THE BUCKLING OF THIN CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION*

by

D. M. A. LEGGETT (King's College, London)

SUMMARY — The value of the compressive stress at which a thin circular cylindrical shell becomes unstable was worked out theoretically by Southwell. Subsequent experimental results, however, indicated that this result was appreciably too high, and von Karman and Tsien [4] have shown that a thin cylindrical shell can be maintained in a buckled state by a compressive load considerably smaller than that previously predicted by theory.

The present paper is an extension of the work of von Karman and Tsien, and shows that the smallest load which will keep a thin circular cylindrical shell in a buckled condition is about one third of that given by Southwell.

1. *INTRODUCTION*

The value of the compressive stress at which a thin circular cylindrical shell becomes unstable has been worked out theoretically by Southwell [1]. Subsequent experimental results, however, have indicated that this value is appreciably too high and that the form of distortion which occurs in practice differs from that assumed in theory. In the latter the buckles are alternately inwards and outwards and of equal size, whereas in experiment, buckling is always directed inwards and the buckles are of diamond shape.

In recent years much work has been done on this problem in America, where Lundquist [2] and Donnell [3] have concluded that the buckling of a cylindrical shell is greatly influenced by initial irregularities, and the value obtained by Southwell is therefore only correct for a perfectly formed cylinder. As in practice no cylinder is completely free from

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* This paper was presented at the Sixth International Congress of Applied Mechanics held in Paris in September 1946, but has never been published. Much of the material in this paper has appeared as *A. R. C. R. & M.* № 2190. Leggett & Tones, The behaviour of a cylindrical shell under axial compression when the buckling load has been exceeded.

slight imperfections, they further concluded that the way to obtain a more satisfactory criterion for instability was to investigate the behaviour of the cylinder after buckling. Such an investigation has since been made by von Karman and Tsien [4,5] who have indicated theoretically that a thin cylindrical shell can be maintained in a buckled state by a compressive load considerably smaller than that previously predicted by theory. This paper extends the work of von Karman and Tsien.

2. *STATEMENT OF PROBLEM AND METHOD OF SOLUTION*

The problem investigated is the initial buckling and post buckling behaviour of a thin and unstiffened circular cylindrical shell subjected to uniformly distributed axial compression. It is assumed that the thickness of the cylinder wall $2h$ is constant, and that the ratio of cylinder length l to cylinder radius r, is sufficiently large for the conditions of support at the ends of the cylinder to be unimportant. A further and very important assumption is that the stresses in the shell are always within the elastic range of the material of which the shell is made.

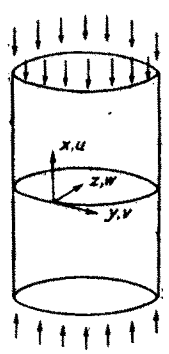

Fig. 1

The generator, line of curvature, and inward drawn normal through a point in the middle surface of the undeformed shell are taken as axes. Referred to these, the coordinate displacements of any point in the middle surface are denoted by u, v, w; and the components of strain, components of stress, and corresponding stress resultants by $\varepsilon_x, \varepsilon_y, \gamma_{xy}$; $\sigma_x, \sigma_y, \tau_{xy}$; and N_x, N_y, N_{xy}.

Referred to these same axes, the equilibrium conditions which the stress resultants must satisfy are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \tag{1}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \tag{2}$$

$$D\nabla^4\omega = N_x \frac{\partial^2 \omega}{\partial x^2} + 2N_{xy}\frac{\partial^2\omega}{\partial x\,\partial y} + N_y\left(\frac{1}{r} + \frac{\partial^2\omega}{\partial y^2}\right), \tag{3}$$

where E is Young's modulus, ν is Poisson's ratio, and D is the flexural rigidity, i. e. $2Eh^3/3\ (1-\nu^2)$. Moreover the following relations hold between the displacements, components of strain, and components of stress,

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial \omega}{\partial x}\right)^2,$$
$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial \omega}{\partial y}\right)^2 - \frac{\omega}{r} \tag{4}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial \omega}{\partial x}\frac{\partial \omega}{\partial y},$$

$$E\,\varepsilon_x = \sigma_x - \nu\sigma_y,$$
$$E\,\varepsilon_y = \sigma_y - \nu\sigma_x, \tag{5}$$
$$E\,\gamma_{xy} = 2(1+\nu)\,\tau_{xy}.$$

