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BIPRODUCTS IN MONOIDAL CATEGORIES

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Abstract. In 2016, Garner and Schäppi gave a criterion for existence of finite biproducts in a specific class of monoidal categories. We provide an elementary proof of (a slight generalization of) their result. Also, we explain how to prove, by using the same technique, an analogous result including infinite biproducts.

1. Introduction

Products and coproducts are among the most important constructions in category theory. Categorical product is a generalization of various notions of product in mathematics (Cartesian product of sets, direct product of vector spaces), while coproduct generalizes the notion of sum (disjoint union of sets, direct sum of vector spaces). Biproduct in a category behaves simultaneously as a product and a coproduct. For example, finite biproducts exist in the category of vector spaces, because direct product and direct sum coincide for finite collections of objects.

Biproducts in a category interact with the rest of its structure. The common case is when the monoidal structure is endowed with biproducts. For example, in homological algebra, abelian groups (as a $\mathbb{Z}$-modules) together with tensor product and direct sum are an example of a monoidal category with biproducts.

Houston in [**3**] gave a criterion for existence of finite biproduct in certain monoidal categories. Precisely, if $(\mathcal{A}, \otimes, I)$ is a monoidal category with finite products and coproducts, such that for every object a of $\mathcal{A}$, the functor $a \otimes \_$ preserves products and the functor $\_ \otimes a$ preserves coproducts, then $\mathcal{A}$ has finite biproducts. An important corollary of this result is that compact closed categories with finite products (or coproducts) necessarily have a biproduct structure.

Garner and Schäppi in [**2**] generalized this result by showing that a monoidal category $(\mathcal{A}, \otimes, I)$ with finite coproducts (including the initial object 0), preserved by the functors $a \otimes \_$, has a zero object and finite biproducts if and only if the initial object 0 and the coproduct $I + I$ have right duals. In the proof they use the fact that a category with finite biproducts is necessarily semi-additive, and one of the

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main ingredients in that proof is the characterization of a semi-additive category by existence of counital comagma structure on each object.

In this paper, we give an elementary proof of the main result of [**2**]. It is elementary in the sense that it involves neither semi-additive structure of a category with biproducts nor its characterization via existence of counital comagmas. Actually, we prove a slightly more general result than the one provided in [**2**]. In fact, the assumption concerning right duals can be a little weakened, i.e., for existence of finite biproducts it is enough that 0 and the coproduct $I + I$ admit only right *semi-duals* (precise coherence conditions will be considered in the next section, see Definition 2.1).

Additionally, we consider an analogous result concerning arbitrary (not only finite) biproducts. Note that a criterion for existence of arbitrary biproducts in semi-additive categories is given in [**4**, Theorem 1.4]. Monoidal categories with infinite biproducts play a significant role in *quantitative models of computation* (see [**5**, Section III] and [**6**, Section 3]). These models refer to semantics of linear logic that interprets proofs as linear maps between vector spaces (or more generally, modules), and standard lambda terms as power series. Infinite biproducts in these models are required for construction of linear logic exponential $!A$. Since quantitative semantics has applications in quantum computing, infinite biproducts find their use in this field also (see e.g. [**8**, Section 4]).

2. Notation and background

In this section we briefly recall some categorial notions relevant for the rest of the paper. A *monoidal category* is a category $(\mathcal{A}, \otimes, I)$ equipped with a unit object I, a bifunctor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and natural isomorphisms

$$\alpha_{a,b,c} : a \otimes (b \otimes c) \to (a \otimes b) \otimes c, \quad \lambda_a : I \otimes a \to a \quad \text{and} \quad \rho_a : a \otimes I \to a,$$

subject to standard coherence conditions, [**7**, Section VII.1]. A monoidal category is *strict* when all the components of α, λ and ρ are identities. Explicitly, the following associativity and unit axioms hold

$$a \otimes (b \otimes c) = (a \otimes b) \otimes c, \qquad a \otimes I = a = I \otimes a,$$
$$f \otimes (g \otimes h) = (f \otimes g) \otimes h, \qquad f \otimes \mathbb{1}_I = f = \mathbb{1}_I \otimes f,$$

for all objects a, b, c and arrows f, g, h of $\mathcal{A}$. Throughout this paper, we denote the coproduct $I + I$ by 2.

