}$ is an unconditional basis in $MF^{\omega,\beta}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

The proof of Theorem 5.3 is the same as in Theorem 5.2, so, the proof will be omitted.

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