I the components of stress are related to a stress function ϕ so that

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}, \quad \tau_{xy} = -\frac{\partial^2\phi}{\partial x\,\partial y}, \quad \sigma_y = \frac{\partial^2\phi}{\partial x^2}, \tag{6}$$

equations (1) and (2) are satisfied identically, and the problem reduces to that of solving, subject to appropriate boundary conditions, the two equations

$$D\,\nabla^4\omega = 2h\left\{\frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\omega}{\partial x^2} - 2\frac{\partial^2\phi}{\partial x\,\partial y}\frac{\partial^2\omega}{\partial x\,\partial y} + \frac{\partial^2\phi}{\partial x^2}\left(\frac{\partial^2\omega}{\partial y^2} + \frac{1}{r}\right)\right\}, \tag{7}$$

$$\nabla^4\phi = E\left\{\left(\frac{\partial^2\omega}{\partial x\,\partial y}\right)^2 - \frac{\partial^2\omega}{\partial x^2}\frac{\partial^2\omega}{\partial y^2}\right\} - \frac{E}{r}\frac{\partial^2\omega}{\partial x^2}. \tag{8}$$

To obtain an exact solution of these two equations is extremely difficult, so recourse is had to the following approximate method which uses a minimum strain energy condition in place of equation (7). A form for ω is chosen, which includes a number of parameters and which is a good approximation to the type of distortion observed in practice. The stress function ϕ is obtained from equation (8), and the parameters are then found by applying the static analogue of Kelvin's minimum energy theorem.

The expression assumed for ω is

$$\omega = A\cos\lambda(y+mx)\cos\lambda(y-mx) + B\{\cos\lambda(y+mx) + \cos\lambda(y-mx)\} + C, \tag{9}$$

where A, B, C, λ and m are parameters to be determined. The reason for choosing this particular form is that, when A and C are zero, (9) reduces to

$$\omega = 2B\cos\lambda mx\cos\lambda y, \tag{10}$$

which represents a system of axial and circumferential waves, and is the theoretically exact form of distortion for the case of initial buckling. When A is equal to B, (9) reduces to

$$\omega = 4B\cos^2\frac{\lambda}{2}(y+mx)\cos^2\frac{\lambda}{2}(y-mx) + \text{a constant}, \tag{11}$$

which represents a system of diamond shaped buckles of a kind which is known to occur when buckling is well developed. The experimental fact that the cylinder always buckles inwards is taken into account by the periodic terms in (11) being squared.

Substituting for ω in (8), a particular integral for ϕ is

$$\begin{aligned} -E\Bigg[& A^2\frac{m^2}{4(1+m^2)^2}\cos 2\lambda mx\cos 2\lambda y + AB\left\{\frac{4m^2}{(1+m^2)^2}\cos\lambda mx\cos\lambda y\right. \\ & \left.+\frac{2m^2}{(9+m^2)^2}\cos\lambda mx\cos 3\lambda y + \frac{2m^2}{(1+9m^2)^2}\cos 3\lambda mx\cos\lambda y\right\} \\ & + B^2\left\{\frac{1}{8m^2}\cos 2\lambda mx + \frac{m^2}{8}\cos 2\lambda y\right\} - \frac{A}{r}\frac{1}{8\lambda^2 m^2}\cos 2\lambda mx \\ & - \frac{B}{r}\frac{2m^2}{\lambda^2(1+m^2)^2}\cos\lambda mx\cos\lambda y\Bigg], \end{aligned} \tag{12}$$

and the complementary function is

$$\begin{aligned} & \frac{1}{2}x^2 p_y - xy\,q_{xy} + \frac{1}{2}y^2 p_x + \\ & + \sum_s \{(A_s\cosh sy + B_s\sinh sy) + y\,(E_s\cosh sy + F_s\sinh sy)\}\cos sx \\ & + \sum_s \{(C_s\cosh sy + D_s\sinh sy) + y\,(G_s\cosh sy + H_s\sinh sy)\}\sin sx, \end{aligned} \tag{13}$$

where p_x, p_y, and q_{xy} are the average values of the direct and shear stresses, and the A's, B's, C's, D's, E's, F's, G's and H's are arbitrary constants.