An *initial object* of a category $\mathcal{A}$ is an object k in $\mathcal{A}$ such that for every object a in $\mathcal{A}$ there exists a unique arrow $\kappa_a : k \to a$. Dually, an object t is *terminal* if for every object a there exists a unique arrow $\tau_a : a \to t$. A *zero object* is both initial and terminal. If 0 is a zero object, there is a unique arrow $0_{a,b} : a \to 0 \to b$ between any pair of objects a and b.

Let $\mathcal{A}$ be a category with zero object and suppose that $\iota = \{\iota^i : a_i \to a \mid i \in I\}$ is a coproduct cocone in $\mathcal{A}$. Let $\pi = \{\pi^i : a \to a_i \mid i \in I\}$ be a family of arrows such that

$$\pi^i \circ \iota^j = \begin{cases} \mathbb{1}_{a_i}, & i = j, \\ 0_{a_j, a_i}, & i \neq j. \end{cases}$$

If π is a universal cone in $\mathcal{A}$, we say that ι is a *biproduct* in $\mathcal{A}$. Observe that this definition is slightly different than the one given in [**7**, Section VIII.2].

DEFINITION 2.1. Let x be an object in a strict monoidal category $\mathcal{A}$. An object y is called a *right dual* of x if there exist two arrows $\eta : I \to y \otimes x$ and $\varepsilon : x \otimes y \to I$ such that the following *triangular equations* hold

$$(\varepsilon \otimes x) \circ (x \otimes \eta) = \mathbb{1}_x \quad \text{and} \quad (y \otimes \varepsilon) \circ (\eta \otimes y) = \mathbb{1}_y.$$

If only the left triangular equation holds, we say that y is a right *semi-dual* of x.

Let $a + b$ be a coproduct with injections $\iota^1_{a,b} : a \to a + b$ and $\iota^2_{a,b} : b \to a + b$. For arrows $f : a \to c$ and $g : b \to c$ we denote by $[f, g]$ the unique arrow such that $[f, g] \circ \iota^1_{a,b} = f$ and $[f, g] \circ \iota^2_{a,b} = g$. Similarly, for the coproduct $a = \sum_{j\in J} a_j$ of an arbitrary family of objects with injections $\iota^j_a : a_j \to a$ and for arrows $f_j : a_j \to b$, we denote by $[f_j]_{j\in J}$ the unique arrow such that $[f_j]_{j\in J} \circ \iota^k_a = f_k$ for all $k \in J$.

Dually, let $a\times b$ be a product with projections $\pi^1_{a,b}$ and $\pi^2_{a,b}$. Then for arrows $f : c \to a$ and $g : c \to b$ we denote by $\langle f, g\rangle$ the unique arrow such that $\pi^1_{a,b} \circ \langle f, g\rangle = f$ and $\pi^2_{a,b} \circ \langle f, g\rangle = g$. Similarly, for the product $a = \prod_{j\in J} a_j$ of an arbitrary family of objects with projections $\pi^j_a : a \to a_j$ and for arrows $f_j : b \to a_j$, we denote by $\langle f_j\rangle_{j\in J}$ the unique arrow such that $\pi^k_a \circ \langle f_j\rangle_{j\in J} = f_k$ for all $k \in J$. It is not hard to see that for arrows u and v of the appropriate type we have

$$u \circ [f_j]_{j\in J} = [u \circ f_j]_{j\in J}, \quad \langle f_j\rangle_{j\in J} \circ v = \langle f_j \circ v\rangle_{j\in J},$$

and consequently

$$u \circ [f, g] = [u \circ f, u \circ g], \quad \langle f, g\rangle \circ v = \langle f \circ v, g \circ v\rangle.$$

For any set J, the *copower* $J \cdot a$ denotes the coproduct of $|J|$ copies of a, i.e., $J \cdot a = \sum_{j\in J} a$. A functor F is said to *preserve coproducts* if the image under F of a coproduct cocone is always a coproduct cocone.

3. The existence of biproducts

In this section we give certain criteria for existence of finite and infinite biproducts in monoidal categories. We begin with the following lemma.

LEMMA 3.1. *Let $(\mathcal{A}, \otimes, I)$ be a monoidal category with a zero object 0, preserved by the functors $a \otimes \_ : \mathcal{A} \to \mathcal{A}$, for every object a of $\mathcal{A}$. For arrows $f : a \to b$ and $g : c \to d$ of $\mathcal{A}$ we have*

$$f \otimes 0_{c,d} = 0_{a\otimes c,b\otimes d} = 0_{a,b} \otimes g.$$

PROOF. We have that $0_{c,d} = u \circ v$, for the unique arrows $u : 0 \to d$ and $v : c \to 0$. Then $f \otimes 0_{c,d} = (f \otimes u) \circ (a \otimes v)$ for $f \otimes u : a \otimes 0 \to b \otimes d$ and $a \otimes v : a \otimes c \to a \otimes 0$. Since $a \otimes \_$ preserves 0, we have that $a \otimes 0$ is a zero object. Now the uniqueness of the zero arrow between the two objects implies that $f \otimes 0_{c,d} = 0_{a\otimes c,b\otimes d}$. The other part of the equality follows in a similar way. □

Remark 3.1. Let $(\mathcal{A}, \otimes, I)$ be a strict monoidal category with a coproduct 2, preserved by the functors $a \otimes \_ : \mathcal{A} \to \mathcal{A}$ for every object a of $\mathcal{A}$. Then we can define a coproduct $a + a$ to be $a \otimes 2$, and for arrows $f, g : a \to b$ we can define $f + g : a + a \to b + b$ as the unique arrow such that the diagram

$$\begin{array}{ccccc} a & \xrightarrow{a\otimes\iota^1_{I,I}} & a+a & \xleftarrow{a\otimes\iota^2_{I,I}} & a \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f+g} & & \downarrow{\scriptstyle g} \\ b & \xrightarrow[b\otimes\iota^1_{I,I}]{} & b+b & \xleftarrow[b\otimes\iota^2_{I,I}]{} & b \end{array}$$

commutes. In particular, $f + f$ coincides with $f \otimes 2$. Also, it is straightforward to check that for an object c and arrows $u, v : b \to c$ we have

$$[u, v] \circ (f + g) = [u \circ f, v \circ g],$$
$$(u + v) \circ (f + g) = (u \circ f) + (v \circ g),$$
$$c \otimes (f + g) = c \otimes f + c \otimes g.$$

Remark 3.2. If a category $\mathcal{A}$ has an initial object 0, then we can define a coproduct $a + 0$ to be a, since

$$a \xrightarrow{\mathbb{1}_a} a \xleftarrow{\kappa_a} 0$$

is a coproduct diagram (κ_a is the unique arrow from the initial object 0). Similarly, we can define a coproduct $0 + a$ to be a. It follows that for $f : a \to b$ we have $[f, \kappa_b] = f = [\kappa_b, f]$. If 0 is a zero object and τ_a the unique arrow to 0, we can define $\mathbb{1}_a + \tau_a$ to be $[\mathbb{1}_a, \kappa_a \circ \tau_a]$, and similarly, $\tau_a + \mathbb{1}_a$ to be $[\kappa_a \circ \tau_a, \mathbb{1}_a]$.

The following lemma is a key ingredient in the proof of our criterion for existence of finite biproducts in certain class of monoidal categories.

Lemma 3.2. *Let $(\mathcal{A}, \otimes, I)$ be a strict monoidal category with a zero object 0 and a coproduct 2, preserved by the functors $a \otimes \_ : \mathcal{A} \to \mathcal{A}$ for every object a of $\mathcal{A}$. Suppose that 2 has a right semi-dual. Then there exists an arrow $\delta_a : a \to a + a$ such that:*

(1) *for the unique arrow $\tau_a : a \to 0$, we have $(\mathbb{1}_a + \tau_a) \circ \delta_a = \mathbb{1}_a = (\tau_a + \mathbb{1}_a) \circ \delta_a$;*
(2) *for $f : a \to b$ we have $(f + f) \circ \delta_a = \delta_b \circ f$;*
(3) *for a coproduct $a + b$, we have $[\iota^1_{a,b} \circ \pi^1_{a,b}, \iota^2_{a,b} \circ \pi^2_{a,b}] \circ \delta_{a+b} = \mathbb{1}_{a+b}$, where $\pi^1_{a,b} = [\mathbb{1}_a, 0_{b,a}]$ and $\pi^2_{a,b} = [0_{a,b}, \mathbb{1}_b]$.*