The expression (13) can now be greatly simplified, for since ϕ must be periodic and the externally applied load consists solely of forces acting parallel to the x axis, p_y is the only constant which does not vanish, From (6) the stresses in the middle surface are as follows:

$$\sigma_x = \frac{E m^2 \lambda^2 A^2}{(1+m^2)^2} \cos 2\lambda mx \cos 2\lambda y + EAB \left\{ \frac{4 m^2 \lambda^2}{(1+m^2)^2} \cos \lambda mx \cos \lambda y \right.$$

$$\left. + \frac{18 m^2 \lambda^2}{(9+m^2)^2} \cos \lambda mx \cos 3\lambda y + \frac{2 m^2 \lambda^2}{(1+9m^2)^2} \cos 3\lambda mx \cos \lambda y \right\}$$

$$+ \frac{E \lambda^2 m^2 B^2}{2} \cos 2\lambda y - \frac{2 E m^2 B}{(1+m^2)^2 r} \cos \lambda mx \cos \lambda y + p_x,$$

$$\tau_{xy} = \frac{E \lambda^2 m^3 A^2}{(1+m^2)^2} \sin 2\lambda mx \sin 2\lambda y + EAB \left\{ \frac{4 \lambda^2 m^3}{(1+m^2)^2} \sin \lambda mx \sin \lambda y \right.$$

$$\left. + \frac{6 \lambda^2 m^3}{(9+m^2)^2} \sin \lambda mx \sin 3\lambda y + \frac{6 \lambda^2 m^3}{(1+9m^2)^2} \sin 3\lambda mx \sin \lambda y \right\}$$

$$- \frac{2 E m^3 B}{(1+m^2)^2 r} \sin \lambda mx \sin \lambda y,$$

$$\sigma_y = \frac{E \lambda^2 m^4 A^2}{(1+m^2)^2} \cos 2\lambda mx \cos 2\lambda y + EAB \left\{ \frac{4 \lambda^2 m^4}{(1+m^2)^2} \cos \lambda mx \cos \lambda y \right.$$

$$\left. + \frac{2 \lambda^2 m^4}{(9+m^2)^2} \cos \lambda mx \cos 3\lambda y + \frac{18 \lambda^2 m^4}{(1+9m^2)^2} \cos 3\lambda mx \cos \lambda y \right\}$$

$$+ \frac{E \lambda^2 B^2}{2} \cos 2\lambda mx - \frac{EA}{2r} \cos 2\lambda mx - \frac{2 E m^4 B}{(1+m^2)^2 r} \cos \lambda mx \cos \lambda y .$$

From (4), (5) and the above expressions for the stresses, it is possible to deduce expressions for u and v. They are, however, extremely cumbersome, so only those terms which are not periodic in x and y are set down in full in what follows:

$$u = x \left\{ \frac{p_x}{E} - (A^2 + 2B^2) \frac{\lambda^2 m^2}{4} \right\} + \text{terms periodic in } x \text{ and } y, \tag{14}$$

$$v = y \left\{ \frac{C}{r} - \frac{\nu p_x}{E} - (A^2 + 2B^2) \frac{\lambda^2}{4} \right\} + \text{terms periodic in } x \text{ and } y. \tag{15}$$

From (14) and (15) there follow two equations. Denoting by e the average overall compressive strain

$$\frac{p_x}{E} - (A^2 + 2B^2) \frac{\lambda^2 m^2}{4} = - e, \tag{16}$$

and since v is essentially periodic

$$\frac{C}{r} - \frac{\nu p_x}{E} - (A^2 + 2B^2)\frac{\lambda^2}{4} = 0. \tag{17}$$

Equation (16) gives an essential relation betwen p_x and e, while (17) determines C.

The next step is to evaluate the mean strain energy W per unit area of middle surface of cylinder. According to well established theory this is given by

$$W = \frac{Eh}{(1-\nu^2)}\int\int\left\{\varepsilon_x^2 + 2\nu\,\varepsilon_x\,\varepsilon_y + \varepsilon_y^2 + \frac{(1-\nu)}{2}\gamma_{xy}^2\right\}dx\,dy + $$
$$+\frac{D}{2}\int\int\left\{\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2\omega}{\partial x^2}\frac{\partial^2\omega}{\partial y^2} - \left(\frac{\partial^2\omega}{\partial x\,\partial y}\right)^2\right\}dx\,dy. \tag{18}$$

Then from (5) and (6), and since

$$\int\int\left[\frac{\partial^2\omega}{\partial x^2}\frac{\partial^2\omega}{dy^2} - \left(\frac{\partial^2\omega}{dx\,dy}\right)^2\right]dx\,dy = 0,$$