Proof. Let d be a right semi-dual of 2, and let $\eta : I \to d \otimes 2$ and $\varepsilon : 2 \otimes d \to I$ be the arrows such that $(\varepsilon \otimes 2) \circ (2 \otimes \eta) = \mathbb{1}_2$. We first define arrows $\chi_i : d \to I$ as $\varepsilon \circ (\iota^i_{I,I} \otimes d)$, for $i \in \{1, 2\}$, and then we define δ_a as $a \otimes ((\chi_1 + \chi_2) \circ \eta)$. Before proving (1)–(3), let us show that

$$(\chi_i + \chi_i) \circ \eta = \iota^i_{I,I}, \quad \text{for } i \in \{1, 2\}. \tag{3.1}$$

Using Remark 3.1 and the triangular equation, we have

$$(\chi_i + \chi_i) \circ \eta = (\chi_i \otimes 2) \circ \eta = (\varepsilon \otimes 2) \circ (\iota^i_{I,I} \otimes d \otimes 2) \circ (I \otimes \eta)$$

$$= (\varepsilon \otimes 2) \circ (\iota^i_{I,I} \otimes \eta) = (\varepsilon \otimes 2) \circ ((2 \circ \iota^i_{I,I}) \otimes (\eta \circ I))$$
$$= (\varepsilon \otimes 2) \circ (2 \otimes \eta) \circ \iota^i_{I,I} = \iota^i_{I,I}.$$

Now we return to the statements of the lemma.

(1) Using Remark 3.1, we have

$$\begin{aligned}(\mathbb{1}_a + \tau_a) \circ \delta_a &= (\mathbb{1}_a + \tau_a) \circ (a \otimes (\chi_1 + \chi_2)) \circ (a \otimes \eta)\\ &= (\mathbb{1}_a + \tau_a) \circ (a \otimes \chi_1 + a \otimes \chi_2) \circ (a \otimes \eta)\\ &= (a \otimes \chi_1 + \tau_a \circ (a \otimes \chi_2)) \circ (a \otimes \eta)\\ &= (a \otimes \chi_1 + \tau_a \circ (a \otimes \chi_1)) \circ (a \otimes \eta),\end{aligned}$$

where the last equality holds because 0 is a terminal object. Now, using Remark 3.1 again and equality (3.1), we have

$$\begin{aligned}(\mathbb{1}_a + \tau_a) \circ \delta_a &= (\mathbb{1}_a + \tau_a) \circ (a \otimes \chi_1 + a \otimes \chi_1) \circ (a \otimes \eta)\\ &= (\mathbb{1}_a + \tau_a) \circ (a \otimes ((\chi_1 + \chi_1) \circ \eta)) = (\mathbb{1}_a + \tau_a) \circ (a \otimes \iota^1_{I,I}) = \mathbb{1}_a.\end{aligned}$$

The other part of the statement (1) follows in a similar way.

(2) From the bifunctoriality of $\otimes$, we have

$$\begin{aligned}(f + f) \circ \delta_a &= (f \otimes 2) \circ (a \otimes ((\chi_1 + \chi_2) \circ \eta))\\ &= (b \otimes ((\chi_1 + \chi_2) \circ \eta)) \circ (f \otimes I) = \delta_b \circ f.\end{aligned}$$

(3) Using previous statements of this lemma, we have

$$\begin{aligned}[\iota^1_{a,b} \circ \pi^1_{a,b}, \iota^2_{a,b} \circ \pi^2_{a,b}] \circ \delta_{a+b} \circ \iota^1_{a,b} &= [\iota^1_{a,b} \circ \pi^1_{a,b}, \iota^2_{a,b} \circ \pi^2_{a,b}] \circ (\iota^1_{a,b} + \iota^1_{a,b}) \circ \delta_a\\ &= [\iota^1_{a,b}, 0_{a,a+b}] \circ \delta_a = [\iota^1_{a,b}, \kappa_{a+b}] \circ (\mathbb{1}_a + \tau_a) \circ \delta_a = \iota^1_{a,b}.\end{aligned}$$

Similarly, $[\iota^1_{a,b} \circ \pi^1_{a,b}, \iota^2_{a,b} \circ \pi^2_{a,b}] \circ \delta_{a+b} \circ \iota^2_{a,b} = \iota^2_{a,b}$, and the claim follows. □

Now we can prove our criterion for existence of finite biproducts in certain type of monoidal categories.