W can be expressed in the simplified form

$$\frac{h}{E}\int\int\left\{\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)^2 - 2(1+\nu)\left[\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial y^2} - \left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2\right]\right\}dx\,dy$$
$$+\frac{D}{2}\int\int\left(\frac{\partial^2\omega}{\partial x^2} + \frac{\partial^2\omega}{\partial y^2}\right)^2 dx\,dy. \tag{19}$$

Introducing the non-dimensional quantities α, β, γ, θ, $\bar{e}$ and $\bar{p}_x$, defined by the relations

$$A = \alpha h, \qquad \lambda^2 = \frac{\theta}{rh},$$

$$B = \beta h, \qquad e = \frac{\bar{e}h}{r},$$

$$C = \gamma h, \qquad \frac{p_x}{E} = \frac{\bar{p}_x h}{r},$$

equation (16) is

$$\bar{p}_x - (\alpha^2 + 2\beta^2)\frac{\theta m^2}{4} = -\bar{e}, \tag{20}$$

and the expression (19) is

$$\frac{Eh^3}{r^2}\left\{\theta^2\left[\alpha^4\frac{m^4}{4(1+m^2)^2}+\alpha^2\beta^2\left(\frac{4m^4}{(1+m^2)^2}+\frac{m^4}{(9+m^2)^2}+\frac{m^4}{(1+9m^2)^2}\right)\right.\right.$$

$$\left.+\beta^4\frac{(1+m^4)}{8}+\frac{16}{45}\left(2(1+m^4)\alpha^2+(1+m^2)^2\beta^2\right)\right] \qquad (21)$$

$$\left.-\theta\alpha\beta^2\left[\frac{1}{4}+\frac{4m^4}{(1+m^2)^2}\right]+\left[\frac{\alpha^2}{8}+\beta^2\frac{m^4}{(1+m^2)^2}\right]+\bar{p}_x^{\,2}\right\}$$

where the value 0.25 has been given to ν throughout. Substituting for $\bar{p}_x$ from (20) in (21), W is finally given by

$$W=\frac{Eh^3}{r^2}\left\{\theta^2\left[\alpha^4\left\{\frac{m^4}{16}+\frac{m^4}{4(1+m^2)^2}\right\}\right.\right.$$

$$+\alpha^2\beta^2\left\{\frac{m^4}{4}+\frac{4m^4}{(1+m^2)^2}+\frac{m^4}{(9+m^2)^2}+\frac{m^4}{(1+9m^2)^2}\right\}$$

$$\left.+\beta^4\left\{\frac{1+3m^4}{8}\right\}+\frac{16}{45}\left\{2(1+m^4)\alpha^2+(1+m^2)\beta^2\right\}\right] \qquad (22)$$

$$-\theta\left[\alpha\beta^2\left\{\frac{1}{4}+\frac{4m^4}{(1+m^2)^2}\right\}+\frac{m^2\bar{e}^2}{2}\left\{\alpha^2+2\beta^2\right\}\right]$$

$$\left.+\frac{\alpha^2}{8}+\beta^2\frac{m^4}{(1+m^2)^2}+\bar{e}^2\right\}.$$

Apart from known constants, W is expressed in terms of the imposed compressive strain $\bar{e}$, and the variable parameters α, β, θ and m: it now remains to find these parameters in terms of $\bar{e}$. For a given compressive strain, the static analogue of Kelvin's minimum energy theorem requires the cylinder to take up that particular form which makes the strain energy a minimum. For small changes in the parameters it follows that δW is zero, and hence that

$$\frac{\partial W}{\partial \alpha}=0, \quad (23) \qquad \frac{\partial W}{\partial \theta}=0, \quad (25)$$

$$\frac{\partial W}{\partial \beta}=0, \quad (24) \qquad \frac{\partial W}{\partial m}=0. \quad (26)$$

The detailed solution of these equations is extremely laborious and attention is confined to outlining the method. An essential point to notice is that it is not necessary to solve equations (23) to (25) for α, β, θ and m in terms of $\bar{e}$. To solve for $\bar{e}, \beta, \theta$ and m in terms of α amounts to the same thing and is considerably easier. From (23) to (25) it is possible to eliminate $\bar{e}$ and β and thus obtain an equation in α, θ, and m, and the problem now reduces to finding the value of m which corresponds to any particular value of α. This is done by taking a number of values for m, and substituting the corresponding values of β, θ and $\bar{e}$ in the left hand side of equation (26). After a little trial and error it is possible to find the approximate value of m, and the correct value — that for which the left hand side of (26) vanishes — can then be found by interpolation.