Theorem 3.1. *Let* $(\mathcal{A}, \otimes, I)$ *be a strict monoidal category with an initial object* 0 *and a coproduct* 2*, preserved by the functors* $a \otimes \_ : \mathcal{A} \to \mathcal{A}$ *for every object* a *of* $\mathcal{A}$*. Then the following statements are equivalent:*

(1) 0 *is a zero object, and any finite coproduct that exists in* $\mathcal{A}$ *is a biproduct;*
(2) 0 *is a zero object, and* $2 \otimes 2$ *is a biproduct;*
(3) *objects* 0 *and* 2 *are self-dual;*
(4) *there exists an arrow from* I *to* 0*, and* 2 *has a right semi-dual.*

Proof. (1) ⇒ (2) and (3) ⇒ (4) are obvious. In order to show (2) ⇒ (3) let us first prove that 0 is self-dual. The functor $0 \otimes \_$ preserves the initial object 0, so $0 \otimes 0$ is an initial object and thus isomorphic to 0. Therefore, $0 \otimes 0$ is a terminal object as well. We define $\eta : I \to 0 \otimes 0$ as the unique arrow to the terminal object $0 \otimes 0$, and $\varepsilon : 0 \otimes 0 \to I$ as the unique arrow from the initial object $0 \otimes 0$. Then the equations $(0 \otimes \varepsilon) \circ (\eta \otimes 0) = \mathbb{1}_0$ and $(\varepsilon \otimes 0) \circ (0 \otimes \eta) = \mathbb{1}_0$ hold because $\mathbb{1}_0$ is the only arrow from 0 to 0.

Let us prove that 2 is self-dual. Since $2 \otimes \_$ preserves coproducts, we have that

$$2 \xrightarrow{2\otimes\iota^1_{I,I}} 2 \otimes 2 \xleftarrow{2\otimes\iota^2_{I,I}} 2 \tag{3.2}$$

is a coproduct diagram. Also, we claim that

$$2 \xleftarrow{2\otimes\pi^1_{I,I}} 2 \otimes 2 \xrightarrow{2\otimes\pi^2_{I,I}} 2 \tag{3.3}$$

is a product diagram, where $\pi^1_{I,I} = [\mathbb{1}_I, 0_{I,I}]$ and $\pi^2_{I,I} = [0_{I,I}, \mathbb{1}_I]$. Since $2 \otimes 2$ is a biproduct, it is enough to show

$$(2 \otimes \pi^1_{I,I}) \circ (2 \otimes \iota^1_{I,I}) = \mathbb{1}_2, \qquad (2 \otimes \pi^2_{I,I}) \circ (2 \otimes \iota^2_{I,I}) = \mathbb{1}_2,$$
$$(2 \otimes \pi^1_{I,I}) \circ (2 \otimes \iota^2_{I,I}) = 0_{2,2}, \qquad (2 \otimes \pi^2_{I,I}) \circ (2 \otimes \iota^1_{I,I}) = 0_{2,2}.$$

These equalities follow from the bifunctoriality of $\otimes$, definition of biproduct and Lemma 3.1, so (3.3) is indeed a product diagram.

Now, we can define $\eta : I \to 2\otimes 2$ as $\langle \iota^1_{I,I}, \iota^2_{I,I} \rangle$ (since (3.3) is a product diagram), and $\varepsilon : 2 \otimes 2 \to I$ as $[\pi^1_{I,I}, \pi^2_{I,I}]$ (since (3.2) is a coproduct diagram). Let us check the triangular equations. By Remark 3.1, we have

$$\begin{aligned}(2 \otimes \varepsilon) \circ (\eta \otimes 2) &= (2 \otimes [\pi^1_{I,I}, \pi^2_{I,I}]) \circ (\eta + \eta) = [2 \otimes \pi^1_{I,I}, 2 \otimes \pi^2_{I,I}] \circ (\eta + \eta) \\ &= [(2 \otimes \pi^1_{I,I}) \circ \eta, (2 \otimes \pi^2_{I,I}) \circ \eta] = [\iota^1_{I,I}, \iota^2_{I,I}] = \mathbb{1}_2.\end{aligned}$$

We prove the second triangular equation analogously.