3. *COMPARISON WITH METHOD OF SOLUTION OF VON KARMAN AND TSIEN*

Throughout the first part of this paper, the method used is similar to von Karman and Tsien's; the same basic form is assumed for the distortion of the middle surface, and an analogous expression is obtained for the strain energy. Thereafter, however, the present treatment is more general. For von Karman gives arbitrary values to the two parameters (λ, m) which fix the shape of the buckles, and then determines the two remaining parameters (α, β) which decide the amplitude of the buckles by making the strain energy stationary. Whereas here all four parameters are found by making the strain energy stationary. The importance of this difference in method is due to the fact that the form of distortion assumed for the middle surface is only approximately correct, so that the equation of equilibrium for motion normal to the middle surface is not satisfied exactly. Accordingly, in addition to the uniformly distributed end load, constraining forces must be applied to the curved surface of the cylinder in order to maintain the assumed form of distortion. For a given amount of overall strain, the magnitude and distribution of these constraining forces, which vary with the parameters λ and m, will affect the magnitude of the end load; and von Karman's results suggest that for a given value of the overall strain, the correct values to take for λ and m are those which make the end load a minimum.

From Fig. 5 of von Karman's paper (reproduced here except for the heavy curve as Fig. 4), it can be deduced that this criterion is inconsistent with the condition that the strain energy is stationary, and that it is therefore unsound. The figure referred to shows how, for a given value of m, and for a number of particular values of λ, the load applied to the two ends of the cylinder varies with the average overall strain. There is a different stress strain curve for each λ, and $OACD$ in Fig. 2 represents a typical member of the family, the straight and curved portions referring to the unbuckled and buckled states respectively. In the latter state the buckles are always constrained to be of the same shape (by the action of the constraining forces already mentioned) but corresponding to each value of the overall strain, the amplitudes of the buckles are determined from the condition that the strain energy is stationary. This means that no work is done by the constraining forces as the overall strain is increased, and that the strain energy in the cylinder corresponding to any particular value of the overall strain is given by the area enclosed between the stress strain curve $OACD$, the strain axis and the appropriate ordinate.

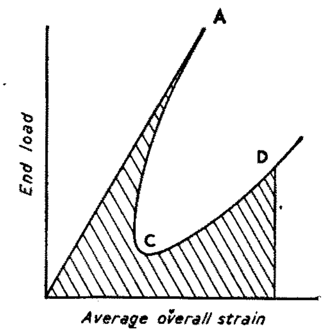

Fig. 2

It is now necessary to consider the physical meaning of the two parameters λ and m. The former is inversely proportional to the number of circumferential buckles, and hence is limited to the set of discrete values λ_r; the later measures the angle of the diamond shaped buckles and so is free to take any value.

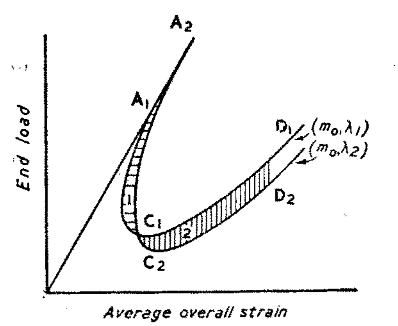

Fig. 3

Fig. 3 shows the two stress strain curves corresponding to (m_0, λ_1) and $(m_0\ \lambda_2)$, where λ_1 and λ_2 are assumed to be successive values of λ. Denoting the strain energy corresponding to (m, λ) by $W\ (m, \lambda)$, it follows that $W\ (m_0, \lambda_1)$ is not equal to $W\ (m_0, \lambda_2)$ for the value of the overall strain at which the two stress strain curves intersect, but for some considerably larger value, corresponding to which the areas 1 and 2 in the figure are equal. This means that for increasing end load the stress strain curve is $A_1\ C_1\ D_1$, because over this range of overall strain $W(m_0\ \lambda_1) < W\ (m_0, \lambda_2)$. But once D_1 is reached, a change takes place. For if the overall strain is increased further, $W(m_0, \lambda_1) > W(m_0, \lambda_2)$, and there will be a sudden change in the type of buckling, indicated in the figure by the 'jumps' $D_1\ D_2$.