It remains to show (4) $\Rightarrow$ (1). We first prove that 0 is a zero object. Let φ be an arrow from I to 0. For an object a of $\mathcal{A}$, we have the arrow $a \otimes \varphi : a \to a \otimes 0$, and since $a \otimes 0$ is isomorphic to 0, we have an arrow f from a to 0. Let us prove that f is unique. From the bifunctoriality of $\otimes$, we have $(0 \otimes \varphi) \circ f = (f \otimes 0) \circ (a \otimes \varphi)$. Since $a \otimes 0$ is an initial object, the arrow $f \otimes 0$ is equal to the unique arrow $\kappa : a \otimes 0 \to 0 \otimes 0$. Also, the arrow $0 \otimes \varphi$ is invertible since both 0 and $0 \otimes 0$ are initial objects. If we denote the inverse of $0 \otimes \varphi$ by ψ, we have $f = \psi \circ \kappa \circ (a \otimes \varphi)$, so f is obviously unique.

To prove that all finite coproducts are biproducts, it is enough to prove that all binary coproducts are biproducts. Suppose that $a + b$ is a coproduct, and let us show that it is also a product. Projections $\pi^1_{a,b}$ and $\pi^2_{a,b}$ are defined as in Lemma 3.2. For arrows $f : c \to a$ and $g : c \to b$, we define $h : c \to a + b$ with

$$h = [\iota^1_{a,b} \circ f, \iota^2_{a,b} \circ g] \circ \delta_c,$$

where δ_c is the arrow from Lemma 3.2. To show the commutativity of the product diagram, we use Remarks 3.1 and 3.2 and Lemma 3.2, so we have

$$\begin{aligned}\pi^1_{a,b} \circ h &= \pi^1_{a,b} \circ [\iota^1_{a,b} \circ f, \iota^2_{a,b} \circ g] \circ \delta_c = [f, 0_{c,a}] \circ \delta_c \\ &= [f, \kappa_a] \circ (\mathbb{1}_c + \tau_c) \circ \delta_c = [f, \kappa_a] = f.\end{aligned}$$

The equality $\pi^2_{a,b} \circ h = g$ follows in a similar way. To prove the uniqueness of h, let us assume that $h' : c \to a + b$ is another arrow such that $\pi^1_{a,b} \circ h' = f$ and $\pi^2_{a,b} \circ h' = g$. Thus, we have $\iota^1_{a,b} \circ \pi^1_{a,b} \circ h' = \iota^1_{a,b} \circ f$ and $\iota^2_{a,b} \circ \pi^2_{a,b} \circ h' = \iota^2_{a,b} \circ g$, which implies $[\iota^1_{a,b}\circ\pi^1_{a,b}\circ h', \iota^2_{a,b}\circ\pi^2_{a,b}\circ h']\circ\delta_c = h$. By Remark 3.1 and Lemma 3.2, we have

$$\begin{aligned}[\iota^1_{a,b}\circ\pi^1_{a,b}\circ h', \iota^2_{a,b}\circ\pi^2_{a,b}\circ h']\circ\delta_c &= [\iota^1_{a,b}\circ\pi^1_{a,b}, \iota^2_{a,b}\circ\pi^2_{a,b}]\circ(h'+h')\circ\delta_c\\ &= [\iota^1_{a,b}\circ\pi^1_{a,b}, \iota^2_{a,b}\circ\pi^2_{a,b}]\circ\delta_{a+b}\circ h' = h',\end{aligned}$$

so it follows that $h' = h$. □

In the rest of the section we consider a version of Theorem 3.1 that includes arbitrary (not only finite) biproducts. We first adapt our notation and reformulate preliminary results.