It is now possible to form a picture of what happens when the cylinder is constrained to distort into diamond shaped buckles of angle defined by m_0. The final stress strain curve will consist of portions of the stress strain curves in the figure and a number of 'jumps'. On theoretical grounds these 'jumps' occur whenever, for the same overall strain, the strain energies corresponding to two successive values of λ are equal. In this paper the assumption is made that λ is continuous, but owing to the considerable number of circumferential waves into which the cylinder buckles, this is not likely to make any serious difference to the results obtained. In practice, the exact moment when these 'jumps' occur depends upon initial irregula-

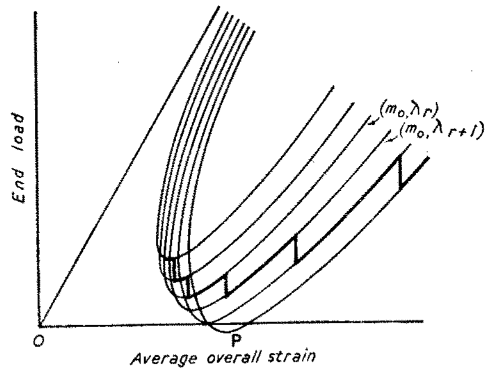

Fig. 4

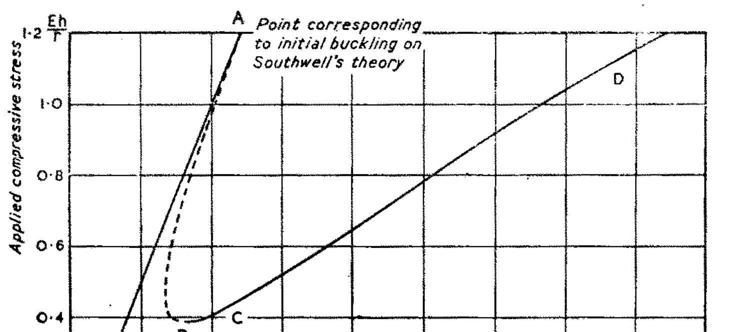

Fig. 5 — Variation of Overall Cempressive Strain with Applied Compressive Stress.

rities and any vibration which may be present, and so varies from one cylinder to another.

The theoretical stress strain curve, in which the 'jumps' are assumed to occur as soon as the strain energies corresponding to successive values of λ are equal, is the curve drawn heavily in Fig. 4; and it is clear from the figure that this is very different from the envelope of the family of stress strain curves. This is because at points on the envelope, the strain energy is not stationary with respect to λ although it is with respect to α and β. So that as the end load is increased, work must continually be done by some external agency other than the uniformly distributed end load. Consequently, the area enclosed between the envelope and the strain axis does not represent the strain energy stored in the cylinder. In particular, the configuration corresponding to the point P in Fig. 4 is not one in which the strain energy is stationary.

From the above argument two points may be gathered. Firstly, on the basis of pure theory and on the assumption that changes in strain are completely controlled, the correct stress strain curve in Fig. 4 is the heavily drawn curve, not the envelope. (The stress strain curve in Fig. 5 corresponds to the heavily drawn curve in Fig. 4, except that m is no longer constant and λ has been regarded as a continuous variable). Secondly, if the stress strain curve in Fig. 4 is to follow the envelope, work must be done by some external agency other than the uniformly distributed end load.

4. *DESCRIPTION OF RESULTS*

In what follows it is important to keep in mind the assumption made throughout this paper that the stresses in the shell remain within the elastic range of the material of which the shell is made. As soon as this is exceeded a new factor is introduced which will modify the results obtained. For most shells of aluminium alloy or steel, theory and experiment indicate that this occurs very soon after buckling.

4.1. *THE RELATION BETWEEN AVERAGE STRESS AND OVERALL STRAIN*

The variation of the average compressive stress and the average overall compressive strain is shown in Fig. 5. Before buckling the stress strain relationship is represented by the straight line OA where A represents the critical stress p_{cr} at which buckling first takes place according to past theory. In practice the buckling stress is considerably less than that given by A, although how much less depends on the extent of initial irregularities and therefore varies from one cylinder to another. For the perfectly formed cylinder which does not buckle until the compressive stress is p_{cr}, the theoretical stress strain curve is $OABCD$. As the part AB does not correspond to any physically possible condition for equilibrium, it is indicated by a broken line in Fig. 5, and the practical stress strain curve consists of the two parts OA, and BCD. The later indicates the very important result that the cylinder can be maintained in a buckled condition of stable equilibrium by a compressive stress which is only one-third of p_{cr}.