In a strict monoidal category $(\mathcal{A},\otimes,I)$ with a copower $J\cdot I$ preserved by the functors $a\otimes\_$ we can define a copower $J\cdot a$ to be $a\otimes(J\cdot I)$. For arrows $f_j : a\to b$, $j\in J$, we can define $\sum_{j\in J} f_j : J\cdot a\to J\cdot b$ as the unique arrow such that

$$\Big(\sum_{j\in J} f_j\Big)\circ(a\otimes\iota^k_{J\cdot I}) = (b\otimes\iota^k_{J\cdot I})\circ f_k$$

holds for every $k\in J$. If $f_j = f_k = f$ for every $j,k\in J$, then we denote $\sum_{j\in J} f_j$ by $J\cdot f$, and it coincides with $f\otimes(J\cdot I)$.

For an object c and arrows $g_j : b\to c$, we have

$$[g_j]_{j\in J}\circ\sum_{j\in J} f_j = [g_j\circ f_j]_{j\in J},$$

$$\Big(\sum_{j\in J} g_j\Big)\circ\Big(\sum_{j\in J} f_j\Big) = \sum_{j\in J} g_j\circ f_j,\quad c\otimes\sum_{j\in J} f_j = \sum_{j\in J} c\otimes f_j.$$

Let $\mathcal{A}$ be a category with an initial object 0, and let $k\in J$ be a fixed index. Then we can define a coproduct

$$\sum_{j\in J} a_j,\quad \text{where}\quad a_j = \begin{cases} a, & j=k,\\ 0, & \text{otherwise,}\end{cases}$$

to be a. For $f : a\to b$, we denote by $[f,\kappa_b]_k$ the following arrow

$$[f_j]_{j\in J} : a\to b,\quad \text{where}\quad f_j = \begin{cases} f, & j=k,\\ \kappa_b, & \text{otherwise,}\end{cases}$$

and we have $[f,\kappa_b]_k = f$ for all $k\in J$. Also, if 0 is a zero object and $\tau_a : a\to 0$, we denote by $(\mathbb{1}_a+\tau_a)_k$ the following arrow

$$\sum_{j\in J} f_j : J\cdot a\to a,\quad \text{where}\quad f_j = \begin{cases} \mathbb{1}_a, & j=k,\\ \tau_a, & \text{otherwise.}\end{cases}$$

Now, along the lines of the proof of Lemma 3.2, one can prove the following analogue of that lemma.

Lemma 3.3. *Let $(\mathcal{A},\otimes,I)$ be a strict monoidal category with a zero object* 0 *and a copower of $J\cdot I$, preserved by the functors $a\otimes\_ : \mathcal{A}\to\mathcal{A}$, for every object a of $\mathcal{A}$. Suppose that $J\cdot I$ has a right semi-dual. Then there exists an arrow $\delta_a : a\to J\cdot a$ such that:*

(1) *for* $\tau_a : a \to 0$ *and for all* $k \in J$ *we have* $(\mathbb{1}_a + \tau_a)_k \circ \delta_a = \mathbb{1}_a$*;*
(2) *for* $f : a \to b$ *we have* $(J \cdot f) \circ \delta_a = \delta_b \circ f$*;*
(3) *for a coproduct* a*, we have* $[\iota_a^j \circ \pi_a^j]_{j\in J} \circ \delta_a = \mathbb{1}_a$*, where* π_a^j *for* $j \in J$ *are defined analogously as in Lemma* 3.2*.*

Finally, using Lemmas 3.1 and 3.3 and performing the same steps as in the proof of Theorem 3.1, one can prove the following analogous result concerning arbitrary biproducts.

THEOREM 3.2. *Let* $(\mathcal{A}, \otimes, I)$ *be a strict monoidal category with an initial object* 0 *and arbitrary copowers of the unit object* I*, preserved by the functors* $a \otimes \_ : \mathcal{A} \to \mathcal{A}$*, for every object* a *of* $\mathcal{A}$*. Then the following statements are equivalent:*

(1) 0 *is a zero object, and arbitrary coproducts that exist in* $\mathcal{A}$ *are biproducts;*
(2) 0 *is a zero object, and for arbitrary copower* $J \cdot I$*, we have that* $(J \cdot I) \otimes (J \cdot I)$ *is a biproduct;*
(3) 0 *and all copowers of* I *are self-dual;*
(4) *there exists an arrow from* I *to* 0*, and all copowers of* I *have right semi-duals.*

We conclude the paper with some examples.

EXAMPLE 3.1. The category **Vect** of vector spaces over the field $\mathbb{R}$ is monoidal with respect to the usual tensor product $\otimes$ of vector spaces. The initial object in **Vect** is the zero vector space $\{0\}$, the unit object is $\mathbb{R}$ and the coproduct is the direct sum $\oplus$. From $V \otimes \{0\} \cong \{0\}$ and $V \otimes (\mathbb{R} \oplus \mathbb{R}) \cong (V \otimes \mathbb{R}) \oplus (V \otimes \mathbb{R})$, we have that the functor $V \otimes \_$ preserves the initial object $\{0\}$ and the coproduct $\mathbb{R} \oplus \mathbb{R}$. Let $\{e_1, e_2\}$ be the basis of $\mathbb{R} \oplus \mathbb{R} = \mathbb{R}^2$. Then, if we define $\eta : \mathbb{R} \to \mathbb{R}^2 \otimes \mathbb{R}^2$ with

$$\eta(1) = e_1 \otimes e_1 + e_2 \otimes e_2,$$

and $\varepsilon : \mathbb{R}^2 \otimes \mathbb{R}^2 \to \mathbb{R}$ with

$$\varepsilon(e_i \otimes e_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

it is not hard to see that the coproduct $\mathbb{R} \oplus \mathbb{R}$ is self-dual. Also, it is easy to see that the zero vector space $\{0\}$ is self-dual. Now Theorem 3.1 implies that $\{0\}$ is a zero object and that **Vect** is a category with finite biproducts.

For an infinite set J, the copower $V = \bigoplus_{j\in J} \mathbb{R}$ is an infinite dimensional vector space. Let us show that V does not have a right semi-dual. Indeed, if we suppose that W is a right semi-dual of V, then we have linear maps $\eta : \mathbb{R} \to W \otimes V$ and $\varepsilon : V \otimes W \to \mathbb{R}$, such that the composition

$$V \xrightarrow{\mathbb{1}_V \otimes \eta} V \otimes W \otimes V \xrightarrow{\varepsilon \otimes \mathbb{1}_V} V \tag{3.4}$$

is equal to $\mathbb{1}_V$. Let V' be the subspace of V spanned by the second component of $\eta(1)$. The subspace V' is finite dimensional, and it contains the image of composition (3.4). Consequently, the image of composition (3.4) is finite dimensional (cf. [**1**, Example 2.10.12]). This contradicts the assumption that composition (3.4) is equal to $\mathbb{1}_V$. Thus, V does not have a right semi-dual, so by Theorem 3.2 we can conclude that the category **Vect** does not have infinite biproducts.

EXAMPLE 3.2. The category **Rel** of sets and binary relations is monoidal with respect to the Cartesian product $\times$ of sets. The initial object in **Rel** is the empty set $\emptyset$, the unit object is any singleton $\{*\}$ and the coproduct is the disjoint union $\sqcup$. From $X \times \emptyset \cong \emptyset$ and $X \times (Y \sqcup Z) \cong (X \times Y) \sqcup (X \times Z)$, we have that the functor $X \times \_$ preserves the initial object $\emptyset$ and all copowers of the unit object $\{*\}$. It is not hard to see that every object X in the category **Rel** is self-dual where the arrow $\eta : \{*\} \to X \times X$ is the relation $\{(*, (x, x)) \mid x \in X\}$ and $\varepsilon : X \times X \to \{*\}$ is its converse $\{((x, x), *) \mid x \in X\}$. In particular, the object $\emptyset$ and all copowers of $\{*\}$ are self-dual. Now from Theorem 3.2 it follows that $\emptyset$ is a zero object and that the category **Rel** has arbitrary (finite and infinite) biproducts.

REMARK 3.3. By Mac Lane's strictification theorem, we know that every monoidal category is equivalent to a strict one (see [**7**, XI.3, Theorem 1]). Note that in the previous examples, we actually considered a strictifications of the usual monoidal categories **Vect** and **Rel**.

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