The effective stiffness of the cylinder is measured by the gradient of the stress strain curve in Fig. 5. During buckling the conception of stiffness is meaningless, but once C is reached, the gradient is practically constant, and the corresponding stiffness is $E/4$. With increasing strain there is a slight tendency for the stiffness to fall off, but below D this is unimportant.

4.2. *FORM OF DISTORTION AFTER BUCKLING*

When buckling first starts α and γ are zero, and w takes the form

$$2h\,\beta \cos \lambda\, mx \cos \lambda y.$$

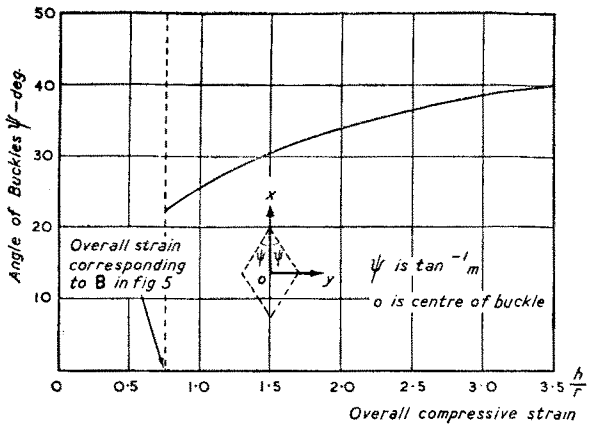

Fig. 6 — **Variation of Angle of Diamond Shaped Buckles with Overall Compressive Strain**

This agrees with past theory and indicates that the cylinder buckles into a system of axial and circumferential waves. During buckling the form of distortion varies rapidly, but once buckling has taken place, changes occur more slowly and subsequent alterations in the parameters are shown in Figs. 6 to 8.

Fig. 7 gives the variation in α, β and γ, and the most significant point here is the ratio of α to β. For, if α/β were unity, w would be

$$4h\,\beta \cos^2 \frac{\lambda}{2}(y + mx) \cos^2 \frac{\lambda}{2}(y - mx)$$

apart from a constant, and this represents a system of diamond shaped buckles of a kind which is known to occur when buckling is far advanced. As shown in Fig. 7 this condition is never quite attained, but once buck-

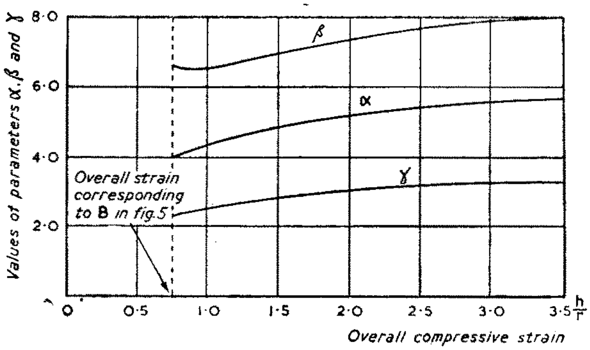

Fig. 7 — Variation of Parametres in w with Overall Compressive Strain.

ling has taken place it is a condition which is gradually approached. Fig. 6 shows how the shape of the diamond buckles change and, in particular,

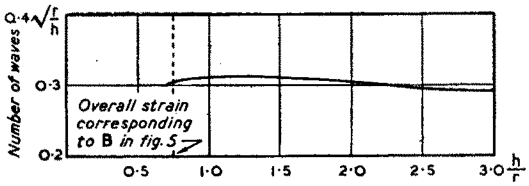

Fig. 8 — Variation in Number of Circumferential Waves with Overall Compressive Strain.

how they start by being very narrow and then become wider as the strain is increased. Fig. 8 gives the number of circumferential waves and indicates that once buckling has taken place any variation is small.

4.3. *COMPARISON WITH EXPERIMENT*

When account is taken of the considerable scatter among the experimental results of Lundquist and Donnell, a comparison between these and the theoretical results obtained in this paper indicates very fair agreement between experiment and theory.

5. *CONCLUSIONS*

Two main conclusions can be drawn from the above results so long as the stresses in the cylinder remain within the elastic range of the material of which it is made. The first is that an axially loaded cylindrical shell can be maintained in a buckled condition by a load which is approximately one-third of the critical compressive load obtained by Southwell. The second is that once the cylinder has buckled it has only one-quarter of its original stiffness.

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